EXCEPTIONAL AND WEAKLY EXCEPTIONAL QUOTIENT SINGULARITIES

DMITRIJS SAKOVICS

Contents

An updated version of these notes is available at

`http://cgp.ibs.re.kr/~dmitry85/notes/singularities.pdf`

For simplicity, assume that all varieties are $\mathbb{Q}$-factorial.

1. Singularities and blowups

KLT, LC varieties. Given a variety V and a point $O \in V$, let $f : \tilde{V} \to V$ be a resolution of the singularity of V at 0. Then have

$$K_{\tilde{V}} + \sum a_i E_i =_{\mathbb{Q}} f^* (K_V)$$

Then

- If $a_i < 1\ \forall i$, $(V \ni O)$ is said to be *Kawamata Log Terminal (KLT)*
- If $a_i \leq 1\ \forall i$, $(V \ni O)$ is said to be *Log Canonical (LC)*

Note 1. *This is independent of the choice of resolution.*

KLT, LC, plt for pairs. Now, assume $(V \ni O)$ is KLT, and $D = \sum d_i D_i$ is a $\mathbb{Q}$-Cartier divisor on V, where $0 < d_i \in \mathbb{Q}$ and D_i are distinct prime divisors. Given a resolution

$$K_{\tilde{V}} + \tilde{D} + \sum a_i E_i =_{\mathbb{Q}} f^* (K_V + D)$$

- (V, D) is *KLT* if $a_i < 1$, $d_j < 1\ \forall i, j$.
- (V, D) is *LC* if $a_i \leq 1$, $d_j \leq 1\ \forall i, j$.
- (V, D) is *Purely Log Terminal (plt)* if $a_i < 1$, $d_j \leq 1\ \forall i, j$.

Note 2. *It is worth mentioning other related terms: a pair is called ϵ-KLT (ϵ-LC) if the respective definitions hold with 1 replaced by $1 - \epsilon$.*

Definitions of singularity types.

Definition 1.1. *Let* $(V \ni O)$ *be a germ of a Kawamata log terminal singularity. The singularity is said to be* exceptional *if, given an effective* $\mathbb{Q}$*-divisor* D *on* V *one of the following hold:*

- (V, D) *is not log canonical.*
- *There exists at most one exceptional divisor* E *over* 0*, with* $a_E = 1$*. (Discrepancy* -1 *w.r.t.* (V, D)*).*

Let $(V \ni O)$ be a germ of a KLT singularity. Then a birational morphism $\pi : W \to V$, such that:

- the exceptional locus of π consists of one irreducible divisor E such that $0 \in \pi(E)$,
- the log pair (W, E) has purely log terminal singularities.
- the divisor $-E$ is a π-ample $\mathbb{Q}$-Cartier divisor.

is called a *plt blowup* of $(V \ni O)$.

Theorem 1.2 (see [3, Theorem 3.7], [9])**.** *If* $(V \ni 0)$ *is a germ of a KLT singularity, then it does have a plt blowup.*

Definition 1.3. *Let* $(V \ni 0)$ *be a germ of a Kawamata log terminal singularity. The singularity is said to be* weakly exceptional *if its plt blowup is unique.*

Some properties.

Lemma 1.4 (see [14, Theorem 4.9])**.** *If* $(V \ni O)$ *is exceptional, then* $(V \ni O)$ *is weakly exceptional.*

Example 1.5. *Du Val singularities. The singularity types can be determined directly from the intersection graphs of the resolution. For details, see Example A.1.*

2. Log Fano Varieties

Definitions.

Definition 2.1. *A variety* V *is* log Fano *if there exists a divisor* D *on* V*, such that* (X, D) *is KLT and* $-(K_V + D)$ *is ample.*

Definition 2.2. *Let* $(V \ni 0)$ *be a germ of a KLT singularity, let* E *be the exceptional divisor of a plt blowup of this singularity. Then the* different *is defined by the relation*

$$K_E + \mathit{Diff}_E(0) = (K_X + E + 0)|_E$$

Theorem 2.3 ([9], [14], [17])**.** *Let* $f : \tilde{V} \to V \ni 0$ *be a plt blowup with exceptional divisor* E*. Then* $(E, \mathit{Diff}_E(0))$ *is log Fano. The variety is exceptional (weakly exceptional) if and only if for any effective* $\mathbb{Q}$*-divisor* D *on* E *with*

$$D \sim_{\mathbb{Q}} -(K_E + \mathit{Diff}_E(0)),$$

the pair $(E, \mathit{Diff}_E(0) + D)$ *is KLT (Log Canonical).*

Log Canonical Threshold (Tian's α-invariant). Assume that $(V \ni O)$ is KLT, and D is an effective $\mathbb{Q}$-Cartier divisor on V. Now take $\lambda \in \mathbb{R}$ and consider the pair $(V, \lambda D)$. Intuitively, one can see that as the value of λ grows, the singularities of this pair become worse. This idea can be transformed into the following definition:

Definition 2.4. *Given a KLT variety V and an effective $\mathbb{Q}$-Cartier divisor D on V, the number*

$$lct(V, D) = \sup \{\lambda \in \mathbb{R} \mid \text{the pair } (V, \lambda D) \text{ is log canonical}\}$$

is the Log Canonical Threshold *of the pair (V, D).*

Note 3. *The log canonical threshold of a pair is often referred to as Tian's α-invariant, written as $\alpha(V, D)$.*

It should be clear that (at least, in case of log Fanos) the (weak) exceptionality of a variety can be considered in terms of the corresponding log canonical thresholds:

Proposition 2.5. *Let $f : \tilde{V} \to V \ni 0$ be a plt blowup with exceptional divisor E. Let*

$$\alpha = \inf \{lct(E, Diff_E 0 + D) \mid D \sim_{\mathbb{Q}} -(K_E + Diff_E(0))\}$$

Then

- *V is weakly exceptional if and only if $\alpha \geq 1$.*
- *V is exceptional whenever $\alpha > 1$. Note: not "if and only if".*

3. Quotient Singularities

Take a finite group $G \subset \mathrm{GL}_n(\mathbb{C})$ and consider the variety $V = \mathbb{C}^n/G$. Let $0 \in V$ be the image of the origin in $\mathbb{C}^n$ under the natural projection. Then it is clear that (under certain conditions on G) V has a singularity at 0. Call singularities of this form *quotient singularities.* From now on, assume that all the singularities mentioned here are quotient singularities, and fix the group G. Furthermore, take the natural projection $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$, and let $\bar{G}$ be the image of G under this projection.

Note 4. *The inclusion $G \subset GL_n(\mathbb{C})$ fixes not only the isomorphism class of G, but also its conjugacy class in $GL_n(\mathbb{C})$, i.e. its action on $\mathbb{C}^n$. So from now on G can be considered to be generated by explicit $n \times n$ matrices (up to a change of basis). As a consequence, it will be assumed throughout that the action of G on $\mathbb{C}^n$ is faithfull. Note that this also fixes the action of $\bar{G}$ on $\mathbb{P}^{n-1}$.*

Definition 3.1. *Take M to be an $n \times n$ diagonal matrix matrix with $(n-1)$ diagonal entries being equal to 1, and the remaining one being equal to a primitive k-th root of unity (for $k > 1$). Such a matrix M is called a* pseudoreflection.

This type of element is important due to the following theorem:

Theorem 3.2 (Chevalley–Shephard–Todd theorem, see [19, Theorem 4.2.5])**.** *The following properties of a finite group G are equivalent:*

- *G is a finite reflection group.*
- *S is a free graded module over S^G with a finite basis.*
- *S^G is generated by n algebraically independent homogeneous elements.*

In other words, if the group G is generated by pseudoreflections, then the variety V is smooth. This means that when considering the singularities of V, one can disregard the pseudoreflections in G. Therefore, from now on assume that G contains no pseudoreflections.

Looking at the previous sections, it should be clear that the types of singularities are actually dependent on the group $\bar{G} \subset \mathrm{PGL}_n(\mathbb{C})$, rather than $G \subset \mathrm{GL}_n(\mathbb{C})$. Therefore, it is often advantageous to use the following:

Proposition 3.3. *Let $G \subset GL_n(\mathbb{C})$ be a finite subgroup, and let $\pi : GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$ be the natural projection. Then there exists a group $G' \subset SL_n(\mathbb{C}) \subset GL_n(\mathbb{C})$, such that $\pi(G) = \pi(G')$.*

Proof. Easy exercise. □

Using this proposition, assume from now on that $G \subset \mathrm{SL}_n(\mathbb{C})$.

Of course, it is possible to use all the definitions above in the special case of these quotient varieties. The following turns out to be particularly useful:

Definition 3.4. *The* global $\bar{G}$-invariant log canonical threshold *is*

$$lct_{\bar{G}}\left(\mathbb{P}^{n-1}\right) = \sup\left\{\lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{the pair } (V, \lambda D) \text{ is log canonical for every} \\ \bar{G}\text{-invariant } D \sim_{\mathbb{Q}} -K_{\mathbb{P}^{n-1}} \end{array}\right\}$$

This is also called the global $\bar{G}$-invariant α-invariant.

With this definition, have:

Proposition 3.5. *The following hold:*

- *$0 \in \mathbb{C}^n/G$ is weakly exceptional if and only if $lct_{\bar{G}}\left(\mathbb{P}^{n-1}\right) \geq 1$.*
- *$0 \in \mathbb{C}^n/G$ is exceptional whenever $lct_{\bar{G}}\left(\mathbb{P}^{n-1}\right) > 1$.*

Definition 3.6. *A polynomial $F(x_1, \ldots, x_n)$ is a* semiinvariant *of a group $G \subset GL_n(\mathbb{C})$ if for every $g \in G$, $g(F) = \lambda_g F$, for some $\lambda_g \in \mathbb{C}$.*

Proposition 3.7. *If G has a semiinvariant of degree $d < n$ ($d \leq n$) then the singularity $0 \in \mathbb{C}^n/G$ is not weakly exceptional (resp. not exceptional).*

Proof. Let F be this semiinvariant, and take the divisor $D = \{F = 0\} \subset \mathbb{P}^{n-1}$. If $H \subset \mathbb{P}^{n-1}$ is a hyperplane, then $D \sim dH$ and $-K_{\mathbb{P}^{n-1}} \sim nH$. D is $\bar{G}$-invariant, so

$$\mathrm{lct}_{\bar{G}}\left(\mathbb{P}^{n-1}\right) \leq \mathrm{lct}\left(\mathbb{P}^{n-1}, \frac{n}{d}D\right) \leq \frac{d}{n}$$

If $d < n$, then the result follows by Proposition 3.5. If $d = n$, then the result is an easy consequence of Theorem 2.3. □

The converse of this result is, unfortunately, false. However, a somewhat modified version of it does hold:

Theorem 3.8 ([3, Theorem 1.12]). *Let G be a finite group in $GL_{n+1}(\mathbb{C})$ that does not contain reflections. If $\mathbb{C}^{n+1}/G$ is not weakly exceptional, then there is a $\bar{G}$-invariant, irreducible, normal, Fano type projectively normal subvariety $V \subset \mathbb{P}^n$ such that*

$$\deg V \leq \binom{n}{\dim V}$$

and for every $i \geq 1$ and for every $m \geq 0$ one has

$$h^i\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\dim V + 1) \otimes \mathcal{I}_V\right) = h^i\left(V, \mathcal{O}_V(m)\right) = 0,$$

$$h^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(\dim V+1\right)\otimes\mathcal{I}_V\right)\geq\left(\begin{array}{c}n\\ \dim V+1\end{array}\right),$$

where $\mathcal{I}_V$ is the ideal sheaf of the subvariety $V\subset\mathbb{P}^n$. Let Π be a general linear subspace of $\mathbb{P}^n$ of codimension $k\leq\dim V$. Put $X=V\cap\Pi$. Then $h^i\left(\Pi,\mathcal{O}_\Pi\left(m\right)\otimes\mathcal{I}_X\right)=0$ for every $i\geq 0$ and $m\geq k$, where $\mathcal{I}_X$ is the ideal sheaf of the subvariety $X\subset\Pi$. Moreover, if $k=1$ and $\dim V\geq 2$, then X is irreducible, projectively normal, and $h^i\left(X,\mathcal{O}_X\left(m\right)\right)=0$ for every $i\geq 1$ and $m\geq 1$.

Before proceeding, it is necessary to recall some things from basic representation theory:

Definition 3.9. *Let $G\subset GL_n\left(\mathbb{C}\right)$ be a finite group. Then G acts on $\mathbb{C}^n$. A* partition of imprimitivity *for (this action of) G is a set $\{V_1,\ldots,V_k\}$, where $V_i\subseteq\mathbb{C}^n$ are subspaces, such that $V_i\cap V_j=\{0\}$ whenever $i\neq j$,*

$$\mathbb{C}^n=V_1\oplus\cdots\oplus V_k$$

and for every $g\in G$ and every $i\in\{1,\ldots,k\}$, $g\left(V_i\right)=V_j$ for some $j\in\{1,\ldots,k\}$.

Fact 3.10. *Every action of a group G on $\mathbb{C}^n$ has at least one partition of imprimitivity, namely, $\{\mathbb{C}^n\}$.*

Definition 3.11. *If the action of G has exactly one partition of imprimitivity, then this action is called* primitive.

Lemma 3.12 (Jordan's theorem — see, for example, [7])**.** *For any given N, there are only finitely many finite primitive subgroups of $SL_N\left(\mathbb{C}\right)$.*

Definition 3.13. *If for every partition of imprimitivity $\{V_1,\ldots,V_k\}$, the action of G permutes the subspaces V_i transitively, then the action is called* irreducible.

Note 5. *This permutation action defines a map $G\mapsto\mathbb{S}_k$. This map will be very useful later on.*

Proposition 3.14. *If the action of G is irreducible, then for any given partition of imprimitivity of G, all the subspaces have the same dimension.*

Definition 3.15. *If G has a partition of imprimitivity with all the subspaces having dimension 1, then G is called* monomial.

Proposition 3.16. *If the singularity $0\in\mathbb{C}^n/G$ is weakly exceptional, then the action of G is irreducible.*

The proof is an easy (although messy) generalisation of the following:

Example 3.17. *Say that $n=4$, and a partition of imprimitivity consists of two 2-dimensional G-invariant subspaces. So, $G\subset SL_4\left(\mathbb{C}\right)$, and*

$$L_1=\{u=v=0\}, L_2=\{x=y=0\}\subset\mathbb{P}^3=(x:y:u:v)$$

with $\bar{G}\left(L_i\right)=L_i$. Consider

$$\mathcal{M}_1=\{\lambda u+\mu v\}\sim\mathcal{O}(1)$$

Take some divisor $M\in\mathcal{M}_1$. Then

$$N=Orb\left(M\right)=\sum_{i=1}^{k}M_i\subset\mathcal{M}_1$$

for some distinct divisors $M_1,\ldots,M_k\in\mathcal{M}_1$. Note that $N\in\mathcal{O}(k)$.

Now consider the pair $\left(\mathbb{P}^3, 4\mathcal{M}_1\right)$ *and a blowup* $f : V \to \mathbb{P}^3$.

$$\begin{aligned} K_V &= f^*\left(K_{\mathbb{P}^3}\right) + E \\ \mathcal{M}_V &= f^*\left(K_{\mathcal{M}_1}\right) - E \\ K_V + 4\mathcal{M}_V &= f^*\left(K_{\mathbb{P}^3} + 4K_{\mathcal{M}_1}\right) - 3E \end{aligned}$$

Thus, taking $D = \frac{4}{k}N$, *get a* $\bar{G}$*-invariant divisor with the pair* $\left(\mathbb{P}^3, D\right)$ *not log canonical.*

Proposition 3.18. *If the singularity* $0 \in \mathbb{C}^n/G$ *is exceptional, then the action of* G *is primitive.*

The proof is almost identical to the previous one:

Example 3.19. *Say that* $n = 4$, *and a partition of imprimitivity consists of two* 2*-dimensional* G*-invariant subspaces. So,* $G \subset SL_4(\mathbb{C})$, *and*

$$L_1 = \{u = v = 0\}, L_2 = \{x = y = 0\} \subset \mathbb{P}^3 = (x : y : u : v)$$

with $\bar{G}(L_i) = L_j$. *Consider*

$$\begin{aligned} \mathcal{M}_1 &= \{\lambda u + \mu v\} \sim \mathcal{O}(1) \\ \mathcal{M}_2 &= \{\lambda x + \mu y\} \sim \mathcal{O}(1) \end{aligned}$$

Take some divisor $M \in \mathcal{M}_1$. *Then*

$$N = Orb(M) = \sum_{i=1}^{k} M_i \subset \mathcal{M}_1 + \mathcal{M}_2$$

for some distinct divisors $M_1, \ldots, M_l \in \mathcal{M}_1$, $M_{l+1}, \ldots, M_k \in \mathcal{M}_2$. *Note that* $N \in \mathcal{O}(k)$.

Now consider the pair $\left(\mathbb{P}^3, 2\left(\mathcal{M}_1 + \mathcal{M}_2\right)\right)$ *and a blowup* $f : V \to \mathbb{P}^3$.

$$\begin{aligned} K_V &= f^*\left(K_{\mathbb{P}^3}\right) + E \\ \mathcal{M}_V &= f^*\left(K_{\mathcal{M}_1} + K_{\mathcal{M}_2}\right) - E \\ K_V + 2\mathcal{M}_V &= f^*\left(K_{\mathbb{P}^3} + 2\left(K_{\mathcal{M}_1} + K_{\mathcal{M}_2}\right)\right) - E \end{aligned}$$

Thus, taking $D = \frac{2}{k}N$, *get a* $\bar{G}$*-invariant divisor with the pair* $\left(\mathbb{P}^3, D\right)$ *not KLT.*

Example 3.20. *Now go back to the example of ADE singularities. It is well-known that these are in fact quotient singularities,* A_n *corresponding to the case* $\bar{G} = \mathbb{Z}_{n-1}$, D_n *corresponding to* $\bar{G} = \mathbb{D}_{2(n-2)}$, *and* E_6, E_7, E_8 *corresponding to* $\bar{G}$ *being* $\mathbb{A}_4$, $\mathbb{S}_4$ *and* $\mathbb{A}_5$ *resp. (in order to get* G*, these need to be lifted with a central extension of order* 2*). For more details, see Example A.1.*

It is worth noting that here the weakly exceptional singularities are exactly those coming from irreducible group actions (of the dihedral, alternating and symmetric groups), and the exceptional singularities are those coming from primitive actions (of $\mathbb{A}_4$, $\mathbb{S}_4$ *and* $\mathbb{A}_5$*).*

Question 3.21. *Does the irreducibility and primitivity of a group action determine the (weak) exceptionality of the corresponding singularity?*

4. Dimension 3 ($\mathbb{C}^3$, $\mathbb{P}^2$)

Throughout this section, $G \subset \mathrm{SL}_3(\mathbb{C})$ is a finite group. Such groups in low dimensions have been classified, so it makes sense to look at what G might be. In the case of G being primitive, the classical source for the classification is H. Blichfeldt's book [1]. However, the result in that book is known to have a mistake (it is missing one of the central extension groups). This has long been known, and a modern result corrects this omition (and also clasifies the imprimitive subgroups):

Theorem 4.1 (Consequence of [20])**.** *Let $G \subset SL_3(\mathbb{C})$ be a finite irreducible group. Then G is one of the following:*

- *If G is monomial, then it is either $D \rtimes \mathbb{Z}_3$ or $D \rtimes \mathbb{S}_3$, where, in some basis, D is the subgroup of diagonal matrices and $\mathbb{Z}_3$ (or $\mathbb{S}_3$) permutes the basis elements.*
- *G is one of the primitive groups $E_{108} \lhd F_{216} \lhd \mathbb{H}_{648}$, where $\mathbb{H}_{648}$ is the Hessian group of order 648 and the other two are the normal subgroups of order 108 and 216 respectively.*
- *G is isomorphic to Klein's simple group $\mathbb{K}_{168}$ of order 168.*
- *G is isomorphic to $\mathbb{A}_5$ or the non-trivial central extension $3\mathbb{A}_6$.*
- *G is isomorphic to one of $\mathbb{Z}_3 \times \mathbb{A}_5$ or $\mathbb{Z}_3 \times \mathbb{K}_{168}$, where $\mathbb{Z}_3$ is gnerated by scalar matrices (i.e. is the center of $SL_3(\mathbb{C})$).*

Proof. See Section B.2 □

First of all, consider the exceptional quotient singularities in this dimension.

Theorem 4.2 (see [11, Section 3.12])**.** *Given a finite group $G \subset SL_3(\mathbb{C})$, the singularity $\mathbb{C}^3/G$ is exceptional if and only if G has no semiinvariants of degree at most 3.*

This easily implies:

Theorem 4.3 (see [11, Theorem 3.13])**.** *The group G induces an exceptional singularity if and only if $\bar{G}$ is isomorphic to $\mathbb{A}_6$, Klein's simple group $\mathbb{K}_{168}$ of size 168, Hessian group $\mathbb{H}_{648}$ of size 648 or its normal subgroup F_{216} of size 216.*

Remark 4.4. *This means that if G is primitive, but the singularity is not exceptional, then G is isomorphic to either E_{108} or $\mathbb{A}_5$.*

Now, a similar result for the weakly exceptional singularities can be stated as:

Theorem 4.5 ([3, Theorem 3.18])**.** *Let $GSL_3(\mathbb{C})$ be a finite irreducible group. Then the quotient singularity $\mathbb{C}^3/G$ is weakly exceptional if and only if there is no $\bar{G}$-invariant conic curve on $\mathbb{P}^2$.*

Assume the singularity is not weakly exceptional. Then there is a $\bar{G}$-invariant conic C.

Note 6. *It is easy to prove that the group must act faithfully on C. The same statement is true more generally (for $\bar{G}$-invariant varieties in arbitrary dimensions) under mild conditions on the variety.*

Remark 4.6. *The conic C must be smooth: otherwise either it is a double line (defining a G-invariant plane in $\mathbb{C}^3$) or has exactly one singular point (defining a G-invariant line in $\mathbb{C}^3$). In either case, this contradicts the irreducibility of G.*

Proposition 4.7. *The group $\bar{G}$ is isomorphic to $\mathbb{A}_4$, $\mathbb{S}_4$ or $\mathbb{A}_5$.*

Proof. A smooth conic in $\mathbb{P}^3$ is rational. Therefore, $\mathrm{Aut}(C) = \mathrm{Aut}(\mathbb{P}^1)$. So $\bar{G}$ is is either cyclic or dihedral or is isomorphic to one of the three given groups. But cyclic and dihedral group don't have irreducible lifts from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{SL}_3(\mathbb{C})$. □

Locating the relevant isomorphism classes in the classification of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$, get the following result:

Theorem 4.8. *Let $G \subset SL_3(\mathbb{C})$ be a finite subgroup. Then G induces a weakly-exceptional but not an exceptional singularity if and only if one of the following holds:*

- *G is a monomial group, and $\bar{G}$ is not isomorphic to $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_3$ or $(\mathbb{Z}_2)^2 \rtimes \mathbb{S}_3$.*
- *G is isomorphic to the normal subgroup $E_{108} \lhd F_{216}$ of size 108.*

Note 7. *This section proves that G being irreducible (primitive) is not a sufficient condition for the sigularity being weakly exceptional (resp. exceptional).*

5. Dimension 4 ($\mathbb{C}^4$, $\mathbb{P}^3$)

In this dimension, one first meets several types of groups that will be important in the later study. Thus, it is worth it to consider them in detail.

Example 5.1. *Take two integers $a, b > 1$, such that $n = ab$. Write the coordinates of $\mathbb{C}^n$ in a $a \times b$ grid. In this form, any point of $\mathbb{C}^n$ can be viewed as a $a \times b$ matrix M. Let $G_a \subset SL_a(\mathbb{C})$ and $G_b \subset SL_b(\mathbb{C})$. Now, for any $g_a \in G_a$, $g_b \in G_b$ define:*

$$\alpha(g_a, g_b)(M) = g_a M g_b^T$$

It is clear that this defines an action of $G \subset SL_{ab}(\mathbb{C})$ (where $G \subseteq G_a \times G_b$ as abstract groups).

Example 5.2. *In the above example, let $a = b$ and let G be the group defined by α. Let the element t be defined by transposing the matrix form M. Then let $G_1 = \langle G, t \rangle$.*

Lemma 5.3. *In the two examples above, if $a = b$ then G and G_1 have a semiinvariant of degree $a < n$.*

Proof. Take the determinant of the matrix form. □

Example 5.4. *Let $TSL_4(\mathbb{C})$ be the group generated by the following matrices:*

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and the scalar matrix with all diagonal entries equal to $\sqrt{-1}$. This is the Extra Special Group of order 2^5. It belongs to the family of Heisenberg groups, which appear as subgroups of primitive groups in many dimensions.

Note 8. *The group T constructed above has a 5 linearly independent T-invariant degree 4 polynomials.*

Example 5.5 (see [13])**.** *Take the group T in the above example and consider two matrices*

$$\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & -1 & -1 \\ i & 1 & 0 & 0 \end{pmatrix} \quad \rho = e^{2\pi i/8} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then the group N generated by T, ν and ρ. Then N the normaliser of T is $SL_4(\mathbb{C})$ Take the degree 4 polynomials left invariant by T. It is possible to rearrange them to get 6 (linearly dependent) T-invariant degree 4 polynomials $t_1, \ldots, t_6$, such that N acts by permuting the t_i. This defines a map $\alpha : N \mapsto \mathbb{S}_6$, and whenever $T \subset G \subseteq N$, G can be determined by its image $\alpha(G)$.

Proposition 5.6. *If the group G defined in Example 5.5 has no semiinvariants of degree $d \leq 4$, then $\alpha(G)$ is one of $\mathbb{A}_5$, $\mathbb{S}_5$, $\mathbb{A}_6$, $\mathbb{S}_6$.*

Proof. By direct computation, T (and thus any groups containing T) has no semiinvariants of degree $d < 4$. The degree 4 semiinvariants are excluded by direct computation on the permutation representations of the relevant subgroups $\alpha(G) \subseteq \mathbb{S}_6$. □

Coming back to computing the singularity types, a good starting point is the following theorem:

Theorem 5.7 ([1], [6])**.** *Let $G \subset SL_4(\mathbb{C})$ be a finite primitive subgroup. Then G is one of the following*

- *A group of the type constructed in Example 5.1 or 5.2 (for $a = b = 2$)*
- *A primitive group $T \subset G \subset N$ constructed in Example 5.5*
- *$\mathbb{A}_5$ or $\mathbb{S}_5$*
- *$SL_2(\mathbb{F}_5)$ or $SL_2(\mathbb{F}_7)$*
- *Central extensions $2\mathbb{A}_6$, $2\mathbb{S}_6$, $2\mathbb{A}_7$.*
- *$\tilde{\mathbb{O}}'(5, \mathbb{F}_3)$, a central extension of the commutator of $\mathbb{O}'(5, \mathbb{F}_3)$*

By Proposition 3.18, if the singularity $\mathbb{C}^4/G$ is exceptional, then G is one of these groups. Moreover, by Proposition 3.7, such a group G cannot have semiinvariants of degree $d \leq 4$. As it turns out, these two conditions are almost sufficient:

Theorem 5.8 ([3, Theorem 4.3])**.** *The singularity $\mathbb{C}^4/G$ is exceptional whenever the following are satisfied:*

- *The group G is primitive,*
- *The group G has no semiinvariants of degree $d \leq 4$,*
- *$|\bar{G}| > 168$.*

The next step is to check the semiinvariant condition for these groups. Looking back at Theorem 5.7, thefirst two group types can be checked using Lemma 5.3 and Proposition 5.6. As for the other groups, this can be done in different ways, but the simplest (although most tedious) way is to just construct the character table of the relevant groups ant look at the characters of the first n symmetric powers of relevant representations. I would strongly recommend to do this by hand once, and then to find a computer algebra program that can do that for you. In any case, the results should be as follows:

Proposition 5.9. *For a given G, let d be the smallest degree of a G-semiinvariant polynomial. Then the following hold:*

- *If $G = \mathbb{A}_5$ then $d \leq 3$*
- *If $G = \mathbb{S}_5$ then $d \leq 3$*
- *If $G = SL_2(\mathbb{F}_5)$ then $d = 4$*
- *If $G = SL_2(\mathbb{F}_7)$ then $d = 4$*
- *If $G = 2\mathbb{A}_6$ then $d \geq 5$*
- *If $G = 2\mathbb{S}_6$ then $d \geq 5$*
- *If $G = 2\mathbb{A}_7$ then $d \geq 5$*
- *If $G = \tilde{\mathbb{O}}'(5, \mathbb{F}_3)$ then $d = 12$*

Proof. By direct computation or otherwise. □

Looking at the sizes of the remaining groups, it is clear that they all satisfy the required lower bound on the group size. Thus the result is:

Theorem 5.10. *The singularity $\mathbb{C}^4/G$ is exceptional whenever G is one of $2\mathbb{A}_6$, $2\mathbb{S}_6$, $2\mathbb{A}_7$ $\tilde{\mathbb{O}}'(5, \mathbb{F}_3)$ and the four groups described in Proposition 5.6.*

Now look for the weakly exceptional singularities. Unfortunately, although all the possible finite irreducible subgroups of $\mathrm{SL}_4(\mathbb{C})$ have been classified, the classification is too complicated for the same method as above to be practical. On the other hand, a similar sufficient condition still exists:

Theorem 5.11 ([3, Theorem 4.1])**.** *The singularity $\mathbb{C}^4/G$ is weakly exceptional if and only if the following are satisfied:*

- *The group G is irreducible,*
- *The group G has no semiinvariants of degree $d \leq 3$,*
- *There is no $\bar{G}$-invariant smooth rational cubic curve in $\mathbb{P}^3$.*

The last condition is implied by the condition "$\left|\bar{G}\right| > 60$", so, looking at the exceptional case, one can hope that this condition may be unnecessary. Unfortunately, this turns out to be false:

Proposition 5.12. *Let $G \subset SL_4(\mathbb{C})$ be a finite irreducible subgroup, such that $\bar{G}$ preserves a smooth rational cubic curve $C \subset \mathbb{P}^3$. Assume that G has no semiinvariants of degree $d \leq 3$. Then $\bar{G} = \mathbb{A}_5$.*

Proof. Since C is smooth, $\mathrm{Aut}(C) = \mathrm{Aut}\left(\mathbb{P}^1\right)$. Therefore, $\bar{G}$ is isomorphic to one of $\mathbb{Z}_n$, $\mathbb{D}_{2n}$, $\mathbb{A}_4$, $\mathbb{S}_4$, $\mathbb{A}_5$.

Central extensions of cyclic and dihedral groups do not have any irreducible 4-dimensional representations. If $\bar{G}$ is equal to $\mathbb{A}_4$ or $\mathbb{S}_4$, then it is easy to check that G has a semiinvariant of degree 2 ($\bar{G}$ preserves a smooth quadric surface). $\bar{G}$ has to be isomorphic to $\mathbb{A}_5$.

It needs to be noted, that there are two irreducible groups G with $\bar{G} = \mathbb{A}_5$. One of them has a semiinvariant of degree 2, but the other one has no semiinvariants of degree $d < 4$ (it fact, in this case $G = \mathrm{SL}_2(\mathbb{F}_5) = 2\mathbb{A}_5$) and does indeed preserve such C. □

Remark 5.13. *This shows that in general, the condition of a group not having any low-degree semiinvariants is not sufficient to deduce that the corresponding singularity is (weakly) exceptional, and that one should not expect such a condition to work in a higher dimension.*

Since it does not seem feasible to classify the weakly exceptional quotient singularities by starting with the list of finite subgroups of $\mathrm{SL}_4(\mathbb{C})$, one has to find a different approach. The theorem above implies the following:

Proposition 5.14. *Let $G \subset SL_4(\mathbb{C})$ be a finite subgroup, such that the singularity $\mathbb{C}^4/G$ is* not *weakly exceptional. Then at least one of the following holds:*

- *G is not irreducible.*
- *$\bar{G} = \mathbb{A}_5$, preserving a smooth rational cubic curve $C \subset \mathbb{P}^3$.*
- *G has a semiinvariant of degree 2 or 3.*

Thus, it is sufficient to classify the cases where G has a semiinvariant of degree 2 or 3. Let $S \subset \mathbb{P}^3$ be the subvariety defined by this semiinvariant. Then it must be smooth:

Proposition 5.15. *If G is irreducible, then the $\bar{G}$-orbit of any point $p \in \mathbb{P}^3$ has at least 4 points.*

Proof. Say a point p has an orbit of $k < 4$ points. Then $\mathbb{C}^4$ contains a G-invariant union of k lines, which define a proper G-invariant subspace of $\mathbb{C}^4$, contradicting irreducibility of G. ☐

Lemma 5.16. *Let $G \subset SL_4(\mathbb{C})$ be a finite irreducible group, $\bar{G} \subset PGL_4(\mathbb{C})$ its projection, and let $S \subset \mathbb{P}^3$ be a $\bar{G}$-invariant surface of degree minimal among the degrees of all $\bar{G}$-invariant surfaces. Then either $\deg S \geq 4$ or S is smooth.*

Proof. Since G is irreducible, $\deg S \geq 2$. If $\deg S = 2$ and S is singular, then either S has exactly one isolated singularity (which has to be a $\bar{G}$-fixed point, i.e. an orbit consisting of 1 point), or S is a union of two planes and thus has a singular line (defining a plane in $\mathbb{C}^4$), which must then be $\bar{G}$-invariant. Both of these contradict the irreducibility of G. Therefore, if $\deg S = 2$ then S must be smooth.

If $\deg S = 3$, and S is not irreducible, then either it is the union of a plane and an irreducible quadric surface (each of which must thus be a $\bar{G}$-invariant surface of smaller degree, contradicting the minimality of the degree of S) or S is the union of 3 distinct planes, whose intersection gives either a point or a line fixed by all of $\bar{G}$ (stopping G from being irreducible). Hence S is irreducible.

Assume that $\deg S = 3$ and S has non-isolated singularities, with C being the union of all singular curves on S. Then, one can see that C is a line. Since $\bar{G}(S) = S$, must have $\bar{G}(C) = C$, and so there exists a $\bar{G}$-invariant line, contradicting irreducibility of G. Therefore if $\deg S = 3$ then S must have at worst isolated singularities.

If $\deg S = 3$ and S is singular with only isolated singularities, then by [2], the singularity types form one of the following collections: (A_1), $(2A_1)$, (A_1, A_2), $(3A_1)$, (A_1, A_3), $(2A_1, A_2)$, $(4A_1)$, (A_1, A_4), $(2A_1, A_3)$, $(A_1, 2A_2)$, (A_1, A_5). Given any type of singularity, the set of such singularities on S must be preserved by the action of $\bar{G}$ (as it will be a union of a number of orbits). Therefore, it must either be empty or have size at least 4. Therefore, S has to have exactly four A_1 singularities. Since there is only one such surface (see, for example, [2, proof of Lemma 3]), S must be the Cayley cubic, defined (in some basis) by

$$S = \left\{(x : y : u : v) \in \mathbb{P}^3 \mid xyu + xyv + xuv + yuv = 0\right\}$$

This surface, contains exactly 9 lines, six of which pass through pairs of singular points and the other three are defined by

$$x + y = 0 = u + v$$
$$x + u = 0 = y + v$$
$$x + v = 0 = y + u$$

These last three lines must therefore be mapped to each other by all of $\bar{G}$. But since they are coplanar, $\bar{G}$ preserves the plane they lie in, contradicting the irreducibility assumption for G. Thus if S is a cubic surface, then it must be smooth. □

Now consider the two different degrees separately:

Lemma 5.17. *If $G \subset SL_4(\mathbb{C})$ is a finite irreducible subgroup, and $\bar{G}$ its projection to $PGL_4(\mathbb{C})$. Also assume that there is no $\bar{G}$-invariant quadric surface, and $S \subset \mathbb{P}^3$ is a smooth $\bar{G}$-invariant cubic surface. Then G must be isomorphic to a central extension of one of:*

- $\left((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$.
- $(\mathbb{Z}_3)^3 \rtimes \mathbb{S}_4$.

by scalar elements, acting as described below. Both these cases produce monomial actions.

Proof. Since $\bar{G} \subset \mathrm{Aut}(S)$ is a finite subgroup, by [8] $\bar{G}$ must be isomorphic to one of:

(1) Cyclic groups $\{\mathrm{Id}_{\bar{G}}\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_8$.
(2) Dihedral groups $(\mathbb{Z}_2)^2 \cong \mathbb{D}_4$, $\mathbb{S}_3 \cong \mathbb{D}_6$, $\mathbb{S}_3 \times \mathbb{Z}_2 \cong \mathbb{D}_{12}$.
(3) $\mathbb{S}_4$.
(4) $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$ or $\left((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$.
(5) $\mathbb{S}_5$.
(6) $(\mathbb{Z}_3)^3 \rtimes \mathbb{S}_4$.

These cases will be considered separately:

(1) The group $\bar{G}$ cannot be cyclic, since central extensions of cyclic groups do not have any irreducible 4-dimensional representations.
(2) Dihedral groups and their extensions by scalar elements do not have any irreducible 4-dimensional representations, so these groups cannot act irreducibly.
(3) The group $\mathbb{S}_4$ by itself has no 4-dimensional irreducible representations, but its central extension has (up to a choice of a root of unity) only one such. This representation preserves a quadric surface.
(4) For convenience, write $\bar{G}' = (\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$ and $\bar{G}'' = \left((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$, with all the notation following in the obvious manner (i.e. write G' for the lift of $\bar{G}'$ to $\mathrm{SL}_4(\mathbb{C})$, etc.).

   Using the notation from [8], define the group G_{54}^9 generated by elements $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ with $\bar{\alpha}^3 = \bar{\beta}^3 = \bar{\gamma}^3 = \bar{\delta}^2 = \mathrm{id}$, $\bar{\alpha}$ generating the centre of G_{54}^9 and $G_{54}^9 / C\left(G_{54}^9\right) \cong (\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$.

   Then one can see that $\bar{G}' = G_{54}^9 / C\left(G_{54}^9\right)$ and $\bar{G}'' = G_{54}^9 \rtimes \mathbb{Z}_2$ (with additional generator $\bar{\epsilon}$, such that $\bar{\epsilon}^2 = \mathrm{id}$). Let $\alpha, \beta, \gamma, \delta, \epsilon$ be lifts of $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ (respectively) to $\mathrm{SL}_4(\mathbb{C})$.

Let $h_1 := \alpha^3$, $h_2 := \beta^3$. $\bar{\alpha}, \bar{\beta}$ commute, so say $\beta\alpha = \alpha\beta h_3$. By the structure of the lift, h_i are scalar matrices of order 1, 2 or 4. Then

$$h_1^3 h_2 = \left(\alpha^2\beta\alpha\right)^3 = \left(\beta h_1 h_3\right)^3 = h_2 h_1^3 h_3^3$$

and so $h_3 = \mathrm{id}$. Similarly, get α, β, γ all commuting. Hence the corresponding matrices can all be taken to be diagonal (by choosing a suitable basis). It is then easy to see that δ and ϵ must act as elements of a central extension of $\mathbb{S}_4$ permuting the basis.

Since $\bar{G}''$ has only one normal subgroup of index 2, and $\bar{G}''$ has no centre (otherwise $\bar{G}''/C\left(\bar{G}''\right)$ would be on the list of groups acting on a cubic surface), $\bar{\delta}\bar{\epsilon} \neq \bar{\epsilon}\bar{\delta}$. Therefore, up to conjugation, δ interchanges the first and the second basis vectors, and ϵ interchanges the first basis vector with the third one and the second basis vector with the fourth one.

This means that G' is not irreducible, while G'' is irreducible, monomial and (up to conjugation) is generated by

$$\begin{pmatrix} \zeta_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta_3^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta_3^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & \zeta_3^{-1} \end{pmatrix},$$

$$\zeta_8 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

This group leaves (for example) the cubic polynomial $x^3+y^3+z^3+w^3$ (in coordinates (x, y, z, w) for $\mathbb{C}^4$) semi-invariant, and by direct computation, one sees that the group does not have a semi-invariant quadric surface.

(5) The group $\mathbb{S}_5$ is, according to [15, §100], the automorphism group of the irreducible diagonal cubic surface

$$S = \left\{ (x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^4 \;\middle|\; \begin{array}{c} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0 \end{array} \right\}$$

which immediately implies that there exists a $\bar{G}$-invariant quadric surface.

(6) As stated in [15, §100], the monomial group $\left(\mathbb{Z}_3\right)^3 \rtimes \mathbb{S}_4$ acts by permuting the basis vectors of $\mathbb{C}^4$ arbitrarily and multiplying them by arbitrary cube roots of unity. Hence (up to conjugation) G is a central extension of such a group by scalar elements.

This group clearly leaves the cubic polynomial $x^3 + y^3 + z^3 + w^3$ (in coordinates (x, y, z, w)) semi-invariant, and by direct computation, one sees that the group does not have a semi-invariant quadric surface.

□

All that is left to consider is the case where $\bar{G}$ preserves a smooth quadric surface $S \subset \mathbb{P}^3$. By Lemma 5.16, one can assume that S is smooth. This means that S can be taken to be the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding. Therefore, in the notation of Examples 5.1 and 5.2 (for $a = b = 2$), S is defined by $\det(M) = 0$.

Thus, G can be any irreducible group constructed as in Examples 5.1 and 5.2, or any subgroup of such a group. All such G have been classified in [5]. However, that paper was not concerned with the irreducibility or primitivity of the group action, so this needs to be determined for the different groups on the list.

The bad news is that these groups form several infinite families, so there is no hope that the concept of weak exceptionality in a general dimension is particularly close to the irreducibility of the group action.

6. Higher dimensions

Having classified the exceptionality of quotient singularities in low dimensions, it is interesting to see how the singularities $\mathbb{C}^n/G$ behave for higher values of n. In this case, the exceptional quotient singularities are somewhat less interesting: the group giving rise to such a singularity has to be primitive (by Proposition 3.18), and by Jordan's Theorem (Lemma 3.12), for any fixed dimension there are only finitely many such groups. Furthermore, as the dimension grows, the exceptional quotient singularities occur less and less often, so there is hope that there is a bound on the dimension, beyond which no such singularities exist.

Therefore, one can concentrate on studying weakly exceptional quotient singularities. Obviously, there is no meaningful classification for finite subgroups of $\mathrm{SL}_n(\mathbb{C})$ for a general value of n, so one can only hope to classify the irreducible groups G, such that the corresponding singularities are not weakly exceptional. Judging by the previous results, one can suspect that the situation will depend on wheter or not the dimension n is a prime number.

6.1. **WE in prime dimensions.** For now, assume that $n = q \geq 3$, a prime number, and $\Gamma \subset \mathrm{SL}_q(\mathbb{C})$ is a finite irreducible subgroup, such that the singularity $\mathbb{C}^q/\Gamma$ is not weaky exceptional.

Recalling the definitions, it becomes clear that Γ is either primitive or monomial. Since there are only finitely many finite primitive subgroups of $\mathrm{SL}_q(\mathbb{C})$, can assume that G is monomial, i.e. there are q 1-dimensional subspaces $\{V_1, \ldots, V_q\}$, such that V_i span $\mathbb{C}^q$, and for any $g \in \Gamma$ and any i, $g(V_i) = V_j$. This defines a map $\pi : \Gamma \mapsto \mathbb{S}_q$, and, taking $T \subset \mathbb{S}_q$ to be the image of Γ and D the kernel, have the exact sequence

$$1 \to D \to \Gamma \to T \to 1$$

Furthermore, $T \subseteq \mathbb{S}_q$ must be transitive — otherwise the the orbit of V_1 would span a proper invariant subspace of $\mathbb{C}^q$.

Lemma 6.1. *Assume $G \subset SL_q(\mathbb{C})$ is a finite irreducible monomial subgroup. Setting $G \cong D \rtimes T$ as above, there exists $\tau \in G \setminus D$ and a basis $e_1, \ldots, e_q$ for $\mathbb{C}^q$, such that $\tau^q = Id_G$, and τ acts by*

$$\tau(e_i) = e_{i+1}\ \forall i < q;\ \tau(e_q) = e_1$$

Proof. Since G is irreducible, T must be a transitive subgroup of $\mathbb{S}_q$, and must thus contain a cycle of length q (since q is prime). Take $\tau \in \Gamma$, such that $\pi(\tau)$ is a generator of this cycle. Let $e_1 \in V_1$ be a non-zero vector. Then, renaming the V_i-s if necessary, $\tau^i(e_1) \in V_{i+1}$ (for $1 \leq i < q$). Set $e_i = \tau^{i-1}(e_1)$ $(2 \leq i \leq q)$. Clearly, $\tau(e_q) = \alpha e_1$ for some $\alpha \in \mathbb{C}$.

Since all the subspaces V_i are disjoint and one-dimensional, e_i must generate V_i, and so $e_1, \ldots, e_q$ must form a basis for $\mathbb{C}^q$. Also, since $g \in D = \ker\pi$, and τ permutes the subspaces V_i non-trivially, $\tau \notin D$. Since $\tau \in G \subseteq \mathrm{SL}_q(\mathbb{C})$ and q odd, one also observes that $\alpha = 1$, and so τ acts as stated above. □

Corollary 6.2. *There exists a subgroup $G = D \rtimes \mathbb{Z}_q \subseteq \Gamma$ generated by D and τ. The singularity of $\mathbb{C}^q/G$ is not weakly exceptional, and $|\Gamma| \leq (q-1)!\,|G|$.*

Proof. Take G generated by D and the element $\tau \in \Gamma$ obtained in Lemma 6.1. Clearly, $G \subseteq \Gamma$ and, looking at the action of τ, $G \cong D \rtimes \mathbb{Z}_q$. Let $\bar{\Gamma}$ and $\bar{G}$ be projections of Γ and G (respectively) to $\mathrm{PGL}_q(\mathbb{C})$. Then $\bar{G} \subseteq \bar{\Gamma}$, and any

$\bar{\Gamma}$-invariant variety is also $\bar{G}$-invariant. Thus, using Theorem 3.8, the singularity induced by G is not weakly exceptional. Finally,

$$|\Gamma| \leq \frac{|\mathbb{S}_q|}{|\mathbb{Z}_q|}\,|G| = (q-1)!\,|G|$$

□

From now on, fix the group G constructed above, the subgroup $D \subset G$, the element $\tau \in G$ and the basis $e_1, \ldots, e_q$ for $\mathbb{C}^q$ constructed in Lemma 6.1. It should be clear that obtaining a bound on the possible types of such groups G does the same for the groups Γ (as each G can only be extended in very few ways). On the other hand, the groups of this type turn out to be much easier to work with.

It is now necessary to obtain a specialised criterion for determining whether or not such groups induce a weakly exceptional singularity. For that, a small classical result is needed:

Proposition 6.3. *Any irreducible representation of G (given above) over $\mathbb{C}$ is either* 1*-dimensional or q-dimensional.*

Proof. See [16, §8.1]: here, $A = D$, $(G : D) = q$, which is only divisible by 1 or itself. □

Lemma 6.4 (generalising [3, Theorem 3.4])**.** *Let q be an odd prime and assume $G \subset SL_q(\mathbb{C})$ is a finite imprimitive subgroup isomorphic to $A \rtimes \mathbb{Z}_q$ for some abelian A. Then the singularity of $\mathbb{C}^q/G$ is not weakly exceptional if and only if G has a (non-constant) semi-invariant of degree $d < q$.*

Proof. If G does have a semi-invariant of degree at most $q-1$, then the singularity is not weakly exceptional by Proposition 3.7. Suppose that G does not have any such semi-invariants, but the singularity is not weakly exceptional.

Then, by Theorem 3.8, there exists a $\bar{G}$-invariant irreducible normal Fano type variety $V \subset \mathbb{P}^{q-1}$, such that $\deg V \leq \begin{pmatrix} q-1 \\ \dim V \end{pmatrix}$ and $h^i(V, \mathcal{O}_V(m)) = 0$ $\forall i \geq 1$ $\forall m \geq 0$ (where $\mathcal{O}_V(m) = \mathcal{O}_V \otimes \mathcal{O}_{\mathbb{P}^{q-1}}(m)$).

Let $n = \dim V$. Then, since G has no semi-invariants of degree less than q, have $n \leq q-2$. Let $\mathcal{I}_V$ be the ideal sheaf of V. Then

$$h^0(V, \mathcal{O}_V(m)) = h^0\left(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(m)\right) - h^0\left(\mathbb{P}^{q-1}, \mathcal{I}_V(m)\right)$$

For instance, $h^0(V, \mathcal{O}_V) = 1$.

Take any $m \in \mathbb{Z}$ with $0 < m < q$. Let $W_m = H^0\left(\mathbb{P}^{q-1}, \mathcal{I}_V(m)\right)$. This is a linear representation of G, so $q | \dim W_m$ (by Proposition 6.3, as G has no semi-invariants of degree $m < q$). Since $q | h^0\left(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(m)\right)$,

$$h^0(V, \mathcal{O}_V(m)) \equiv 0 \mod q$$

Since $h^0(V, \mathcal{O}_V(t)) = \chi(V, \mathcal{O}_V(t))$ for any integer $t \geq 0$, there exist integers $a_0, \ldots, a_n$, such that

$$h^0(V, \mathcal{O}_V(t)) = P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

Consider $P(t)$ as a polynomial over $\mathbb{Z}_q$. Since

$$P(m) = h^0(V, \mathcal{O}_V(m)) \equiv 0 \mod q$$

whenever $0 < m < q$, $P(t)$ has at least $q-1$ roots over $\mathbb{Z}_q$. But $\deg P \leq n \leq q-2$, so $P(t)$ must be the zero polynomial over $\mathbb{Z}_q$. In particular, $a_0 \equiv 0 \mod q$. On the other hand, $a_0 = P(0) = h^0(V, \mathcal{O}_V) = 1 \not\equiv 0 \mod q$, leading to a contradiction. ☐

Now let $f(x_1, \ldots, x_q)$ be a semi-invariant of G of degree $d < q$ from Lemma 6.4. Using the chosen basis, let

$$m(x_1, \ldots, x_q) = x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q}$$

be a monomial contained in f (for some $a_i \in \mathbb{Z}_{\geq 0}$). Then $\sum_i a_i = d$ and $\sum_{i=0}^{q} \lambda^i \tau^i(m)$ is a semi-invariant of G whenever $\lambda^q = 1$. So, without loss of generality, assume

$$f(x_1, \ldots, x_q) = \left[m + \lambda\tau(m) + \cdots + \lambda^{q-1}\tau^{q-1}(m)\right](x_1, \ldots, x_q)$$

Note that all the a_i are non-negative integers, not all zero, and $0 < \sum_i a_i = d < q$.

This semi-invariant can now be exploited to obtain a bound for the possible size of D. To do this, the following lemma is necessary:

Lemma 6.5. *Consider the following q by q matrix with integer coefficients:*

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_{q-1} & a_q \\ a_q & a_1 & \cdots & a_{q-2} & a_{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_q & a_1 \end{pmatrix}$$

The determinant of M is not zero.

Proof. Consider the matrix M over $\mathbb{C}$, and assume $\det M = 0$. Then one of the eigenvalues of M must be zero. The eigenvectors and eigenvalues of this matrix are easy to compute, so this implies

$$a_1 + \omega a_2 + \omega^2 a_3 + \ldots + \omega^{q-1} a_q = 0$$

for some ω with $\omega^q = 1$. Since all the a_i-s are non-negative integers, this is a sum of exactly $d = \sum_{i=1}^{q} a_i$ q-th roots of unity. So, by [10], d must be a sum of the prime factors of q. But, by the initial assumptions, q is prime, and $0 < d < q$, producing a contradiction. ☐

This allows to bound the size of cyclic subgroups of D:

Lemma 6.6. *Let $g \in D$, and let n be the smallest positive integer, such that g^n is a scalar matrix. Then $n < q^{2q+1}$.*

Proof. Assume $n > 1$. Since $g \in G \subset \mathrm{SL}_q(\mathbb{C})$, $g^n = \zeta_q \mathrm{I}_q$, where ζ_q is a q-th root of 1 and I_q is the identity matrix. Then, since all the elements of D are diagonal matrices,

$$g = \zeta_q^{\beta_0} \begin{pmatrix} \zeta_n^{\beta_1} & & \\ & \ddots & \\ & & \zeta_n^{\beta_q} \end{pmatrix}$$

where $\beta_i \in \mathbb{Z}$, not all zero, with $0 \leq \beta_i < n$ $\forall i > 0$; $0 \leq \beta_0 < q$. Since n was taken to be minimal, the highest common factor of $\{n, \beta_1, \ldots, \beta_q\}$ is 1.

Now consider the polynomial f of degree $d < q$ described above. Since we know $g \in G$, $g(f) = \lambda f$ for some $\lambda \in \mathbb{C}$. Since $g^{nq} = \mathrm{I}_q$ and all the monomials are g-semi-invariant, $\lambda = \zeta_q^{\beta_0} \zeta_n^C$, some $C \in \mathbb{Z}$. This is equivalent to:

$$\begin{aligned} C &\equiv a_1\beta_1 + a_2\beta_2 + \cdots + a_{q-1}\beta_{q-1} + a_q\beta_q \mod n \\ &\equiv a_1\beta_2 + a_2\beta_3 + \cdots + a_{q-1}\beta_q + a_q\beta_1 \mod n \\ &\equiv a_1\beta_3 + a_2\beta_4 + \cdots + a_{q-1}\beta_1 + a_q\beta_2 \mod n \\ &\dots \\ &\equiv a_1\beta_q + a_2\beta_1 + \cdots + a_{q-1}\beta_{q-2} + a_q\beta_{q-1} \mod n \end{aligned}$$

This can be rewritten as

$$M(\beta_1, \dots, \beta_q)^T \equiv C(1, \dots, 1)^T \mod n$$

where M is the matrix from Lemma 6.5). However, since $\sum_{i=1}^q a_i = d$, M also satisfies

$$M(1, \dots, 1)^T = d(1, \dots, 1)^T$$

Take $v = d(\beta_1, \dots, \beta_q)^T - C(1, \dots, 1)^T$. By linearity, $Mv \equiv 0 \mod n$. Multiplying both sides by the adjugate matrix of M, get:

$$\begin{aligned} (d\beta_1 - C)\det M &\equiv 0 \mod n \\ (d\beta_2 - C)\det M &\equiv 0 \mod n \\ &\dots \\ (d\beta_q - C)\det M &\equiv 0 \mod n \end{aligned}$$

Therefore,

$$d\beta_1 \det M \equiv d\beta_2 \det M \equiv \cdots \equiv d\beta_q \det M \equiv C \det M \mod n$$

This implies that $g^{d \det M}$ is a scalar matrix. By assumption, $0 < d < q$ (in $\mathbb{Z}$), and, by Lemma 6.5, $\det M \neq 0$ (in $\mathbb{Z}$), so $|d \det M| = Kn$ for some positive integer K. Thus, $n \leq |d \det M| \leq q\,|detM|$.

Now look at the entries $M_{i,j}$ of the matrix M. Since $0 \leq a_k \leq d < q$ for all k, $|M_{i,j}| \leq d < q$. Thus,

$$n \leq q\,|\det M| \leq q\left(q \max_{i,j} |M_{i,j}|\right)^q < q^{2q+1}$$

□

This result bounds the size of the biggest possible cyclic subgroup of D. This bounds the order of D, hence the order of G, and therefore the order of Γ. Therefore, this proves the following result:

Theorem 6.7. *For a given prime number q, there are only finitely many finite irreducible subgroups $\Gamma \subset SL_q(\mathbb{C})$, such that the singularity of $\mathbb{C}^n/\Gamma$ is not weakly exceptional.*

Note 9. *Given a prime number q, it is actually possible to compute all possible conjugacy classes for the group Γ: one simply needs to list all the possible monomials of degree d (for each $d < q$, combine them into τ-orbits and then solve the equations that are combined into the matrix M in Lemma 6.6 modulo* $\det M$. *This will produce all the possible subgroups D, which can easily be extended to groups G and then Γ. Here, one needs to keep several things in mind:*

- *This does not take into account the cases where D is trivial. Then $\Gamma = T \subset \mathbb{S}_q$. These groups need to be checked.*
- *The primitive groups need to be checked separately. For small dimensions, the classifications of such groups are known, but for large dimensions one would need to be computed first.*

Example 6.8. *Let $q = 5$, consider monomials of degree 4: the orbits of $x_1^3x_2$, $x_1^3x_3$, $x_1^3x_4$, $x_1^3x^5$, $x_1^2x_2^2$ and $x_1^2x_3^2$. In the first 4 cases the relevant determinant is $4 \cdot 61$, and in the last two cases it is 64. Solving the equations used to build the matrix M, obtain the specific actions of the elements of D. In the first 4 cases, these will be pairwise incompatible, so $|D| = 61$ (or $61 \cdot 5$ if D includes the center of $SL_5(\mathbb{C})$) and $T = \mathbb{Z}_5$. In the last two cases, the actions are compatible, so T can also contain the element $(23) \in \mathbb{S}_5$.*

Going through the calculations for $q = 5$ as in the above example produces the following result (Here, "$[n, a_1, \ldots, a_5]$" denotes the diagonal matrix with the diagonal entries $\zeta^{a_1}, \ldots, \zeta^{a_5}$, where ζ primitive n-th root of unity):

Theorem 6.9. *Let $G \subset SL_5(\mathbb{C})$ be a finite subgroup acting irreducibly. Then the singularity of $\mathbb{C}^5/G$ is weakly-exceptional exactly when:*

1. *The action of G is primitive and G contains a subgroup isomorphic to the Heisenberg group of all unipotent 3×3 matrices over $\mathbb{F}_5$ (for a better classification of all such groups, see [12]).*
2. *The action of G is monomial (making $G \cong D \rtimes T$, with D an abelian group as above and T a transitive subgroup of $\mathbb{S}_5$), and none of the following hold:*
   - *D is central in $SL_5(\mathbb{C})$. In this case, G can be isomorphic to $\mathbb{A}_5$, $\mathbb{S}_5$, or their central extensions by $\mathbb{Z}_5$.*
   - *$|G| = 55$ or $55 \cdot 5$ with $|D| = 11$ or $11 \cdot 5$ resp., $T \cong \mathbb{Z}_5 \subset \mathbb{S}_5$, and there is a $k \in \mathbb{Z}$, $1 \le k \le 4$, such that D is generated by $\left[11, 1, 4^k, 4^{2k}, 4^{3k}, 4^{4k}\right]$ and (in the latter case) also the scalar element $\zeta_5 \cdot Id$. In this case, G is isomorphic to $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ or $(\mathbb{Z}_5 \times \mathbb{Z}_{11}) \rtimes \mathbb{Z}_5$.*
   - *$|G| = 305$ or $305 \cdot 5$ with $|D| = 61$ or $61 \cdot 5$ resp., $T \cong \mathbb{Z}_5 \subset \mathbb{S}_5$, and there is a $k \in \mathbb{Z}$, $1 \le k \le 4$, such that D is generated by $\left[61, 1, 34^k, 34^{2k}, 34^{3k}, 34^{4k}\right]$ and (in the latter case) also the scalar element $\zeta_5 Id$. In this case, G is isomorphic to $\mathbb{Z}_{61} \rtimes \mathbb{Z}_5$ or $(\mathbb{Z}_5 \times \mathbb{Z}_{61}) \rtimes \mathbb{Z}_5$.*
   - *There exists some $d \in \{2, 3, 4\}$ and ω with $\omega^5 = 1$, such that:*
     - *$\forall g \in D$, g^d is a scalar.*
     - *$|D| \in \left\{d^k, 5 \cdot d^k\right\}$ (depending on whether D contains any non-trivial scalar elements) with $1 \le k \le 4$.*
     - *The polynomial $x_1^d + \omega x_2^d + \omega^2 x_3^d + \omega^3 x_4^d + \omega^4 x_5^d$ is G-semi-invariant.*

6.2. **Non-prime dimensions (Abandon all hope, ye who enter here).** Unfortunately, as soon as the assumption of the dimension being a prime number is dropped, the finiteness results disappear. To see that, it is sufficient to recall Example 5.2. It implies that whenever the dimension is a square, $n = k^2$, it is possible to find infinite families of irreducible groups G with semiinvariants of degree k. Also, whenever n is even, it is possible to construct infinite families of irreducible groups giving rise to singularities that are not weakly exceptional (using the construction in Example 5.1 for $a = 2$). So the only result one can hope for is

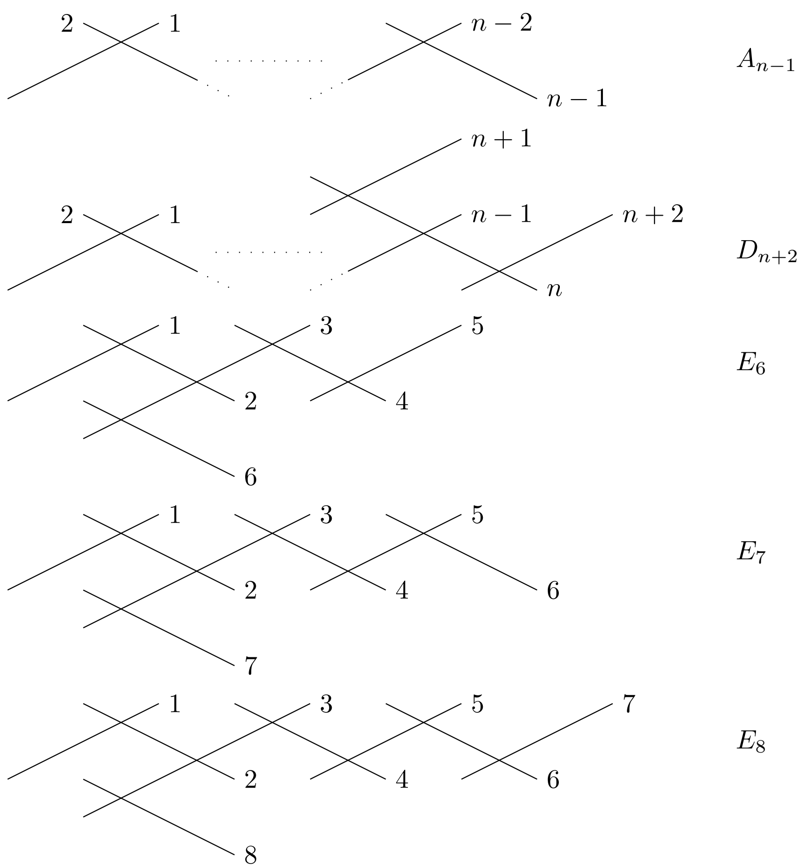
Conjecture 6.10. *All infinite families of groups G giving rise to non-weakly exceptional singularities can be obtain using the construction from Examples 5.1 and 5.2.*

However, even this is unknown, since the proof of Lemma 6.4 also relies on q being a prime.

Appendix A. ADE singularities

Example A.1 (see [18, Section 5.2.3])**.** *Consider the Du Val singularities (i.e.canonical singularitite sin dimension* 2*). It is well known, that these follow the well-known A-D-E classification. It can be seen by taking their minimal resolutions and seeing that the components of the exceptional divisor intersect as seen in Figure 1.*

Figure 1. Blowup graphs of 2-dimensional singularities

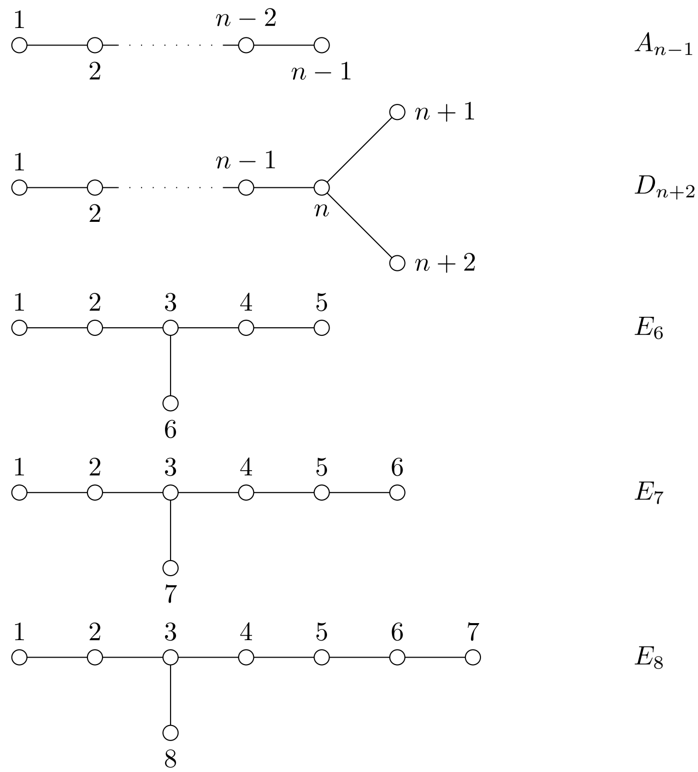


To make these look more familiar, take their dual graphs (where the vertices are the exceptional divisors, and the edges signify their intersection) and compare them with the standard Dynkin diagrams. These dual graphs can be seen in Figure 2.

On these graphs, one can easily see the possible plt blowups: for a singularity of type A_n*, choosing any curve on the resolution and blowing down all others produces a plt blowup. For singularities of types* D *and* E*, the chosen curve needs to be the unique curve that intersects three other curves — otherwise the log pair* (W, E) *(in the definition of plt blowups) will have singularities that are worse than log terminal. This implies that singularities of types* D *and* E *are weakly exceptional, while singularities of type* A *are not. In fact, singularities of type* E *are exceptional.*

It is also known that all these are in fact quotient singularities: they are all of the form $\mathbb{C}^2/G$*, where* $G \subset SL_2(\mathbb{C})$ *is a finite subgroup. Mathcing these up with the*

FIGURE 2. Dual blowup graphs of 2-dimensional singularities



list of finite subgroups of $SL_2(\mathbb{C})$ (see Section B.1), one finds that the groups $\overline{\mathbb{Z}}_n$ (or $\mathbb{Z}_n$ if n is odd), $\overline{\mathbb{D}}_{2n}$, $\overline{\mathbb{A}}_4$, $\overline{\mathbb{S}}_4$ and $\overline{\mathbb{A}}_5$ correspond to singularities of types A_{n-1}, D_{n+2}, E_6, E_7 and E_8 respectively. One can blow these singularities up and look at their blowup graphs.

APPENDIX B. FINITE SUBGROUPS OF $\mathrm{SL}_n(\mathbb{C})$ FOR SMALL VALUES OF n

B.1. **Finite subgroups of $\mathbf{SL}_2(\mathbb{C})$.** This is a well-known classical result, attributed to F. Klein (or sometimes to Plato). A modern treatment can be found in [4]. Note that care must be taking when writing out explicit matrix representations.

Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite group, Let $\bar{G}$ be its image under the projection to $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$. Then $\bar{G}$ belongs to one of the following classes:

- Cyclic: $\mathbb{Z}_n$, $n \geq 1$.
- Dihedral: $\mathbb{D}_{2n} = <a, b \mid a^n = b^2 = \mathrm{id},\ bab = a^{-1}>$ $(n \geq 2)$.
- Polyhedral groups $\mathbb{A}_4$, $\mathbb{S}_4$, $\mathbb{A}_5$.

Lifting the actions of these groups to $\mathrm{SL}_2(\mathbb{C})$, one sees that $\bar{G}$ must be conjugate to one of the following:

- Binary cyclic group

$$\overline{\mathbb{Z}}_n = <a \mid a^{2n} = 1>$$

All its faithful representations are 1–dimensional, and are of the form of $a \rightsquigarrow \zeta_{2n}^{l}$, some $l \in \mathbb{Z}$. Thus a 2–dimensional representation has to be a direct sum of two such.

- Cyclic group $\mathbb{Z}_n$, where n is odd. Similar to the binary cyclic group, this group is abelian, and so its two-dimensional representation must be a direct sum of two one-dimensional representations. When one projects these groups into $\mathrm{PGL}_2(\mathbb{C})$, the kernel is trivial, and a lift of the projection back to $\mathrm{SL}_2(\mathbb{C})$ can be chosen to be either $\mathbb{Z}_n$ or $\overline{\mathbb{Z}}_n$. To simplify notation later on, always choose to lift it as $\overline{\mathbb{Z}}_n$.
- Binary dihedral group
$$\overline{\mathbb{D}}_{2n} = <a, b \mid a^n = b^2, b^4 = 1, aba^{-1} = a^{-1}>$$
The suitable 2–dimensional representations of this group are indexed by different choices of ζ_{2n}. They are:
$$a \rightsquigarrow \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \ b \rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
- Binary tetrahedral group
$$\overline{\mathbb{A}}_4 = <\zeta_4(12)(34), \zeta_4(14)(23), \zeta_4(123)>$$
(using standard notation for elements of the symmetric group). Similarly to above, the suitable 2–dimensional representations of this group are determined by the choice of ζ_8. They are:
$$\zeta_4(12)(34) \rightsquigarrow \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & -\zeta_8^2 \end{pmatrix}, \ \zeta_4(14)(23) \rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\zeta_4(234) \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}$$
- Binary octahedral group
$$\overline{\mathbb{S}}_4 = <\zeta_4(12)(34), \zeta_4(14)(23), \zeta_4(123), \zeta_4(34)>$$
This group only has 2 suitable representations, each having a subrepresentation isomorphic to the representation of $\overline{\mathbb{A}}_4$ that uses the same value of ζ_8. The extra generator acts as
$$\zeta_4(34) \rightsquigarrow \begin{pmatrix} 0 & \zeta_8 \\ -\zeta_8^7 & 0 \end{pmatrix}$$
- Binary icosahedral group
$$\overline{\mathbb{A}}_5 = <\zeta_4(12345), \zeta_4(12)(34)>$$
$$\zeta_4(12345) \rightsquigarrow \begin{pmatrix} \zeta_5^3 & 0 \\ 0 & -\zeta_5^2 \end{pmatrix},$$
$$\zeta_4(12)(34) \rightsquigarrow \frac{1}{\sqrt{5}} \begin{pmatrix} -\zeta_5 + \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix}$$

One can see that these group actions are of the following types:

- The actions of cyclic groups are not irreducible.
- $\overline{\mathbb{A}}_4$, $\overline{\mathbb{S}}_4$, $\overline{\mathbb{A}}_5$ have primitive actions
- Binary dihedral groups have imprimitive monomial actions.

B.2. **Finite subgroups of $\mathbf{SL}_3(\mathbb{C})$.** This result can be seen in H.F. Blichfeldt's book ([1]). However, the classical treatment of this result (including that in this book) misses several of the groups. A more modern (and complete) treatment can be found in [20]).

Define the following matrices:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$U = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix} \quad Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix} \quad V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

where $\omega = e^{2\pi i/3}$, $\epsilon^3 = \omega^2$ and $a, b, c \in \mathbb{C}$ are chosen arbitrarily, as long as $abc = -1$ and Q generates a finite group.

Up to conjugacy, any finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ belongs to one of the following types:

(1) Diagonal abelian group.
(2) Group isomorphic to an irreducible finite subgroup of $\mathrm{GL}_2(\mathbb{C})$, preserving a plane $\mathbb{C}^2 \subset \mathbb{C}^3$, and not conjugate to a group of type (1).
(3) Group generated by the group in (1) and T and not conjugate to a group of type (1) or (2).
(4) Group generated by the group in (3) and Q and not conjugate to a group of types (1)—(3).
(5) Group E_{108} of size 108 generated by S, T and V.
(6) Group F_{216} of size 216 generated by the group E_{108} in (5) and an element $P := UVU^{-1}$.
(7) Hessian group $\mathbb{H}_{648}$ of size 648 generated by the group E_{108} in (5) and U.
(8) Simple group of size 60 isomorphic to alternating group $\mathbb{A}_5$.
(9) Klein's simple group $\mathbb{K}_{168}$ of size 168 isomorphic to permutation group generated by (1234567), (142) (356), (12) (35).
(10) Group of size 180 generated by the group $\mathbb{A}_5$ in (8) and W.
(11) Group of size 504 generated by the group $\mathbb{K}_{168}$ in (9) and W.
(12) Group G of size 1080 with its quotient $G/ <W>$ isomorphic to the alternating group $\mathbb{A}_6$.

One can see that these group actions are of the following types:

- The actions of groups of types (1) and (2) are not irreducible.
- Groups of types (3) and (4) have imprimitive monomial actions
- Groups E_{108}, F_{216}, $\mathbb{H}_{648}$, $\mathbb{A}_5$ and $\mathbb{K}_{168}$, as well as the central extensions of $\mathbb{A}_5$, $\mathbb{A}_6$ and $\mathbb{K}_{168}$ by scalar matrices, have primitive actions.

References

[1] H.F. Blichfeldt. *Finite collineation groups.* The University of Chicago Press, 1917.

[2] J.W. Bruce and C.T.C. Wall. On the classification of cubic surfaces. *J. London Math. Soc. (2)*, 19(2):245–256, 1979.

[3] Ivan Cheltsov and Constantin Shramov. On exceptional quotient singularities. *Geom. Topol.*, 15(4):1843–1882, 2011.

[4] I.V. Dolgachev. McKay correspondence. Winter 2006/07. Lecture notes.

[5] I.V. Dolgachev and V.A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston Inc., Boston, MA, 2009.

[6] W. Feit. The current situation in the theory of finite simple groups. In *Actes du Congres International des Mathématiciens (Nice, 1970), Tome*, volume 1, pages 55–93, 1970.

[7] F.G. Frobenius. *Über den von L. Bieberbach gefundenen Beweis eines Satzes von C. Jordan.* Königliche Akademie der Wissenschaften, 1911.

[8] T. Hosoh. Automorphism groups of cubic surfaces. *J. Algebra*, 192(2):651–677, 1997.

[9] S.A. Kudryavtsev. Pure log terminal blow-ups. *Mathematical Notes*, 69(5):814–819, 2001.

[10] T. Y. Lam and K. H. Leung. On vanishing sums of roots of unity. *J. Algebra*, 224(1):91–109, 2000.

[11] D. Markushevich and Y.G. Prokhorov. Exceptional quotient singularities. *American Journal of Mathematics*, 121(6):1179–1189, 1999.

[12] D. Mumford and G. Horrocks. A rank 2 vector bundle on $\mathbb{P}^4$ with 15,000 symmetries. *Topology*, 12:63–81, 1973.

[13] Isidro Nieto. The normalizer of the level $(2,2)$-Heisenberg group. *Manuscripta Math.*, 76(3-4):257–267, 1992.

[14] Y.G. Prokhorov. Blow-ups of canonical singularities. In *Algebra: proceedings of the International Algebraic Conference on the Occasion of the 90th Birthday of A.G. Kurosh, Moscow, Russia, May 25-30, 1998*, page 301. Walter De Gruyter Inc, 2000.

[15] B. Segre. *The Non-singular Cubic Surfaces.* Oxford University Press, Oxford, 1942.

[16] Jean-Pierre Serre. *Linear representations of finite groups.* Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[17] V. V. Shokurov. Three-dimensional log perestroikas. *Izv. Ross. Akad. Nauk Ser. Mat.*, 56(1):105–203, 1992.

[18] V.V. Shokurov. 3-fold log flips. *Russ. Ac. SC Izv. Math.*, 40(1):95–202, 1993.

[19] T.A. Springer. *Invariant theory.* Springer-Verlag, 1977.

[20] S.S.-T. Yau and Y. Yu. *Gorenstein quotient singularities in dimension three.* American Mathematical Society, Providence, RI, 1993.