P=W CONJECTURES FOR CHARACTER VARIETIES WITH A SYMPLECTIC RESOLUTION

joint work with Camilla Felisetti

C smooth proj curve / $\mathbb{C}$ of genus g

$G = Gl_n, Sl_n$ (complex reductive group)

$M_B(g, G)$ = BETTI MODULI SPACE or CHARACTER VARIETY $= Hom(\pi_1(C), G) /\!/ G$

Interested in $H^\bullet(M_B(g, G))$

$$M_B(g, G) = \text{BETTI MODULI SPACE or CHARACTER VARIETY} = \mathrm{Hom}(\pi_1(C), G) /\!/ G$$

$$= \left\{ (A_1, B_1, \dots A_g, B_g) \in Gl_n^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = I \right\} /\!/ G$$

$$M_{Dol}(C, G) = \text{DOLBEAULT MODULI SPACE} = \{ G\text{-}\overset{\text{semistable}}{\text{Higgs bundle}} \}$$

for $G = GL_n$ $\quad \left\{ (E, \varphi) \;\middle|\; \begin{array}{l} E \text{ vector bundle of rank } n \text{ and degree } 0 \\ \varphi : E \longrightarrow E \otimes K_C \end{array} \right\}$

for $G = SL_n$ $\quad \{ (E, \varphi) + \det E \simeq \mathcal{O}_C ,\ \mathrm{tr}\,\varphi = 0 \}$

NON-ABELIAN HODGE CORRESPONDENCE

$$
\begin{tikzcd}
\mathcal{M}_{\mathrm{Dol}}(C, G) \arrow[rr, "\psi", "\text{real analytic isomorphism}"'] \arrow[d, "\chi"] & & \mathcal{M}_B(g, G) \quad \text{affine} \\
\mathbb{A}^N & &
\end{tikzcd}
$$

(proper) Hitchin fibration

In general these moduli spaces are singular!

Example $\quad C \simeq \mathbb{C}/_{\mathbb{Z}\langle 1, i\rangle}, \quad g = 1$

$$M_B(1, G\ell_n) = \left\{ (A, B) \in G\ell_n^{\,2} \;\middle|\; \prod [A, B] = 1 \right\} /\!\!/ GL_n$$
$$= (\mathbb{C}^\times \times \mathbb{C}^\times)^{(n)}$$

$$M_{Dol}(C, \mathbb{C}^\times) = \left\{ (E, \varphi) \;\middle|\; \begin{array}{l} E \in Pic^0(C) \\ \varphi \in Hom(E, E \otimes K_C) = H^0(\mathcal{O}_C) \end{array} \right\}$$
$$= C \times \mathbb{A}^1$$

$$M_{Dol}(C, G\ell_n) = \left\{ (E_1, \varphi_1) \oplus \ldots \oplus (E_n, \varphi_n) \right\}$$
$$= (C \times \mathbb{A}^1)^{(n)}$$

Example $C \simeq \mathbb{C}/\mathbb{Z}\langle 1, i\rangle$, $g = 1$

$$\begin{tikzcd}
(C \times \mathbb{A}^1)^{(n)} \arrow[rrr, "\psi"] \arrow[d, "\chi = (pr_2)^{(n)}"'] & & & (\mathbb{C}^\times \times \mathbb{C}^\times)^{(n)} \\
(\mathbb{A}^1)^{(n)} \simeq \mathbb{A}^n
\end{tikzcd}$$

$$\psi : (\theta_1, \theta_2, r_1 + i r_2) \longmapsto (e^{-2r_1} e^{i\theta_1}, e^{2r_2} e^{i\theta_2})$$

$$C \times \mathbb{A}^1 \underset{\text{diffeo}}{\simeq} S^1 \times S^1 \times \mathbb{R} \times \mathbb{R} \underset{\text{diffeo}}{\simeq} \mathbb{C}^\times \times \mathbb{C}^\times$$

P = W CONJECTURE

[de Cataldo, Hausel, Migliorini]

$$
\begin{tikzcd}
H^*(M_{Dol}(C,G)) & H^*(M_B(g,G)) \arrow[l, "\psi^*"'] \\
P_k H^*(M_{Dol}(C,G)) \arrow[u, phantom, sloped, "\subset"] & W_{2k} H^*(M_B(g,G)) \arrow[l, "\simeq"'] \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}
$$

perverse Leray filtration associated to χ

weight filtration

MOTIVATION = mixed Hodge structure on $H^\bullet(M_B^{tw}(g, G))$

TWISTED CHARACTER VARIETIES

$d \in \mathbb{Z}$, $(n, d) = 1$, $L \in \mathrm{Pic}^d(C)$

$$M_B^{tw}(g, G) = \left\{ (A_1, \dots, B_g) \in GL_n^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = e^{\frac{2\pi i}{d}} I \right\} /\!/ G$$

$M_{Dol}^{tw}(C, G) =$ for $G = GL_n$ $\left\{ (E, \varphi) \;\middle|\; \begin{array}{l} E \text{ vector bundle of rank } n \text{ and degree } d \\ \varphi : E \longrightarrow E \otimes K_C \end{array} \right\}$

for $G = SL_n$ $\{ (E, \varphi) + \det E \simeq L \,,\, \mathrm{tr}\, \varphi = 0 \}$

MOTIVATION = mixed Hodge structure on $H^\bullet(M_B^{tw}(g, G))$

CURIOUS HARD LEFSCHETZ [Hausel - Villegas, Mellit]

$\exists\, \alpha \in H^2(M_B^{tw}(g, G))$

$$\cup\alpha^{\ell} : Gr^W_{2d-2\ell} H^\bullet(M_B^{tw}) \xrightarrow{\simeq} Gr^W_{2d+2\ell} H^\bullet(M_B^{tw})$$

RELATIVE HARD LEFSCHETZ (for the Hitchin map χ)

$\exists\, \alpha$ χ - ample

$$\cup\alpha^{\ell} : Gr^P_{d-\ell} H^\bullet(M_{Dol}^{tw}) \xrightarrow{\simeq} Gr^P_{d+\ell} H^\bullet(M_{Dol}^{tw})$$

$$P_k H^\bullet(M_{Dol}^{tw}(C, G)) = \psi^* W_{2k} H^\bullet(M_B^{tw}(g, G))$$

Example

$$
\begin{tikzcd}
C \times \{0\} \arrow[r] \arrow[d, phantom, sloped, "\subset"] & S^1 \times S^1 \arrow[d, phantom, sloped, "\subset"] \\
C \times \mathbb{A}^1 \arrow[r, "\psi"] \arrow[d, "\chi = pr_2"'] & \mathbb{C}^\times \times \mathbb{C}^\times \\
\mathbb{A}^1 &
\end{tikzcd}
\qquad (g(C) = 1)
$$

$$
\begin{array}{c}
Gr^P_2 \, H^2(C \times \mathbb{A}^1) = H^0(R^2 \chi_* \mathbb{Q}) = H^2(\chi^{-1}(s)) = H^2(\underline{C \times \{0\}}) \\
\| \\
Gr^W_4 \, H^2(\mathbb{C}^\times \times \mathbb{C}^\times) = H^0(R^2 j_* \mathbb{Q}) = H^2(\underline{S^1 \times S^1})
\end{array}
$$

with $j : \mathbb{C}^\times \times \mathbb{C}^\times \hookrightarrow \mathbb{C}^2$

⚠ In the untwisted case

RHL and CHL may fail for $H^\bullet(M_B)$!

<u>Example.</u> $E(M_B(2, SL_2)) = \sum_{k,d} \dim(Gr^W_{2k} H^d)\, q^k$

$= 1 + q^2 + 17 q^4 + q^6$

CHL $\Rightarrow$ E is palindromic

⚠ In the untwisted case

RHL and CHL may fail for $H^\bullet(M_B)$!

How to restore the symmetries?

take intersection cohomology $\rightarrow$ $P\,I = W\,I$

resolve singularities $\rightarrow$ $P = W$ for resolution

PI = WI CONJECTURE

[de Cataldo, Maulik]

$$\begin{array}{ccc} IH^*(M_{Dol}(C,G)) & \xleftarrow{\ \psi^*\ } & IH^*(M_B(g,G)) \\ \cup\!\!\mid & & \cup\!\!\mid \\ P_k IH^*(M_{Dol}(C,G)) & \xleftarrow{\ \simeq\ } & W_{2k} IH^*(M_B(g,G)) \end{array}$$

perverse Leray filtration associated to χ — weight filtration

1) RHL holds for $IH^\bullet(M_{Dol})$
2) $P_\bullet$ is independent of the complex structure of C

Thm [FH] (Lift of ψ to a resolution)

There exist

- $f_{Dol} : \widetilde{M}_{Dol} \longrightarrow M_{Dol}$
- $f_B : \widetilde{M}_B \longrightarrow M_B$

} resolution of singularities

- $\widetilde{\psi} : \widetilde{M}_{Dol} \longrightarrow \widetilde{M}_B$ diffeomorphism

such that

$$\begin{tikzcd} \widetilde{M}_{Dol} \arrow[r, "\widetilde{\psi}"] \arrow[d, "f_{Dol}"'] & \widetilde{M}_B \arrow[d, "f_B"] \\ M_{Dol} \arrow[r, "\psi"] & M_B \end{tikzcd}$$

commutes UP TO ISOTOPY.

$$\begin{tikzcd} H^\bullet(\widetilde{M}_{Dol}) \arrow[dr, phantom, "\circlearrowleft"] & H^\bullet(\widetilde{M}_B) \arrow[l, "\widetilde{\psi}^*"'] \\ H^\bullet(M_{Dol}) \arrow[u, "f^*_{Dol}"] & H^\bullet(M_B) \arrow[l, "\psi^*"] \arrow[u, "f^*_B"'] \end{tikzcd}$$

P = W CONJECTURE FOR RESOLUTION

$$P_k H^\bullet(\widetilde{M}_{Dol}) \xleftarrow{\widetilde{\psi}^*} W_{2k} H^\bullet(\widetilde{M}_B)$$

Example

$$\begin{tikzcd}
(\mathbb{C} \times \mathbb{A}^1)^{[n]} \arrow[r, dashed, "\tilde{\psi}"] \arrow[d, "f_{Dol}"'] \arrow[dr, phantom, "\circlearrowright \text{ up to isotopy}"] & (\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]} \arrow[d, "f_B"] \\
(\mathbb{C} \times \mathbb{A}^1)^{(n)} \arrow[r, "\psi"'] \arrow[d] & (\mathbb{C}^\times \times \mathbb{C}^\times)^{(n)} \\
\mathbb{A}^{1\,(n)} &
\end{tikzcd}$$

[Trozzo's PhD th]

$g(C) = 1$

(1) $P = W \iff PI = WI$ since $H^\bullet(X) = IH^\bullet(X)$ if X has quotient singul.

(2) de Cataldo - Hausel - Migliorini shows $PI = WI$ and $P = W$ for resolution

Thm [FM] (Lift of ψ to a resolution)

There exist

- $f_{Dol} : \tilde{M}_{Dol} \longrightarrow M_{Dol}$ $\Big\}$ resolution of singularities
- $f_B : \tilde{M}_B \longrightarrow M_B$
- $\tilde{\psi} : \tilde{M}_{Dol} \longrightarrow \tilde{M}_B$ diffeomorphism

such that

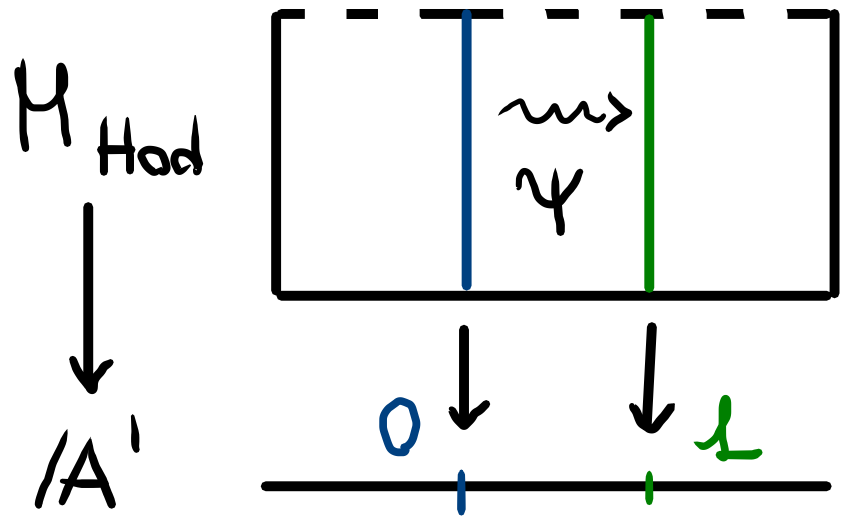
$$\begin{tikzcd} \tilde{M}_{Dol} \arrow[r, "\tilde{\psi}"] \arrow[d, "f_{Dol}"'] & \tilde{M}_B \arrow[d, "f_B"] \\ M_{Dol} \arrow[r, "\psi"] & M_B \end{tikzcd}$$

commutes UP TO ISOTOPY.

Sketch of proof

$\exists$ loc. anal. triv. fibration $M_{Hod} \longrightarrow \mathbb{A}^1$

$M_{Dol} \simeq_{bihol} M_B$



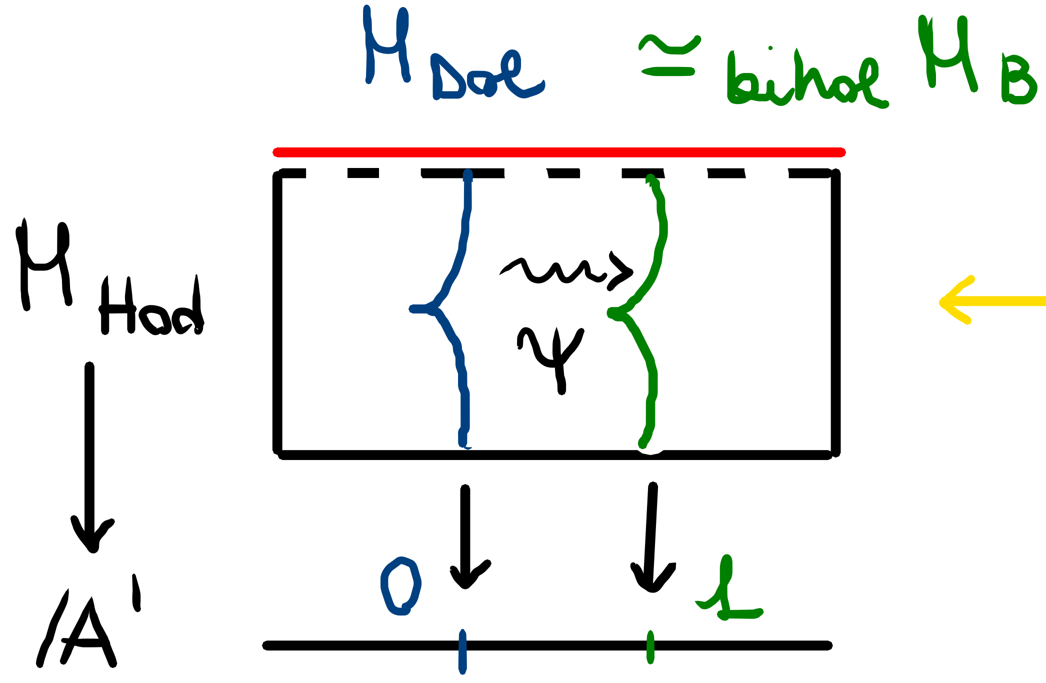
ψ is given by a (topological) trivialization $M_{Hod} \simeq M_{Dol} \times \mathbb{A}^1$

Sketch of proof

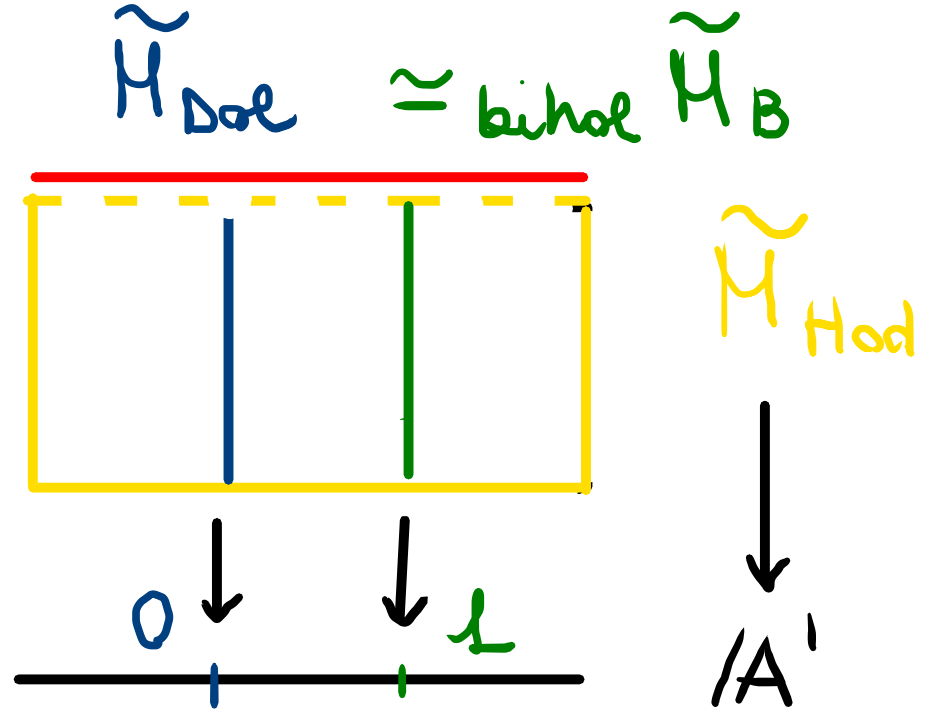
*** RELATIVE COMPACTIFICATION FOR M_{Hod}**

$\exists$ loc. anal. triv. fibration $M_{Hod} \longrightarrow \mathbb{A}^1$

$\exists$ loc. anal. triv. fibration $\widetilde{M}_{Hod} \longrightarrow \mathbb{A}^1$



resolution of singularities



Ψ is given by a (topological) trivialization $M_{Hod} \simeq M_{Dol} \times \mathbb{A}^1$

$\widetilde{\Psi}$ is given by a (topological) trivialization $\widetilde{M}_{Hod} \simeq \widetilde{M}_{Dol} \times \mathbb{A}^1$

MAIN THEOREM

<u>Thm</u> [FM] $P=W$, $PI=WI$, $P=W$ conjecture for resolution hold for character varieties which admit a symplectic resolution, i.e. for $g=1$ and arbitrary rank

$$\boxed{g=2 \text{ and } \operatorname{rank}=2}$$

<u>Def</u>. a resolution $f: \tilde{X} \to X$ is <u>symplectic</u> if a symplectic form on X^{sm} extends to a holomorphic form on $\tilde{X}$.

Rmk

1. Non-trivial evidence for $P=W$

2. A symplectic resolution $\widetilde{M}_{Dol}$
is a degeneration of one of the 4 known
examples of compact hyper-Kähler mfds

Hilbert scheme of K3	OG_{10}	$G = GL_n$
generalized Kummer varieties	OG_6	$G = SL_n$

Rmk ③ A symplectic resolution gives

$$H^\bullet(\widetilde{M}_{\mathrm{Dol}}) = IH^\bullet(M_{\mathrm{Dol}}) \oplus \bigoplus H^\bullet\left(\begin{array}{c}\text{strata of}\\ \text{sing}(M_{\mathrm{Dol}})\end{array}\right)$$

Upshot

$$\boxed{\begin{array}{c} P = W \\ \text{for resolution}\end{array}} = \boxed{P_I = W_I} + \begin{array}{c} P = W \text{ for} \\ \text{lower dim. strata}\end{array}$$

$$\Big\Downarrow \; H^\bullet \subseteq IH^\bullet$$

$$\boxed{P = W}$$

WORKING WITH SINGULARITIES: AN EXTENDED EXAMPLE

From now on $M := M_{Dol}(C, Sl_2)$ $\quad g(C) = 2$

M is a singular 6-fold endowed with actions

① $\Gamma = Pic^\circ(C)[2] \simeq (\mathbb{Z}/2)^4 \curvearrowright M$

$$(\mathcal{L}, (E, \varphi)) \longmapsto (E \otimes \mathcal{L}, \varphi)$$

$$H^\bullet(M) = \underbrace{H^\bullet(M)^\Gamma}_{\text{invariant}} \oplus \underbrace{H^\bullet(M)_{var}}_{\text{variant}}$$

<u>Upshot</u>. $P = W$ for M $\iff$ $\begin{cases} P = W \text{ for invariant part} \\ P = W \text{ for variant part} \end{cases}$

(2) $\mathbb{C}^\times \curvearrowright M$, $(\lambda, (E, \varphi)) \longmapsto (E, \lambda\varphi)$

$$M^{\mathbb{C}^\times} = N \sqcup \bigsqcup_{j=1}^{16} \theta_j$$

$\{(E, 0)\}$ = moduli space of v.b. of rk 2 and deg 0 $\simeq \mathbb{P}^3$

$$H^\bullet(M) = H^\bullet(N) \oplus \bigoplus_{j=1}^{16} H^{\bullet+6}(\theta_j)$$

trivial Γ-module regular Γ-representation

d	0	1	2	3	4	5	6
$\dim H^d(M)^\Gamma$	1	0	1	0	1	0	2

d	0	1	2	3	4	5	6
$\dim H^d(M)^\Gamma$	1	0	1	0	1	0	2

$P = W$ in the twisted case relies on:

$H^\bullet(M)^\Gamma$ is generated in degree ≤ 4

d	0	1	2	3	4	5	6
$\dim H^d(M)^\Gamma$	1	0	1	0	2	0	2

TAUTOLOGICAL CLASSES

⚠ In the twisted case the generators are Künneth components of $c_2(\mathbb{P}(\mathcal{E}))$ where $\mathcal{E}$ is a universal Higgs bundle on $M^{tw} \times C$, i.e.

$$\mathcal{E}\big|_{\{(E,\varphi)\}\times C} \simeq E$$

- no universal bundle [Ramanan]
- $c_2(\mathbb{P}(\mathcal{E})|_{M^{sm}}) = 0$ (main probl: $\operatorname{codim}_M \operatorname{Sing} M = 2$)

d	0	1	2	3	4	5	6
$\dim H^d(M)^\Gamma$	1	0	1	0	1	0	2
$\dim IH^d(M)^\Gamma$	1	0	1	0	2	0	2

<u>Solution</u>: the additional class in $IH^4(M)$ not in $H^\bullet(M)$ is a tautological class of a quasi-étale cover of M

(UNIVERSAL) QUASI-ETALE COVER OF M

$\iota : C \to C$ hyperelliptic involution

M_ι = moduli space of equivariant Higgs bundles

$= \left\{ \begin{matrix} (E, \varphi) \\ \operatorname{rk} E = 2 \\ \deg E = 0 \end{matrix} + h : \begin{tikzcd} E \arrow[r] \arrow[d] & \iota^* E \arrow[d] \\ C \arrow[r, "\iota"'] & C \end{tikzcd} \begin{matrix} \text{lift of the } \iota\text{-action} \\ \text{s.t. } \iota^* h \circ h = id \end{matrix} \right\}$

$$\boxed{\begin{aligned} q : \quad M_\iota &\longrightarrow M \\ (E, \varphi, h) &\longmapsto (E, \varphi) \end{aligned}}$$

GEOMETRY OF M_i

(A) $q : M_i \longrightarrow M$ is a quasi-étale cover branched along $\mathrm{Sing}\, M = \Sigma$

(B) M_i has isolated singularities ($q^{-1}(\Omega)$)

(C) q is the only non-trivial quasi-étale cover of $M(C, Sl_2)$ with $g(C) > 2$.

UNIVERSAL BUNDLE

$\Gamma_i = \langle \Gamma,$ deck transformation of $q : M_i \to M \rangle$

Thm There exists $M_i^\circ \subseteq M_i$ open subset s.t.

- $\operatorname{codim} M_i \setminus M_i^\circ = 3$
- there exists a universal (equivariant Higgs) bundle on $M_i^\circ \times C$

TAUTOLOGICAL CLASS

$$c_2(\mathbb{P}(\mathcal{E})|_{M_i^\circ}) \in H^4(M_i^\circ)^{\Gamma_i}$$

$$\operatorname{codim}_{M_i} \operatorname{Sing} M_i > 4 \quad \left\{ \begin{aligned} &\quad \| \\ &H^4(M_i^{\mathrm{sm}})^{\Gamma_i} \\ &\quad \| \\ &IH^4(M_i)^{\Gamma_i} \end{aligned} \right.$$

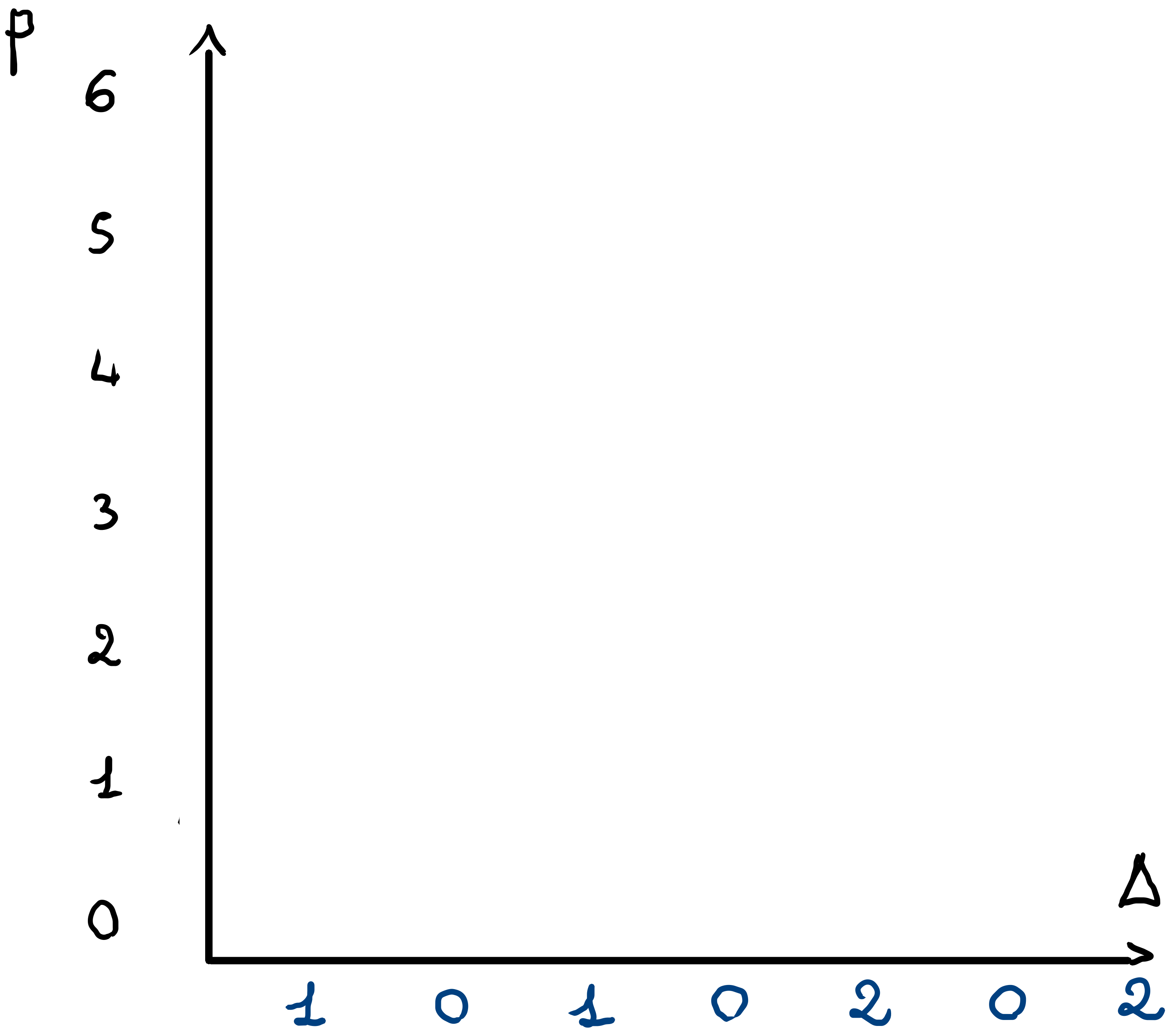
$$\| $$

$$IH^4(M)^{\Gamma}$$

Thm $c_2(\mathbb{P}(\mathcal{E})|_{M_i^\circ})$ has perversity < 4 and $wt = 4$.

PLACING INT. COH. CLASSES IN THE RIGHT PERVERSITY

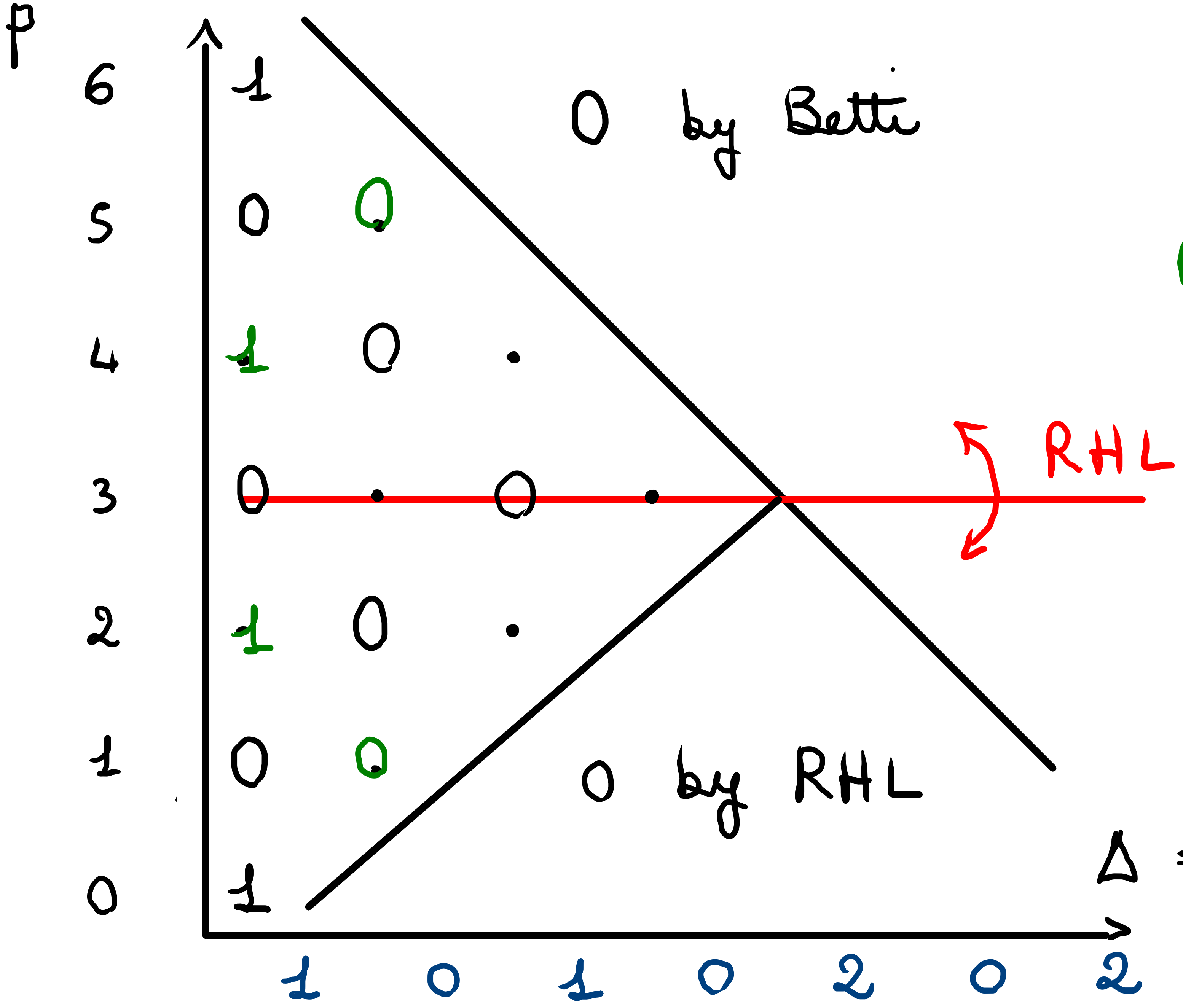
$$p^{\Delta, p} = \dim \mathrm{Gr}^P_p \, IH^{p+\Delta}(M)^\Gamma$$



Δ = column degree $- p$

intersection Betti numbers

PLACING INT. COH. CLASSES IN THE RIGHT PERVERSITY



Δ = column degree $- p$

intersection Betti numbers

$$p^{\Delta, p} = \dim Gr^P_p IH^{p+\Delta}(M)^{\Gamma}$$

(1) $\alpha \in H^2(M)$ is χ-ample

$$P_{d-1} IH^d = \ker\{ H^d(M) \longrightarrow H^d(\chi^{-1}(s)) \}$$

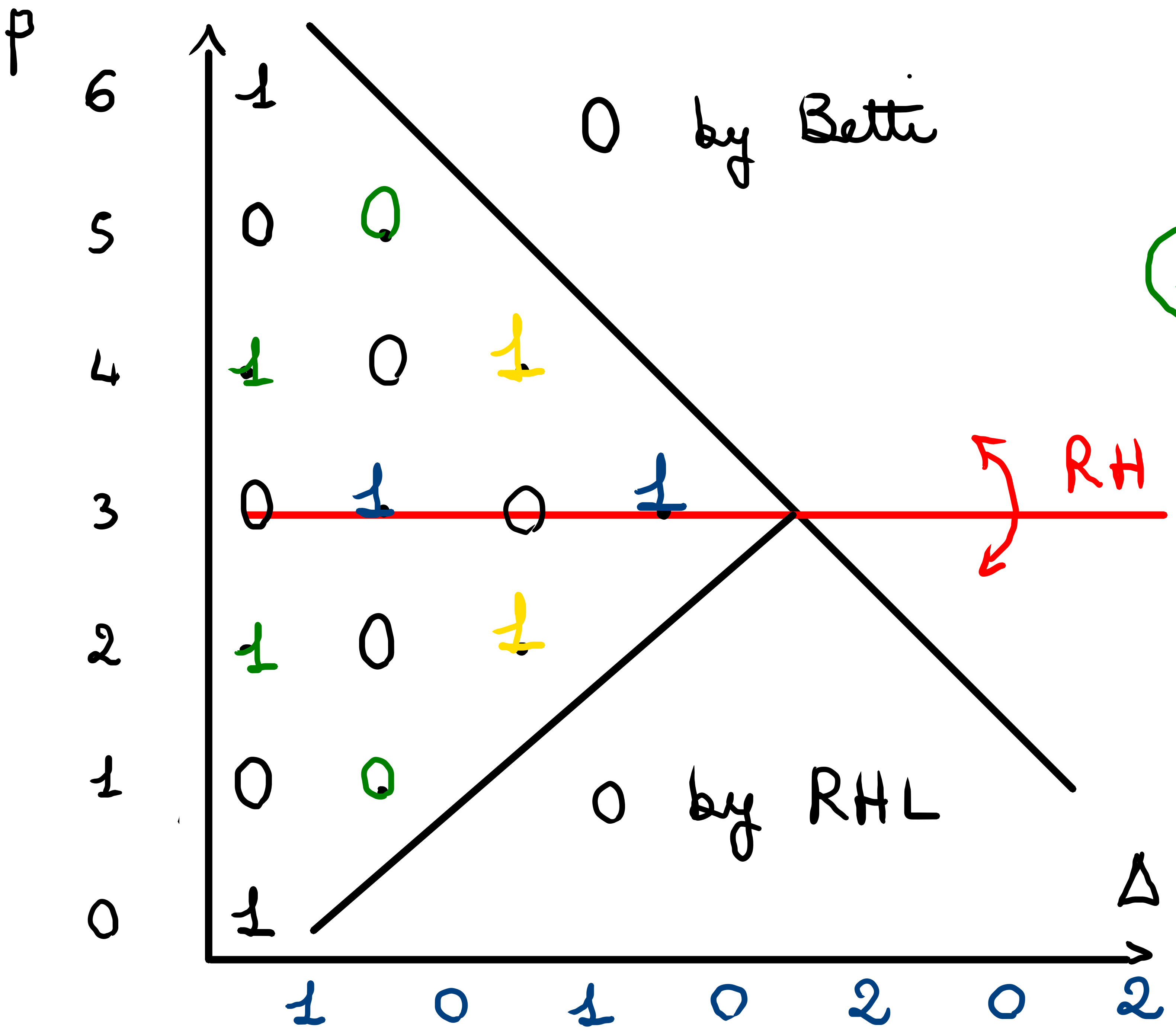
$$\alpha \longmapsto \alpha|_{\chi^{-1}(s)} \neq 0$$

general fiber

$$0 \neq [\alpha] \in Gr^P_d H^d$$

PLACING INT. COH. CLASSES IN THE RIGHT PERVERSITY

$$p^{\Delta, p} = \dim Gr^P_p IH^{p+\Delta}(M)^\Gamma$$



② We are left with 2 options

- either $Gr^P_3 IH^6 = 0$
- or $Gr^P_3 IH^6 \neq 0$

Δ = column degree $- p$

intersection Betti numbers

INTERSECTION FORM

[de Cataldo - Migliorini]

$$\dim \mathrm{Gr}^P_3 H^6(\tilde{M})^\Gamma \leq \text{rank intersection form } H^6_c(\tilde{M})^\Gamma \times H^6_c(\tilde{M})^\Gamma \longrightarrow \mathbb{Q}$$

$$\| $$

$$\dim \mathrm{Gr}^P_3 IH^6(\tilde{M})^\Gamma + 1 \quad \text{since } H^6(\tilde{M})^\Gamma = IH^6(M)^\Gamma \oplus \underbrace{H^2(\Sigma)}_{\text{perversity} > 3} \oplus H^0(\Omega)^\Gamma$$

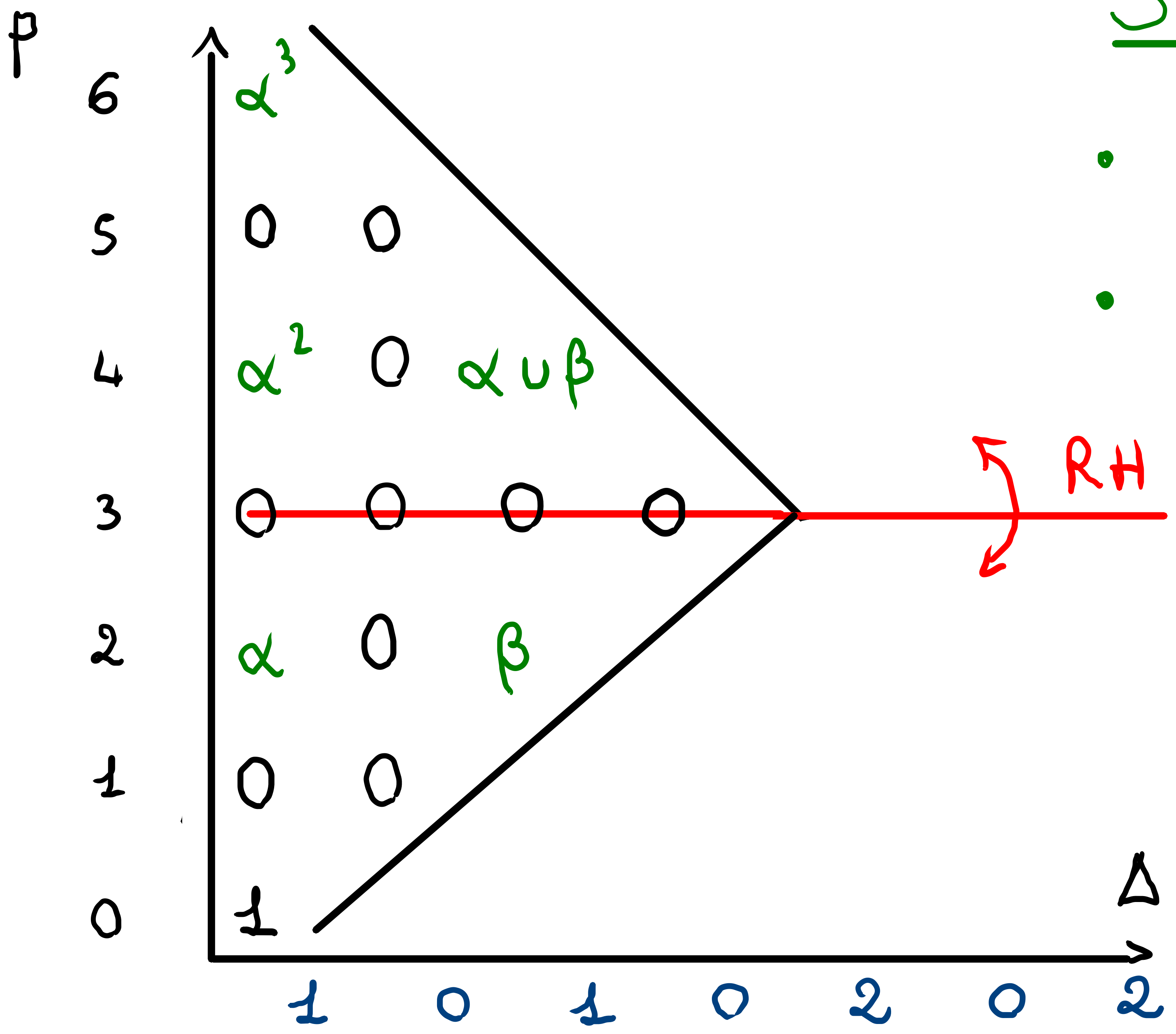
[Poincaré duality] rank inters. $\leq \dim \mathrm{Gr}^W_6 H^6(\tilde{M}) = 1$

$$\Rightarrow \dim \mathrm{Gr}^P_3 IH^6(\tilde{M})^\Gamma = 0$$

<u>Thm</u> [FM] The intersection form on $H^6_c(\tilde{M})^\Gamma$ can be represented by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 16 & -8 \\ 0 & 16 & -64 & 32 \\ 0 & -8 & 32 & -16 \end{bmatrix}$$

in the basis $[(\chi \circ f)^{-1}(0)]$, $[\underset{\substack{\| \\ \text{strict transform of } N}}{\tilde{N}}]$, $[f^{-1}(\Omega)]$, $[f^{-1}(\underset{\substack{\| \\ K}}{\Sigma} \setminus \Omega)]$



Summarizing

- $\beta := c_2(\mathbb{P}(\mathcal{E})|_{M_i^\circ})$ has wt 4
- α has wt 4 (as $H^2 \simeq Gr^W_4 H^2$)

$$\Downarrow$$

PI = WI
(for the invariant part)

Δ = column degree $- p$

intersection Betti numbers

OPEN QUESTIONS

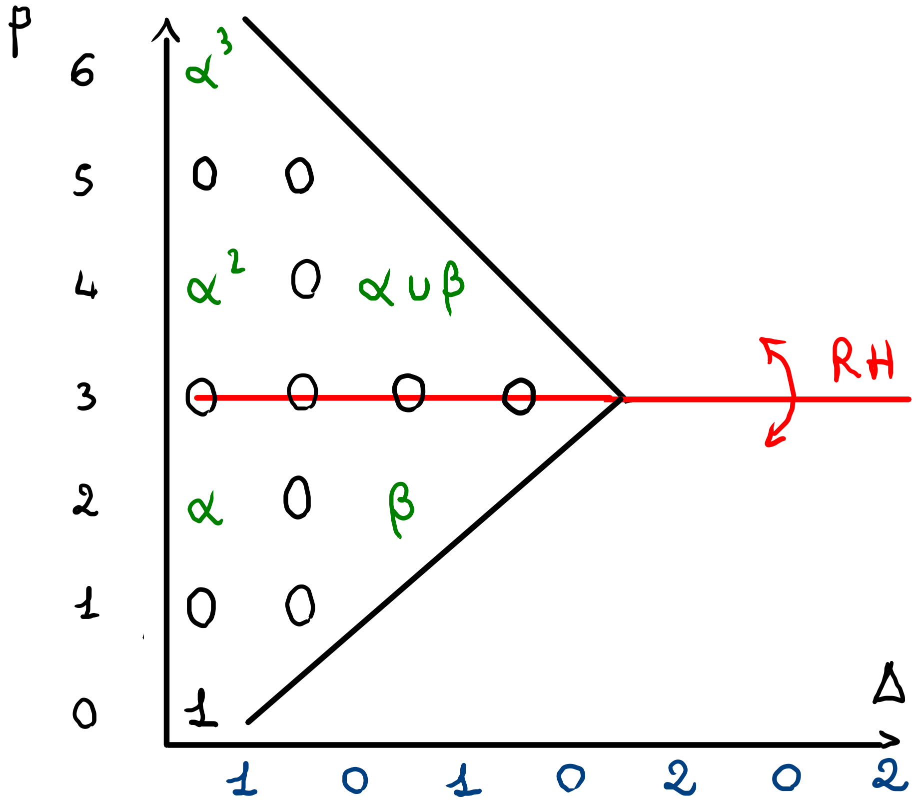
What happens when a symplectic resolution do not exist?

Note that $M_B(g, G)$ have terminal singularities,

and in rank $= 2$ the tautological classes are cohom. classes.

How the tautological ring, $IH^\bullet$ and $H^\bullet$ compare?

Thank you!



p
6
5
4
3
2
1
0
α^3
0 0
α^2 0 $\alpha \cup \beta$
0 0 0 0
RH
α 0 β
0 0
1
Δ
1 0 1 0 2 0 2