LECTURES 3-4: YANG BAXTER ALGEBRAS IN THE CONVOLUTION ALGEBRA

1. Partial flag varieties

Denote by $\mathcal{P}_m(N)$ the set of partions of N with m parts. Let $\lambda \in \mathcal{P}_m(N)$ be a partition and X_λ be the appropriate partial flag variety. Its points are given by flags

$$F_e = (0 \subset F_1 \subset ... \subset F_{m-1} \subset F_m = \mathbb{C}^N)$$

such that $dim(F_i + 1/F_i) = \lambda_i$. There are m natural complex vector bundles M_1,...,M_m over X_λ of the ranks λ_1,...,λ_m respectively.

The action of the torus T on X_λ has a finite set of fixed points $(X_\lambda)^T \subset X_\lambda$. To describe them explicitly let $\{v_1, ..., v_N\}$ be the standard basis in $\mathbb{C}^N$. Let

$$F_e = (0 \subset F_1 \subset ... \subset F_{m-1} \subset F_m = \mathbb{C}^N)$$

be the flag with F_i being the subspace spanned by $\{v_1, ..., v_{\mu_i}\}$, where

$$\mu_i = \sum_{j=1}^{i} \lambda_i.$$

Then F_e is fixed under the action of T. The symmetric group S_N acts on $\mathbb{C}^N$ by permuting the elements of the standard basis, so it acts on the set of all flags. For all $\sigma \in S_N$, $F_\sigma := \sigma(F_e)$ belongs to $(X_\lambda)^T$ and in such a way we get all fixed points. The stabiliser of F_e coincides with the Young subgroup S_λ, so the mapping $\sigma \to F_\sigma$ induces a bijection between S/S_λ and $(X_\lambda)^T$.

Another way to parametrise the fixed points is to identify them with the set of maps $\pi_w : [N] \to [m]$, where $[k]$ is the set $\{1, ..., k\}$. For $w \in S/S_\lambda$ we denote by the same symbol w the corresponding fixed point.

2. Cohomology

Fixed points for us will be a way to introduce the cohomology of flag varieties, namely the T equivariant cohomology. Here is "the definition" of the algebra of cohomology:

Definition 2.1. $H_T(.)$ is a contravariant functor from the category of "nice" algebraic varieties with a T action equivariant maps to the category of algebras over $S := \mathbb{C}[t_1, ..., t_n]$. $H^*(pt) = S$.

The above definition implies that for every T fixed point $x \in X$ the embedding $x \to X$ induces an S module map $H_T X \to H_T(x) = S$.

Example 2.2. Suppose that $\alpha \in Func(\Lambda(n,N), H_T^*(pt))$. Call α a class if it satisfies the following Goresky-Kottwitz-MacPherson (GKM) conditions: For each pair $\mathbf{w}, \mathbf{w}' \in \Lambda(n,N)$ differing only in places i and j (or equivalently, with $\mathbf{w}' = (i \leftrightarrow j)\mathbf{w}$), the difference $\alpha(\mathbf{w}) - \alpha(\mathbf{w}')$ should be a multiple of $t_i - t_j$.

The tangent space $T_w := T(X_\lambda)$ at w inherits the T-action; we will be interested in its Euler class, $\mathcal{E}_w \in H_T^{2d_\lambda}(pt)$, where d_λ is the complex dimension of X_λ. The explicit formula is as follows. Let $\pi_w : [N] \to [m]$ be the map corresponding to w. Then

$$\mathcal{E}_w = \prod_{i>j} \prod_{a \in \pi^1(i),\ b \in \pi^1(j)} (t_a - t_b)$$

Let i_w denote the inclusion $i_w : w \to X_\lambda$ of the fixed point. It is compatible with the T-action. Define the elements $y_w \in H^{2d_\lambda}(X_\lambda)$ as $(i_w)_*(1)$. The explicit formula for them is as follows. For each $i \in [m]$ let y_{ij}, $j \in [\lambda_i]$, denote the Chern roots of M_i the formal symbols such that $c_j(M_i) = \sigma_j(y_{i1}, ..., y_{i\lambda_i})$, the jth elementary symmetric function.

Let π_w be as above. Then

$$y_w = \prod_{i>j} \prod_{a=1}^{\lambda_i} \prod_{b \in \pi^1(j)} (y_{ia} - t_b)$$

Here is the main property of these elements which characterises them:

$$(i_w)^* y_{w'} = \mathcal{E}_w \delta_{w'}^w.$$

The restriction map i_w^* acts as follows: if

$$\pi_w^{-1}(i) = \{k_1, ..., k_{\lambda_i}\}$$

with $k_1 < ... < k_{\lambda_i}$ then

$$(i_w)^*(y_{ij}) = t_{k_j}.$$

The composition $(i_w)^*(i_w)_*$ equals the multiplication by $\mathcal{E}_w$.

3. Subalgebras in the convolution algebra

Here we will be dealing with the partial flag varieties defined by the elements of $\mathcal{P}_2(N)$ and $\mathcal{P}_3(N)$. These are the Grassmanians $Gr(n;N)$ and the three step flag varieties $Gr(n, n+1; N)$.

Introduce the following space

$$G(N) := \coprod_{n=0}^{N} Gr(n, N).$$

It carries a natural action of the group $Gl(n)$ and therefore of its maximal torus T. The diagram

$$\begin{array}{ccccc} & & Gr(n,n+1;N) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ Gr(n;N) & & & & Gr(n+1;N) \end{array}$$

1

defines the standard convolution operators in the $H_T^* Gr(n,N)$

$$b_n(z) = (\pi_2)_\star((\pi_1)^*(z), \;\; c_n(z) = (\pi_1)_\star((\pi_2)^*(z)$$

and therefore of the space $H_T^* G(N)$.

Consider the operators on $H_T^* G(N)$ defined by the multiplication with the Chern classes of the tautological vector bundles $\mathcal{T}_n$ of rank n over $Gr(n;N)$ and the quotient vector bundle $\mathcal{Q}_n$ of rank k over $Gr(n;N)$, $0 \le n \le N-1$.

Introduce the following generating functions

$$\mathcal{A}_n(x) = \sum_{i=0}^{n} (-1)^i c_i(\mathcal{Q}_k) x^{k-i}, \;\; \mathcal{A}'_n(x) = \sum_{i=0}^{k} c_i(\mathcal{T}_k) x^{k-i}$$
$$\mathcal{B}_n(x) = \mathcal{A}_{n+1}(x) b_n, \;\; \mathcal{B}'{}_n(x) = b_n \mathcal{A}'_n(x)$$
$$\mathcal{C}_n(x) = c_n \mathcal{A}_{n+1}(x), \;\; \mathcal{C}'{}_n(x) = \mathcal{A}'_n(x) c_n$$
$$\mathcal{D}_n(x) = c_n \mathcal{A}_{n+1}(x) b_n, \;\; \mathcal{D}'{}_n(x) = b_n \mathcal{A}'_n(x) c_n$$

Consider the following subalgebras of our convolution algebra $\mathcal{C}$:

- YB_N: The subalgebra generated by the coefficients of the polynomials $\mathcal{A}_n(x), \mathcal{B}_n(x), \mathcal{C}_n(x), \mathcal{D}_n(x)$, $0 \le n \le N-1$.
- YB'_N: The subalgebra generated by the coefficients of the polynomials $\mathcal{A}'_n(x), \mathcal{B}'_n(x), \mathcal{C}'_n(x), \mathcal{D}'_n(x)$, $0 \le n \le N-1$.

We will work with the algebra YB_N from now. The algebra YB' can be treated similarly.

4. Yang Baxter algebras again

Theorem 4.1. *YB_N is Yang Baxter algebras defined above with the R matrices and the operators $L(x,t)$ from* `Lectures 1-2`.

Proof. We will prove the statement for the algebra YB by checking all the relations between the generators listed below

$$\begin{aligned} A(x)B(y) &= A(y)B(x), \quad C(x)A(y) = C(y)A(x), \\ B(x)A(y) &- B(y)A(x) = (y-x)A(x)B(y) \\ A(x)C(y) &- A(y)C(x) = (x-y)C(y)A(x) \end{aligned}$$

$$\begin{aligned} (x-y)C(x)B(y) &= A(x)D(y) - A(y)D(x) \\ B(x)C(y) - B(y)C(x) &= (x-y)(D(x)A(y) - A(y)D(x)) \end{aligned}$$

$$\begin{aligned} D(x)B(y) &= D(y)B(x), \quad C(x)D(y) = C(y)D(x) \\ B(x)D(y) &= (x-y)D(x)B(y) + B(y)D(x) \\ D(x)C(y) &= (y-x)C(y)D(x) + D(y)C(x), \end{aligned}$$

in the fixed point basis of $H_T^*G(N)$. This will be a standard fixed point calculation which goes back to the work of Atyih and Bott. We will use the following convinient form of their result from [**?**]. Let M and N be compact oriented manifolds on which T acts, and $f : M \to N$ be a T-equivariant map, then we have a commutative diagram:

Theorem 4.2. [**?**]. *For $a \in H_T(M)$,*

$$(f)_\star a = (i_N^\star)^{-1} \, f_\star^T \, (\frac{(f^T)^\star \, e_N}{e_M} \; i_N^\star \, a) \quad \boxed{\texttt{lian}}$$

where the push-forward and restriction maps are in localized equivariant cohomology and e_N and e_M are the equivariant Euler class of the normal bundle of N_T and M_T.

A fixed point of the T action in $Gr(n; N)$ is represented, as exlained above, by a splitting of the set $[N]$ into two subsets, $\mathbf{w} \subset [N]$ of cardinality n and its complement $C(\mathbf{w}) := [N] \setminus \mathbf{w}$ of cardinality $k = N - n$. We denote such a fixed point by the symbol $\mathbf{w}$.

A fixed point of the T action in $Gr(n, n+1; N)$ is represented by a splitting of $[N]$ into three subsets of cardinalities n, 1, $N - n - 1$. Such a fixed point maps to $\mathbf{w}$ above under π_1 if in the splitting the first subset of cardinality n is exactly $\mathbf{w}$. If the subset of cardinality one is $\{l\}$, $l \in C(\mathbf{w})$ we will denote such a point by $\mathbf{w}_l$.

A fixed point of the T action in $Gr(n+1; N)$ to which $\mathbf{w}_l$ maps is represented by a splitting of $[N]$ into two subsets: $\mathbf{w} \cup l$ of cardinality $n+1$ and its complement in $[N]$.

We have identified earlier the local Euler classes at such fixed points:

$$e_{\mathbf{w}\cup l} = \prod_{j \in C(\mathbf{w}\cup l)} \; \prod_{i \in (\mathbf{w}\cup l)} (t_j - t_i)$$

$$e_{\mathbf{w}_l} = \prod_{j \in C(\mathbf{w}\cup l)} \; \prod_{i \in \mathbf{w}\cup l} (t_l - t_b)(t_j - t_i)(t_j - t_l)$$

Therefore the ratio of these Euler classes which goes into the formula in **??** for the operator $(p_2)_*$ is

$$\frac{e_{\mathbf{w}\cup l}}{e_{\mathbf{w}_l}} = \frac{1}{\prod_{j \in \mathbf{w}} (t_l - t_j)}$$

Similarly the local denominator for the operator $(p_1)_*$ is

$$\frac{e_{\mathbf{w}}}{e_{\mathbf{w}_l}} = \frac{1}{\prod_{j \in C(\mathbf{w}\cup l)} (t_j - t_l)}$$

Recall that the restriction of the Chern class $c_i(Q_n)$ to the fixed point $\mathbf{w}$ is equal to ith elementary symmetric function σ_i in k variables calculated at t_m, $m \in C(\mathbf{w})$, because t_m is the restriction of the Chern root y_m of Q_n to $\mathbf{w}$ as explained above. Likewise the restriction of the Chern class $c_i(T_n)$ to the fixed point $\mathbf{w}$ is equal to ith elementary symmetric function σ_i in n variables calculated at t_m, $m \in \mathbf{w}$, because t_m is the restriction of the Chern root y_m of T_n to $\mathbf{w}$.

Now it is a routine exercise to calculate the matrices of the generators of the algebra YB_N in the fixed point basis.

In this basis the operators $A(x)$ and $A'(x)$ are represented by diagonal matrices with the eigenvalue

$$A(x)(\mathbf{w}) = \prod_{j\in C(\mathbf{w})} (x - t_j)$$

$$A'(\mathbf{w}) = \prod_{j\in \mathbf{w}} (x - t_j)$$

The matrix of the operator $B(x)$ can be calculated as follows: if $\mathbf{w} \subset [N]$, defines a T fixed point in $Gr(n; N)$, then

$$(\pi_1)^{-1}\mathbf{w} = \sum_{l\in C(\mathbf{w})} \mathbf{w}_l$$

Therefore combining it with the above calculation of $(\pi_2)_\star \mathbf{w}_l$ we obtain that in the basis of the fixed points the map $(\pi_2)_\star(\pi_1)^\star$ is given by the matrix with the matrix entry $(\mathbf{w}, \mathbf{w} \setminus l)$ equal to

$$\frac{1}{\prod_{j\in\mathbf{w}}(t_l - t_j)}$$

Putting it together

$$B(x)(\mathbf{w}) = (\pi_2)_\star(\sum_i c_i(Q)x^{k-i}(\pi_1)^\star)(\mathbf{w}) = \sum_{l\in C(\mathbf{w})} \frac{\prod_{i\in C(\mathbf{w}\cup l)}(x - t_i)}{\prod_{j\in\mathbf{w}}(t_l - t_j)}\mathbf{w}^l$$

The matrices of the operators $C(x)$ and $D(x)$ can be obtained easily in the similar way.

We are ready to prove the statement of the theorem.

We split the relations into the three groups for a reason. The fixed point calculation for the equations in each group is similar.

In the first group each identity contains the composition of a diagonal matrix A and either B or C operator. Consider the identities with A and B operators only. These compositions send a fixed point $\mathbf{w}$ to the union of fixed points of the form $\mathbf{w} \cup l)$, $l \in C(\mathbf{w})$. It is easy to see that the denominators of the corresponding matrix entries on the both sides are equal, so it is enough to calculate the numerators of a matrix coefficients $(\mathbf{w}, \mathbf{w} \cup l)$ in the both sides and show that they are equal. For the first identity

$$A(x)B(y) \quad = \quad A(y)B(x)$$

we obtain

$$\prod_{i\in C(\mathbf{w}\cup l)}(x-t_i)(y-t_i) \text{ and } \prod_{i\in C(\mathbf{w}\cup l)}(y-t_i)(x-t_i),$$

hence the identity is proved.

Calculating the numerators of a matrix coefficients $(\mathbf{w},\mathbf{w}\cup l)$ in the both sides of

$$B(x)A(y) \quad - \quad B(y)A(x) = (y-x)A(x)B(y)$$

we obtain:

$$\prod_{j\in C(\mathbf{w})}\prod_{i\in C(\mathbf{w}\cup l)}(y-t_j)(x-t_i) - \prod_{j\in C(\mathbf{w})}\prod_{i\in C(\mathbf{w}\cup l)}(x-t_j)(y-t_i) =$$

$$(y-x)\prod_{i\in C(\mathbf{w})\cup l}(y-t_i)(x-t_i)$$

In the second group of relations involve the compositions of B, C and A, D. Such composition sends a fixed point $\mathbf{w}$ to $(\mathbf{w},\mathbf{w}\cup l\setminus d)$, $l\in C(\mathbf{w})$. The denominares of same matrix coefficients on both sides are equal again, so we need to compare the numerators only. Consider the matrix coefficient $(\mathbf{w},\mathbf{w}\cup l\setminus d)$ in

$$(x-y)C(x)B(y) \quad = \quad A(x)D(y) - A(y)D(x)$$

Calculating the numerators of that matrix coefficients we obtain

$$(x-y)\prod_{i\in C(\mathbf{w}\cup l)}(x-t_i)(y-t_i) =$$

$$\prod_{j\in C(\mathbf{w}\cup l\setminus d)}\prod_{i\in C(\mathbf{w}\cup l)}(x-t_j)(y-t_i) - \prod_{j\in C(\mathbf{w}\cup l\setminus d)}\prod_{i\in C(\mathbf{w}\cup l)}(y-t_j)(x-t_i)$$

The lefthand side of

$$B(x)C(y) - B(y)C(x) \quad = \quad (x-y)(D(x)A(y) - A(y)D(x))$$

at the fixed point $(w\cup l, w\cup d)$ has the numerator

$$\prod_{j\in C(\mathbf{w}\cup d)}\prod_{i\in C(\mathbf{w}\cup l)}(x-t_j)(y-t_i) - \prod_{j\in C(\mathbf{w}\cup d)}\prod_{i\in C(\mathbf{w}\cup l)}(y-t_j)(x-t_i) =$$

$$((x-t_l)(y-t_d) - (y-t_l)(x-t_d))\prod_{j\in C(\mathbf{w}\cup l)\setminus d}(x-t_j)(y-t_j) =$$

$$(x-y)(t_l-t_d)\prod_{j\in C(\mathbf{w}\cup l)\setminus d}(x-t_j)(y-t_j)$$

The righthand side is equal to

$$\prod_{j\in C(\mathbf{w}\cup l\setminus d)}\prod_{i\in C(\mathbf{w}\cup l)}(x-t_j)(y-t_i) - \prod_{j\in C(\mathbf{w}\cup l\setminus d)}\prod_{i\in C(\mathbf{w}\cup l)\setminus d}(y-t_i)(x-t_j) =$$

$$(t_l-t_d)\prod_{j\in C(\mathbf{w}\cup l\setminus d)}(x-t_j)(y-t_j)$$

Finally the last group of the relations. Consider the first relation

$$D(x)B(y) \quad = \quad D(y)B(x)$$

The operators from this identity add two elements to the set representing the fixed point and then removes one from the resulting set. Specifically let $\mathbf{w}$ be a fixed point. The operator B adds an element $l \in C(\mathbf{w})$ to $\mathbf{w}$. The operator D adds further an element $d \in C(\mathbf{w} \cup l)$ to $\mathbf{w} \cup l$ and removes some p from $\mathbf{w} \cup \{l \cup d\}$. Denote the result $\mathbf{w}(l,d,p) := \mathbf{w} \cup \{l \cup d\} \setminus p$. Let us calculate the matrix coefficient $(\mathbf{w}, \mathbf{w}(l,d,p))$ of the composition DB. There is two ways to get from $\mathbf{w}$ to $\mathbf{w}(l,d,p)$, first add l to $\mathbf{w}$ then add d and remove p, or add d to $\mathbf{w}$ and then add l and remove p. Therefore this matrix coefficient will be a sum of two summands. The denominators of these summands differ by a sign, because l and d are added to $\mathbf{w}(l,d,p)$ in a different order so the sum is in fact a differences of two fractions with the same demoninators. Calculating the numerators on the both sides of the first identity

$$\prod_{j \in C(\mathbf{w} \cup \{l,d\})} \prod_{i \in C(\mathbf{w} \cup l)} (x - t_j)(y - t_i) - \prod_{j \in C(\mathbf{w} \cup \{d,l\})} \prod_{i \in C(\mathbf{w} \cup d)} (x - t_i)(y - t_j) =$$

$$(t_d - t_p) \prod_{j \in C(\mathbf{w} \cup \{l,d\})} (x - t_j)(y - t_j) =$$

$$\prod_{j \in C(\mathbf{w} \cup \{l,d\})} \prod_{i \in C(\mathbf{w} \cup l)} (y - t_j)(x - t_i) - \prod_{j \in C(\mathbf{w} \cup \{d,l\})} \prod_{i \in C(\mathbf{w} \cup d)} (y - t_i)(x - t_j)$$

The last two relations to prove are

$$\begin{aligned} B(x)D(y) &= (x - y)D(x)B(y) + B(y)D(x) \\ D(x)C(y) &= (y - x)C(y)D(x) + D(y)C(x), \end{aligned}$$

Above we have calculated the matrix coefficient $(\mathbf{w}, \mathbf{w}(l,d,p))$ of the composition $D(x)B(y)$. Now we calculate this coefficent for the composition

$$B(x)D(y) - B(y)D(x)$$

We need to calculate the numerator only.

$$\prod_{j \in C(\mathbf{w} \cup \{l,d\} \setminus p)} \prod_{i \in C(\mathbf{w} \cup l)} (x - t_j)(y - t_i) - \prod_{j \in C(\mathbf{w} \cup \{l,d\} \setminus p)} \prod_{i \in C(\mathbf{w} \cup l)} (y - t_j)(x - t_i) =$$

$$((x - t_d)(y - t_p) - (y - t_d)(x - t_p)) \prod_{j \in C(\mathbf{w} \cup \{l,d\})} (x - t_j)(y - t_j) =$$

$$(x - y)(t_d - t_p) \prod_{j \in C(\mathbf{w} \cup \{l,d\})} (x - t_j)(y - t_j)$$

The the identities involving the operators C and D can be proved simirlaly. □

References

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