LECTURES 1-2. YANG BAXTER ALGEBRAS.

1. Diagrammatics

In this section we recall the diagrammatics used in the theory of quantum integrable systems. Although this theory is rather standard in the physics literature, we feel it is much less known to mathematicians. We therefore develop it here in quite detail in a way adapted to our situation, but refer also to [**?**] and [**?**].

1.1. **Lax matrices and monodromy matrices.** In the following we consider $\mathbb{C}^2$ with a fixed basis v_0, v_1 and denote by $\mathrm{Mat}_2(\mathbb{C}[x,t])$ the vector space of 2×2-matrices with entries in the polynomial ring $\mathbb{C}[x,t]$ in variables x, t.

We then pick a matrix $L(x,t) \in \mathrm{Mat}_2(\mathrm{Mat}_2(\mathbb{C}[x,t]))$, usually called *Lax matrix*, that is

$$L(x,t) \quad = \quad \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix} \tag{1}$$

with entries in $\mathrm{Mat}_2(\mathbb{C}[x,t])$. We then view the operators $A(x,t)$, $B(x,t)$, $C(x,t)$, $D(x,t)$ as $\mathbb{C}[x,t]$-linear operators acting on $\mathbb{C}^2[x,t] := \mathbb{C}^2 \otimes \mathbb{C}[x,t]$ in the basis v_0, v_1.

Attached to $L(x,t)$ we have the *monodromy matrix*

$$M_N(x,t_1,t_2,\dots,t_N) = L(x,t_1)\overline{\otimes}L(x,t_2)\overline{\otimes}\cdots\overline{\otimes}L(x,t_N)$$

where $\overline{\otimes}$ denotes the *Kronecker product* of matrices defined as follows: Given $L_1, L_2 \in \mathrm{Mat}_2(\mathbb{C}[x,t])$ then $L_1\overline{\otimes}L_2$ is obtained by using the ordinary formulas for the product of matrices, i.e.

$$L(x,t_1)\overline{\otimes}L(x,t_2) = \begin{pmatrix} A(x,t_1) & B_1(x,t_1) \\ C(x,t_1) & D(x,t_1) \end{pmatrix}\begin{pmatrix} A(x,t_2) & B(x,t_2) \\ C(x,t_2) & D(x,t_2) \end{pmatrix} =$$

$$\begin{pmatrix} A(x,t_1)\otimes A(x,t_2) + B_1(x,t_2)\otimes B(x,t_2) & A(x,t_1)\otimes B(x,t_2) + B_1(x,t_2)\otimes D(x,t_2) \\ C(x,t_1)\otimes A(x,t_2) + D(x,t_2)\otimes C(x,t_2) & C(x,t_1)\otimes B(x,t_2) + D(x,t_2)\otimes D_2(x,t_2) \end{pmatrix}$$

Therefore, $M_N(x,t_1,t_2,\dots,t_N)$ is a block 2×2-matrix with blocks of size $2^N \times 2^N$,

$$M_N(x,t_1,t_2,\dots,t_N) \quad = \quad \begin{pmatrix} A_N(x,t_1,t_2,\dots,t_N) & B_N(x,t_1,t_2,\dots,t_N) \\ C_N(x,t_1,t_2,\dots,t_N) & D_N(x,t_1,t_2,\dots,t_N) \end{pmatrix} \tag{2}$$

We will view these blocks as matrices of linear operators acting on

$$\begin{aligned} V^N &:= \mathbb{C}^2[x,t_1] \otimes_{\mathbb{C}[x]} \mathbb{C}^2[x,t_2] \otimes_{\mathbb{C}[x]} \cdots \otimes_{\mathbb{C}[x]} \mathbb{C}^2[x,t_N] \\ &= (\mathbb{C}^2)^{\otimes n}[x,t_1,t_2,\dots,t_N] \end{aligned}$$

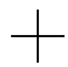
in the standard $\mathbb{C}[x,t_1,t_2,\dots,t_N]$-basis

$$v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_N}, \quad \mathit{where} \quad \epsilon_j \in \{0,1\}. \tag{3}$$

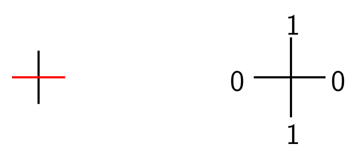
ex **Example 1.1.** Our major examples of Lax matrices are the following

$$L_1(x,t) = \begin{pmatrix} x-t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L_2(x,t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x+t & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

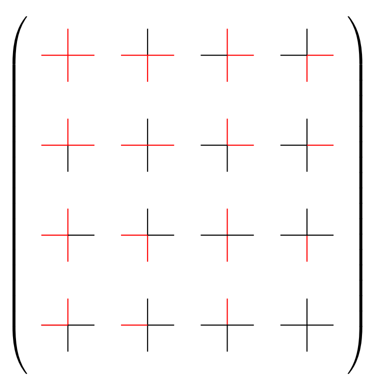
Now we introduce a diagrammatics for calculating the matrix entries of (2) and algebraic relations between them. For this we first identify the standard basis vectors (3) with $\{0,1\}$-words $\epsilon_1\epsilon_2\cdots\epsilon_N$ of length N. A *crossing* is a diagram of the form



built up from 4 line segments, called *edges* meeting in a point. The edge on the right is called *first input*, the bottom the *second input*, the edge on the left the *first output* and the edge on the top *second output*. A *coloured crossing* is a crossing, where all the four edges are labelled by an element from $\{0,1\}$ which we display by colouring the edges with colour red if it is labelled 0 and black if it is labelled 1. For example the following represents the same coloured crossing with inputs red and black and outputs red and black.



The 16 possibilities are displayed in the matrix below.


(5)

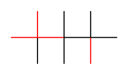
Our given Lax matrix $L(x,t)$ assigns to each labelled crossing a *weight* by assigning to the cross in position i,j above the (i,j)-entry of the matrix $L(x,t)$. For instance the weight of the cross labelled only with 0's equals $x-t$ for $L_1(x,t)$ and equals 1 for $L_2(x,t)$ in case of the Lax matrices from Example **??**.

1.2. **Calculating matrix entries.** It is not hard to check that the matrix entries for the operators

$$A(x, t_1, t_2, \ldots, t_N), B(x, t_1, t_2, \ldots, t_N), C(x, t_1, t_2, \ldots, t_N), D(x, t_1, t_2, \ldots, t_N)$$

can be calculated as follows:

Consider a 1-*row lattice of length* N, that is a diagram obtained by putting N coloured crossing next to each other, and label each external edge by an element from $\{0, 1\}$. We call the resulting diagram a *lattice diagram*:



The *weight* of a lattice diagram is the product of weights of its labelled crosses. This is a polynomial in $\mathbb{C}[x, t_1, t_2, \ldots, t_N]$ of total degree at most N.

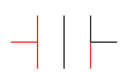
The top set of labels defines a $\{01\}$ word and therefore a vector $v_t \in V^N$ likewise the bottm set of labels defines $v_b \in V^N$. Let us assign to each of the four possible colorings of the two horisontal edges one of our operators as follows:

$$A(x, t_1, t_2, \ldots, t_N) \leftrightarrow \{0, 1\},\ B(x, t_1, t_2, \ldots, t_N) \leftrightarrow \{1, 0\},$$

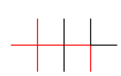
$$C(x, t_1, t_2, \ldots, t_N) \leftrightarrow \{0, 1\},\ D(x, t_1, t_2, \ldots, t_N) \leftrightarrow \{1, 1\}.$$

Proposition 1.2. *Let O be any of the four operators above and $v_t, v_b \in V^N$. The coefficient of the expantion of $O(v_t)$ in front of v_b is equal to the sum of the weights of the lattice diagrams whose external edges labeling is fixed and defined by O, v_t and v_b.*

Example 1.3. Let us calculate the coefficient of $C_3(011)$ defined by $L_1(x, t)$ at 010. The labeling of the external edges therefore is



The only lattice diagram with non zero weight is



and its weight is equal to $x - t_1$.

1.3. **Yang Baxter equation.**

Definition 1.4. Two matrices $R(x, y) \in \mathrm{M}_2(\mathrm{M}_2(\mathbb{C}[x, y]))$ and $L(x, t) \in \mathrm{M}_2(\mathrm{M}_2(\mathbb{C}[x, t]))$ are said to a solution to the Yang Baxter equation if the following holds

$$R^{12}(x, y) L^{13}(x, t) L^{23}(y, t) = L^{23}(y, t) L^{13}(x, t) R^{12}(x, y)$$

and in $V[x, y] \otimes V[t] \otimes V[t]$. The superindeces indicate the factors on which the appropriate operators act.

Proposition 1.5. *If the matrices $R(x, y)$ and $L(x, t)$ give a solution to the Yang Baxter equation then the following identity holds for any N:*

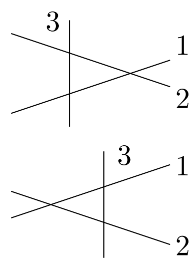
$$R^{12}(x, y)M_N^1(x, t_1, ..., t_N)M_N^2(y, t_1, ..., t_N) = M_N^2(y, t_1, ..., t_N)M_N^1(x, t_1, ..., t_N)R^{12}(x, y)$$

Introduce the following matrices

$$R_1(x, y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & y-x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_2(x, y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x-y & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (6)$$

yangbaxter **Proposition 1.6.** *The pairs $(R_1(x, y), L_1(x, t))$ and $(R_2(x, y), L_2(x, t))$ define solutions to the Yang Baxter equation.*

Using the diagrammatics we introduced above it is easy to write the identities between the matrix entries of the above block 4×4 matrices with blocks of the size 2×2. Consider two types of diagrams pictured below.



By a labeled diagram as before we mean a diagram with $\{0, 1\}$ labels attached to all the edges and the weight of such a diagram is again defined as the product of the weights attached to the vertices. The rule for attaching weights is as follows: the crossing of the lines labeled by 1 and 2 gets weights from the matrix $R[x - y]$, the crossing defined by the lines labeled by 1 and 3 gets weights from the matrix $L[x]$ and the crossing defined by the lanes 2 and 3 gets weights from the matrix $L[y]$. The numbering of the lines which make a crossing defines the order of the inputs and outputs and therefore attaches a weight to every vertex of a labeled diagram uniquely.

Proposition 1.7. *Two matrices $R(x, y)$ and $L(x, t)$ whose entries are identified with labeled crossings as above give a solution to the Yang Baxter equation if the sums of the weights of of all the labeled diagrams with the fixed labeling of the external edges calculated for the two diagrams above are equal.*

Now the proof of **??** is an easy exercise.

1.4. **Yang Baxter algebra.** Define now the main object of our studies . Let as before $O(x)$ to be one of our four operators. Expand it as a polynomial of x

$$O(x) = \sum_i O^i x^i$$

with coefficients in $End\, V^N[t_1, t_2, \ldots, t_N]$.

Definition 1.8. Define an algebra YB_N as the subalgebra of $End\, V^N[t_1, t_2, \dots, t_N]$ generated by the operators

$$A^i_N(t_1, t_2, \dots, t_N),\ B^i_N(t_1, t_2, \dots, t_N),$$
$$C^i_N(t_1, t_2, \dots, t_n),\ D^i_N(t_1, t_2, \dots, t_N)$$

Proposition 1.9. *The identity defines relations in the algebra YB_N*

$$R^{12}(x, y)M^1(x, t)M^2(y, t) = M^2(y, t)M^1(x, t)R^{12}(x, y)$$

Written in terms of the denerators introduced above this amounts to the following

$$\begin{aligned}
C(x)A(y) &= C(y)A(x), \quad A(x)B(y) = A(y)B(x),\\
D(x)B(y) &= D(y)B(x), \quad C(x)D(y) = C(y)D(x),\\
A(x)A(y) &= A(y)A(x), \quad B(x)B(y) = B(y)B(x),\\
D(x)D(y) &= D(y)D(x), \quad C(x)C(y) = C(y)C(x),
\end{aligned}$$

and

$$\begin{aligned}
A(x)C(y) &= (x-y)C(y)A(x) + A(y)C(x)\\
B(x)A(y) &= (y-x)A(x)B(y) + B(y)A(x)\\
B(x)D(y) &= (x-y)D(x)B(y) + B(y)D(x)\\
D(x)C(y) &= (y-x)C(y)D(x) + D(y)C(x)
\end{aligned}$$

$$\begin{aligned}
(x-y)C(x)B(y) &= A(x)D(y) - A(y)D(x)\\
B(x)C(y) - B(y)C(x) &= (x-y)(D(x)A(y) - A(y)D(x))
\end{aligned}$$

Remark 1.10. There are sixteen relations following from the Yang Baxter equation, the ones listed above imply the rest.

Note that the space V^N has a natural decomposition into the sum of its subspaces: $V^N = \sum_{i=0}^N V_n^N$, where V_n^N has a basis generated by the vectors (3)

$$v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \dots \otimes v_{\epsilon_N}$$

such that $\sum_i \epsilon_i = n$.

The representation of the algebra YB_N has the following property.

Proposition 1.11. *For every i and j*

$$A^i_N : V_j^N \to V_j^N,\ D^i_N : V_j^N \to V_j^N,\ C^i_N : V_{j+1}^N \to V_j^N,\ B^i_N : V_j^N \to V_{j+1}^N$$

Proof. For $N = 1$ the matrix $M_1(x, t_1)$ is is just the Lax matrix $L(x, t)$, hence the generators of the algebra YB_1 are the operators given by the coefficients of the expansion in x of the 2×2 blocks of $L(x, t)$:

$$\begin{pmatrix} t & 0\\ 0 & 0\end{pmatrix} \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix} \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} \tag{7}$$

which obviously have the above property. The definition of $M_N(x, t_1, ..., t_N)$ as the Kronecker product implies immediately that this property holds for any N. □

References

Bump [BNN14] D. Bump, P. McNamara, M. Nakasuji, *Factorial Schur functions and the Yang-Baxter equation*. Comment. Math. Univ. St. Pauli **63** (2014), no. 1-2, 2–45.

Haz [HGK10] M. Hazewinkel, N. Gubareni, and V. Kirichenko, *Algebras, rings and modules. Lie algebras and Hopf algebras*, Mathematical Surveys and Monographs, **168**.AMS, 2010.