CYLINDERS IN DEL PEZZO SURFACES

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Contents

1. Cylinders in rational surfaces

Let S be a surface with at most quotient singularities.

Definition 1.1. A Zariski open subset $U \subset S$ is said to be a *cylinder* if $U = \mathbb{C}^1 \times Z$ for some affine curve Z.

If S contains a cylinder, then S is ruled.

Exercise 1.2. Suppose that S is smooth and rational. Show that S contains a cylinder.

Exercise 1.3. Suppose that K_S is pseudo-effective. Show that S does not contain cylinders.

Now we are ready to present examples of rational singular surfaces that do not contain cylinders.

Exercise 1.4. Let $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be the elliptic curve of period $\tau = e^{\frac{2}{3}\pi}$. Its j-invariant is 0 and it is isomorphic to the Fermat cubic curve. Suppose that S is the quotient surface

$$E \times E/\langle \mathrm{diag}(-\tau, -\tau)\rangle.$$

Show that $K_S \sim_{\mathbb{Q}} 0$ and S is rational. Use Exercise 1.3 to conclude that S does not contain cylinders.

Exercise 1.5. Let a_1, a_2, a_3, a_4, w_1, w_2, w_3 and w_4 be positive integers with $\gcd(w_1, w_2, w_3, w_4) = 1$ that satisfy a system of equations

$$a_1w_1 + w_2 = a_2w_2 + w_3 = a_3w_3 + w_4 = a_4w_4 + w_1 = d$$

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with solutions

$$\begin{cases} w_1 = (a_2a_3a_4 - a_3a_4 + a_4 - 1), \\ w_2 = (a_1a_3a_4 - a_1a_4 + a_1 - 1), \\ w_3 = (a_1a_2a_4 - a_1a_2 + a_2 - 1), \\ w_4 = (a_1a_2a_3 - a_2a_3 + a_3 - 1), \\ d = a_1a_2a_3a_4 - 1. \end{cases}$$

Suppose that the surface S is the Klein-type hypersurface in $\mathbb{P}(w_1, w_2, w_3, w_4)$ defined by the quasi-homogeneous equation of degree d

$$x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0.$$

Show that S is a rational surface of Picard number three with 4 cyclic quotient singularities. Furthermore, prove that K_S is ample provided that all numbers a_1, a_2, a_3, a_4 are all greater than 3. Use Exercise 1.3 to conclude that S does not contain cylinders.

The surface in Exercise 1.4 has numerically trivial canonical divisor. The surfaces in Exercise 1.5 have ample canonical divisor. They all do not contain cylinders by Exercise 1.3. However, it is much more interesting to consider the same problem for surfaces whose *anticanonial* divisor is ample. Such surfaces are usually called *del Pezzo surfaces* (see Definition 3.1). They are always rational, but they often contains plenty of cylinders. To construct examples of del Pezzo surfaces without cylinders we need new tools.

2. Singularities of pairs

Let S be a surface with at most quotient singularities, let D be an effective non-zero $\mathbb{Q}$-divisor on the surface S, and let P be a point in the surface S. Put $D = \sum_{i=1}^{r} a_iC_i$, where each C_i is an irreducible curve on S, and each a_i is a non-negative rational number. We assume here that all curves $C_1, \ldots, C_r$ are different. We call (S, D) a *log pair*.

Let $\pi\colon \tilde{S} \to S$ be a birational morphism such that $\tilde{S}$ is smooth. For each C_i, denote by $\tilde{C}_i$ its proper transform on the surface $\tilde{S}$. Let $F_1, \ldots, F_n$ be π-exceptional curves. Then

$$K_{\tilde{S}} + \sum_{i=1}^{r} a_i\tilde{C}_i + \sum_{j=1}^{n} b_jF_j \sim_{\mathbb{Q}} \pi^*(K_S + D)$$

for some rational numbers $b_1, \ldots, b_n$. Suppose, in addition, that $\sum_{i=1}^{r} \tilde{C}_i + \sum_{j=1}^{n} F_j$ is a divisor with simple normal crossings.

Definition 2.1. The log pair (S, D) is said to be *log canonical* at the point P if the following two conditions are satisfied:

- $a_i \leqslant 1$ for every C_i such that $P \in C_i$,
- $b_j \leqslant 1$ for every F_j such that $\pi(F_j) = P$.

This definition is independent on the choice of birational morphism $\pi\colon \tilde{S} \to S$ provided that the surface $\tilde{S}$ is smooth and $\sum_{i=1}^{r} \tilde{C}_i + \sum_{j=1}^{n} F_j$ is a divisor with simple normal crossings.

Exercise 2.2. Let R be any effective $\mathbb{Q}$-divisor on S such that $R \sim_{\mathbb{Q}} D$ and $R \neq D$. Put

$$D_\epsilon := (1+\epsilon)D - \epsilon R$$

for some rational number $\epsilon \geqslant 0$. Then $D_\epsilon \sim_{\mathbb{Q}} D$. Show that there exists the greatest rational number $\epsilon_0 \geqslant 0$ such that the divisor D_{ϵ_0} is effective. Show that $\mathrm{Supp}(D_{\epsilon_0})$ does not contain at least one irreducible component of $\mathrm{Supp}(R)$. Moreover, if (S, D) is not log canonical at P, and (S, R) is log canonical at P, show that the log pair (S, D_{ϵ_0}) is not log canonical at P.

The log pair (S, D) is called *log canonical* if it is log canonical at every point of S.

Exercise 2.3. Suppose that S is smooth at P. Let $f\colon \bar{S} \to S$ be a blow up of the point P, and let E be the f-exceptional curve. Denote by $\bar{D}$ the proper transform of the divisor D on the surface $\bar{S}$ via f. One has

$$K_{\bar{S}} + \bar{D} + \Big(\mathrm{mult}_P(D) - 1\Big)E \sim_{\mathbb{Q}} f^*\big(K_S + D\big).$$

Then the log pair

$$\Big(\bar{S}, \bar{D} + (\mathrm{mult}_P(D) - 1)E\Big)$$

is called *the log pull back* of the log pair (S, D) on the surface $\bar{S}$. Show that it is log canonical at every point of the curve E if and only if the log pair (S, D) is log canonical at the point P. Conclude that (S, D) is not log canonical at P provided that $\mathrm{mult}_P(D) > 2$,

Exercise 2.4. Suppose that S is smooth at P and (S, D) is not log canonical at P. Prove that $\mathrm{mult}_P(D) > 1$.

We can measure how far the pair (S, D) is from being log canonical at P by the positive rational number

$$\mathrm{lct}_P(S, D) := \sup\Big\{\lambda \in \mathbb{Q} \mid \text{the log pair } (S, \lambda D) \text{ is log canonical at } P\Big\}.$$

This number has been introduced by Shokurov and is called the *log canonical threshold* of the pair (S, D) at the point $P \in S$. The log canonical threshold of the pair (S, D) is defined as

$$\mathrm{lct}\big(S, D\big) := \inf_{O \in S} \mathrm{lct}_O(S, D).$$

Exercise 2.5. Suppose that S is smooth at P. Show that

$$\frac{2}{\mathrm{mult}_P(D)} \geqslant \mathrm{lct}_P\big(S, D\big) \geqslant \frac{1}{\mathrm{mult}_P(D)}.$$

The following exercise is a very special case of a much more general result known as *Inversion of Adjunction* (see, for example, [16, Theorem 6.29]).

Exercise 2.6 ([16, Exercise 6.31])**.** Suppose that both S and C_1 is smooth at P, the log pair (S, D) is not log canonical at P, and $a_1 \leqslant 1$. Put $\Delta = \sum_{i=2}^{r} a_i C_i$. Show that $\mathrm{mult}_P(C_1 \cdot \Delta) > 1$.

Exercise 2.7. In the notation and assumptions of Exercise 2.3, suppose that (S, D) is not log canonical at P, and $\mathrm{mult}_P(D) \leqslant 2$. Show that there exists a unique point in E such that $(S, \bar{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at it.

Exercise 2.8. Suppose that S has a singular point of type D_4 at a point P. Let $g\colon \hat{S} \to S$ be the minimal resolution of the point P. Denote by E_1, E_2, E_3 and E_4 the g-exceptional curves, where E_4 is the (-2)-curve intersecting the other three (-2)-curves. Denote by $\hat{D}$ the proper transform of the $\mathbb{Q}$-divisor D on the surface $\hat{S}$. Then

$$\hat{D} \sim_{\mathbb{Q}} g^*(D) - \sum_{i=1}^{4} a_i E_i,$$

for some rational numbers a_1, a_2, a_3 and a_4. Show that the log pair (S, D) is not log canonical at P if and only if $a_4 > 1$.

Exercise 2.9. Suppose that S is smooth at P. Suppose that both curves C_1 and C_2 are also smooth at P and intersect each other transversally at P. Put $\Delta = \sum_{i=3}^{r} a_i C_i$. Suppose that (S, D) is not log canonical at P, and $\mathrm{mult}_P(\Delta) \leqslant 1$. Show that $\mathrm{mult}_P(C_1 \cdot \Delta) > 2(1 - a_2)$ or $\mathrm{mult}_P(C_1 \cdot \Delta) > 2(1 - a_1)$.

Exercise 2.10. Suppose that S is smooth at P. Suppose that both curves C_1 and C_2 are also smooth at P and intersect each other transversally at P. Put $\Delta = \sum_{i=3}^{r} a_i C_i$. Suppose that (S, D) is not log canonical at P, and suppose that there are non-negative rational numbers α, β, A, B, M, and N such that $\alpha a_1 + \beta a_2 \leqslant 1$, $A(B-1) \geqslant 1$, $M \leqslant 1$, $N \leqslant 1$, $\alpha(1-M) + A\beta \geqslant A$ and

$$\alpha(A + M - 1) \geqslant A^2(B + N - 1)\beta.$$

Suppose, in addition, that $2M + AN \leqslant 2$ or

$$\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geqslant AB - 1.$$

Show that $\mathrm{mult}_P(C_1 \cdot \Delta) > M + Aa_1 - a_2$ or $\mathrm{mult}_P(C_2 \cdot \Delta) > N + Ba_2 - a_1$.

All exercises we have considered so far in this section are local. Let us conclude this section by two *global* exercises.

Exercise 2.11. Suppose that S is a smooth surface in $\mathbb{P}^3$, and D is $\mathbb{Q}$-linearly equivalent to its hyperplane section. Prove that each a_i does not exceed 1.

Exercise 2.12. Suppose that S is smooth at P, and there is a double cover $\tau\colon S \to \mathbb{P}^2$ branched over quartic curve C that has at most two ordinary double points. Suppose that D is $\mathbb{Q}$-linearly equivalent to $-K_S$. Show that each a_i does not exceed 1. If (S, D) is not log canonical at P, show that $\tau(P) \in C$.

3. Del Pezzo surfaces without cylinders

Let S be a surface with at most quotient singularities such that the divisor $-K_S$ is ample. By the Nakai–Moishezon criterion, the latter condition is equivalent to $K_S^2 > 0$ and $-K_S \cdot C > 0$ for every curve C on S. Note that K_S^2 is a rational number.

Definition 3.1. We say that S is a *del Pezzo surface* of degree K_S^2.

Del Pezzo surfaces with quotient singularities are indeed rational. This easily follows from Castelnuovo rationality criterion, basic vanishing theorems and the fact the fact that quotient singularities are rational. Moreover, smooth and mildly singular del Pezzo surfaces are completely classified.

Exercise 3.2. Suppose that S is a smooth del Pezzo surface of degree d. Show that either $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $d = 8$, or $d \leqslant 9$ and S a blow up of $\mathbb{P}^2$ in $9 - d$ points such that

- no three of them lie on a one line,
- no six of them lie on a one conic,
- no 8 of them lie on a cubic curve that is singular in one of them.

Exercise 3.3. Suppose that S is a del Pezzo surface of degree d such that S has Du Val singularities. Show that either $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $d = 8$, or S is a quadric cone in $\mathbb{P}^3$ and $d = 8$, or $d \leqslant 9$ and there exists a diagram

$$\begin{array}{ccc} & \tilde{S} & \\ {}^{f}\swarrow & & \searrow^{g} \\ S & & \mathbb{P}^2, \end{array}$$

where f and g are birational morphisms such that $\tilde{S}$ is smooth, $K_{\tilde{S}} \sim f^*(K_S)$, f contracts all curves with self-intersection -2, and g is a blow up of $\mathbb{P}^2$ in $9 - d$ points such that no four of them lie on a one line, and no seven them lie on a one conic. The surface $\tilde{S}$ is a weak del Pezzo surface that corresponds to the surface S.

Furthermore, if S is a del Pezzo surface of degree $d \geqslant 3$ with at worst du Val singularities, then its the anticanonical divisor is very ample, and the anticanonical linear system embeds S into the projective space $\mathbb{P}^d$. In particular, del Pezzo surface of degree 3 with du Val singularities is a cubic surface in $\mathbb{P}^3$. Similarly, del Pezzo surfaces of degree 2 with du Val singularities are hypersurfaces in $\mathbb{P}(1,1,1,2)$ of degree 4, and del Pezzo surfaces of degree 2 with du Val singularities are hypersurfaces in the weighted projective space $\mathbb{P}(1,1,2,3)$ of degree 6. This is all well-known (see, for example, [11] or [15, Theorem 4.4]).

Exercise 3.4. Suppose that S is smooth del Pezzo surface of degree $d \leqslant 3$. Let D be an effective $\mathbb{Q}$-divisor on S, i.e., $D = \sum_{i=1}^{r} a_i C_i$, where every C_i is an irreducible curve on S, and every a_i is a non-negative rational number. Suppose that $D \sim_{\mathbb{Q}} -K_S$. Show that each a_i does not exceed 1. If (S, D) is not log canonical at some point $P \in S$, show that there exists a unique divisor $T \in |-K_S|$ such that T is singular at P, the log pair (S, T) is not log canonical at P, and all irreducible components of T is contained in $\mathrm{Supp}(D)$.

If S is smooth, then it always contains cylinders by Exercise 1.2. If S is singular, this is no longer the case. To see this, we need

Exercise 3.5. Suppose that S contains a cylinder U. Denote by $C_1, \ldots, C_n$ the irreducible curves in S such that $S \setminus U = \sum_{i=1}^{n} C_i$. Show that n is at least the dimension of the vector space $\mathrm{Pic}(S) \otimes \mathbb{Q}$. Suppose that there are non-negative rational numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i C_i \sim_{\mathbb{Q}} -K_S.$$

Show that the singularities of the log pair $(S, \sum_{i=1}^{n} \lambda_i C_i)$ are not log canonical.

Now we can give explicit examples of del Pezzo surfaces with Du Val singularities without cylinders.

Exercise 3.6. Show that there exists a del Pezzo surface of degree 1 with Du Val singularities whose singular locus consists of two singular points of type $\mathbb{D}_4$. Show that there exists a del Pezzo surface of degree 1 with Du Val singularities whose singular locus consists of two singular points of type $\mathbb{A}_3$ and two singular points of type $\mathbb{A}_1$. Show that there exists a del Pezzo surface of degree 1 with Du Val singularities whose singular locus consists of four singular points of type $\mathbb{A}_2$. Suppose that S is one of these surfaces. Show that S contains no cylinders.

One can show that surfaces described in Exercise 3.6 are the only del Pezzo surfaces with du Val singularities that contains no cylinders.

Exercise 3.7. Suppose that for every effective $\mathbb{Q}$-divisor D on S such that $D \sim_{\mathbb{Q}} -K_S$, the log pair (S, D) has log canonical singularities. Suppose, in addition, that S is of of Picard rank 1, i.e., one has $\mathrm{Pic}(S) \otimes \mathbb{Q} \cong \mathbb{Q}$.

Surfaces that satisfy all hypotheses of Exercise 3.7 do exist. One such example has been constructed by Keel and Mckernan in [18, Example 21.3.3]. Moreover, in their example the smooth locus of the surface has trivial algebraic fundamental groups, which provides a counter-example to a conjecture by Miyanishi that smooth locus of every del Pezzo surface of Picard rank 1 with quotient singularities has a finite unramified covering that contains a cylinder.

4. α-invariants of Tian of polarized pairs

Let S be a surface with quotient singularities, and let H be an ample $\mathbb{Q}$-Cartier divisor on it. For the pair (S, H), we define its α-invariant as

$$\alpha(S,H) := \sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (S,\lambda D) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D\sim_{\mathbb{Q}} H\end{array}\right\}\in\mathbb{R}_{>0}.$$

Exercise 4.1. For every ample divisor H on the surfce S, compute $\alpha(S,H)$ in the case when S is one of the following surfaces: $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, or $\mathbb{F}_1$.

The number $\alpha(S,H)$ has been studied intensively by many people who used different notations for it. The notation $\alpha(S,H)$ is due to Tian who defined the number $\alpha(S,H)$ in a different way (see [25, Appendix 2]). Both the definitions match by [8, Theorem A.3]. The α-invariant plays an important role in Kähler geometry, e.g., if S is a del Pezzo surface and $\alpha(S,-K_S)>\frac{2}{3}$, then S admits an orbifold Kähler–Einstein metric (see [24] and [10]).

Exercise 4.2. Suppose that S is a smooth del Pezzo surface of degree $d\leqslant 3$. Compute $\alpha(S,-K_S)$.

The number $\alpha(S,H)$ is usually hard to compute. However, it can be approximated by numbers that are much easier to control. Namely, if nH is a Weil divisor such that $|nH|$ is not empty for some $n\geqslant 1$, then we can define the n-th α-invariant of the pair (S,H) as

$$\alpha_n(S,H) := \sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \text{the pair } \left(S,\frac{\lambda}{n}D\right) \text{ is log canonical for every } D\in|nH|\right\}\in\mathbb{Q}_{>0}.$$

Otherwise we can simply put $\alpha_n(S,H)=+\infty$. Thus, we have $\alpha(S,H)\leqslant\alpha_n(S,H)$ by definition.

Exercise 4.3. Show that

$$\alpha(S,H)=\inf_{n\geqslant 1}\Big\{\alpha_n(S,H)\Big\}.$$

It is natural to expect that $\alpha(S,H)=\alpha_1(S,H)$ provided that H is a very ample Cartier divisor on S (see [25, Conjecture 5.3]). This is indeed true in many cases.

Exercise 4.4. If S is a smooth del Pezzo surface, show that $\alpha(S,-K_S)=\alpha_1(S,-K_S)$.

Exercise 4.5. Suppose that S is a smooth surface in $\mathbb{P}^3$ of degree $d\leqslant 4$, and H is its hyperplane section. Show that $\alpha(S,H)=\alpha_1(S,H)$.

However, this is not true in general:

Exercise 4.6. Suppose that S is a general surface in $\mathbb{P}^4$ of degree $d\geqslant 8$, and H is its hyperplane section. Show that $\alpha(S,H)<\alpha_1(S,H)$.

5. Anticanonical cylinders in del Pezzo surfaces

Let S be a del Pezzo surface with at most quotient singularities.

Definition 5.1. An *anticanonical* cylinder in S is an Zariski open subset U of S such that

(C) $U=\mathbb{A}^1\times Z$ for some affine curve Z, i.e., U is a cylinder in S,
(P) there is an effective $\mathbb{Q}$-divisor D on S with $D\sim_{\mathbb{Q}}-K_S$ and $U=S\setminus\mathrm{Supp}(D)$.

We know that there are singular del Pezzo surfaces without cylinders, so that there are singular del Pezzo surfaces without anticanonical cylinders as well. An easy way to construct infinitely many families of such surfaces is by using

Exercise 5.2. Suppose that S contains an anticanonical cylinder U. Denote by $C_1, \ldots, C_n$ the irreducible curves in S such that $S \setminus U = \sum_{i=1}^{n} C_i$, and let $\lambda_1, \ldots, \lambda_n$ be non-negative rational numbers such that $\sum_{i=1}^{n} \lambda_i C_i \sim_{\mathbb{Q}} -K_S$. Show that the singularities of the log pair $(S, \sum_{i=1}^{n} \lambda_i C_i)$ are not log canonical. Conclude that $\alpha(S, -K_S) < 1$.

If $\alpha(S, -K_S) \geqslant 1$, then S is usually called *weakly-exceptional*. We see that weakly-exceptional del Pezzo surfaces do not contain anticanonical cylinders. By Exercises 4.2 and 4.4, smooth del Pezzo surface is weakly-exceptional if and only if it has degree 1 and its anticanonical linear system does not contain cuspidal curves. Weakly-exceptional del Pezzo surfaces with du Val singularities has been classified in [4]. Many weakly-exceptional del Pezzo surfaces has been constructed in [5] and [9].

Exercise 5.3. Suppose that S is a smooth del Pezzo surface of degree $d \leqslant 3$. Show that S does not contain anticanonical cylinders.

Exercise 5.4. Suppose that S is a smooth del Pezzo surface of degree $d \geqslant 4$. Show that S contains an anticanonical cylinder.

Thus, if S is a smooth del Pezzo surface of degree d, then it does not contain anticanonical cylinders if and only if $d \leqslant 3$. This can be generalized for del Pezzo surface with du Val singularities as follows.

Exercise 5.5. Suppose that S is a del Pezzo surface of degree 1 whose singular points are du Val singular points of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, or $\mathbb{D}_4$. Show that S does not contain anticanonical cylinders.

Exercise 5.6. Suppose that S is a del Pezzo surface of degree 2 with only ordinary double points. Show that S does not contain anticanonical cylinders.

Exercise 5.7. Suppose that S is a singular cubic surface that has du Val singularities. Show that S contains an anticanonical cylinder.

Exercise 5.8. Suppose that S is a del Pezzo surface of degree $d \geqslant 4$ with du Val singularities. Show that S contains an anticanonical cylinder.

Exercise 5.9. Suppose that S is a del Pezzo surface of degree d with du Val singularities. Show that S contains an anticanonical cylinder if and only if one of the following conditions holds:

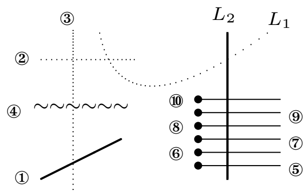
- $d \geqslant 4$,
- $d = 3$ and S is singular,
- $d = 2$ and S has a singular point that is not of type $\mathbb{A}_1$,
- $d = 1$ and S has a singular point that is not of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, or $\mathbb{D}_4$.

We show how to construct an anticanonical cylinder on a del Pezzo surface of degree 2 with a single Du Val singular point of type $\mathbb{A}_2$.

Example 5.10. On the projective plane $\mathbb{P}^2$, take a $\mathbb{Q}$-divisor

$$D_{\mathbb{P}^2} = \frac{7}{4}L_1 + \frac{5}{4}L_2,$$

where L_1 and L_2 are distinct two lines. Let $h\colon \check{S} \to \mathbb{P}^2$ be the composition of these ten blow ups. Denote by $E_{①}$ (resp. $E_{②}, \ldots, E_{⑩}$) the proper transform of the exceptional divisor of the first (resp. second,…, tenth) blow up to the surface $\check{S}$. Suppose that this blow ups follow the depicted instruction:



Here we labeled $E_{\textcircled{1}}$ (resp. $E_{\textcircled{2}}, \ldots, E_{\textcircled{10}}$) by ① (resp. ②, …, ⑩) for simplicity. We then obtain

$$K_{\check{S}} + D_{\check{S}} \sim_{\mathbb{Q}} h^*\Big(K_{\mathbb{P}^2} + D_{\mathbb{P}^2}\Big) \sim_{\mathbb{Q}} 0,$$

where $D_{\check{S}}$ is the divisor

$$\frac{3}{4}E_{\textcircled{1}} + \frac{1}{4}E_{\textcircled{4}} + \frac{1}{4}E_{\textcircled{5}} + \frac{1}{4}E_{\textcircled{6}} + \frac{1}{4}E_{\textcircled{7}} + \frac{1}{4}E_{\textcircled{8}} + \frac{1}{4}E_{\textcircled{9}} + \frac{1}{4}E_{\textcircled{10}} + \frac{5}{4}L_2 + \frac{6}{4}E_{\textcircled{2}} + \frac{5}{4}E_{\textcircled{3}} + \frac{7}{4}L_1$$

Here, the proper transforms of L_1 and L_2 by h are denoted using the same notation. On the surface $\check{S}$, the curve L_2 is a (-5)-curve, the curve $E_{\textcircled{1}}$ is a (-3)-curve, the curves $E_{\textcircled{2}}$, $E_{\textcircled{3}}$ are (-2)-curves and the other eight curves in the second column of the table are (-1)-curves. Starting from the (-1)-curve L_1, we can contract $E_{\textcircled{2}}$ and $E_{\textcircled{3}}$ in turn to the smooth weak del Pezzo surface $\tilde{S}$ corresponding to a del Pezzo surface S of degree 2 with singularity type $\mathbb{A}_2$ (see Exercise 3.3). Denote the composition of these three blow downs by $g\colon \check{S} \to \tilde{S}$. Put

$$D_{\tilde{S}} = g\left(\frac{3}{4}E_{\textcircled{1}} + \frac{1}{4}E_{\textcircled{4}} + \frac{1}{4}E_{\textcircled{5}} + \frac{1}{4}E_{\textcircled{6}} + \frac{1}{4}E_{\textcircled{7}} + \frac{1}{4}E_{\textcircled{8}} + \frac{1}{4}E_{\textcircled{9}} + \frac{1}{4}E_{\textcircled{10}} + \frac{5}{4}L_2\right).$$

This is an effective anticanonical $\mathbb{Q}$-divisor on the surface $\tilde{S}$. Note that the curves $g(E_{\textcircled{1}})$ and $g(L_2)$ are the only (-2)-curves on the surface $\tilde{S}$ and they intersect each other in the form of $\mathbb{A}_2$. Contracting these two (-2)-curves, we obtain a birational morphism $f : \tilde{S} \to S$, where S is a del Pezzo surface of degree 2 with one singular point of type $\mathbb{A}_2$. Put

$$D_S = f \circ g\left(\frac{1}{4}E_{\textcircled{4}} + \frac{1}{4}E_{\textcircled{5}} + \frac{1}{4}E_{\textcircled{6}} + \frac{1}{4}E_{\textcircled{7}} + \frac{1}{4}E_{\textcircled{8}} + \frac{1}{4}E_{\textcircled{9}} + \frac{1}{4}E_{\textcircled{10}}\right).$$

Then D_S an effective $\mathbb{Q}$-divisor on the surface S such that $D_S \sim_{\mathbb{Q}} -K_S$, and

$$S \setminus \mathrm{Supp}(D_S) \cong \mathbb{P}^2 \setminus \mathrm{Supp}(D_{\mathbb{P}^2}) \cong \mathbb{C} \times \mathbb{C}^\star$$

is a cylinder. Note that we have some freedom for the coefficients in the divisor $D_{\check{S}}$. We have fixed its coefficients just for simplicity. Namely, we can replaced consider $D_{\mathbb{P}^2}$ above by

$$(2-\epsilon)\,L_1 + (1+\epsilon)\,L_2$$

Then the proper transform of the divisor $D_{\mathbb{P}^2}$ by the birational morphism h must be replaced by

$$(1-\epsilon)E_{\textcircled{1}} + (1-3\epsilon)E_{\textcircled{4}} + \epsilon E_{\textcircled{5}} + \epsilon E_{\textcircled{6}} + \epsilon E_{\textcircled{7}} + \epsilon E_{\textcircled{8}} + \\ + \epsilon E_{\textcircled{9}} + \epsilon E_{\textcircled{10}} + (1+\epsilon)L_2 + (2-2\epsilon)E_{\textcircled{2}} + (2-3\epsilon)E_{\textcircled{3}} + (2-\epsilon)L_1$$

For this divisor to be effective and to contain the exceptional divisors of the birational morphism h, it is enough to take a rational number ϵ such that $0 < \epsilon < \frac{1}{3}$. In our original $D_{\mathbb{P}^2}$, we have simply chosen $\epsilon = \frac{1}{4}$.

One can use construction in this example to prove the existence of an anticanonical cylinder on *every* del Pezzo surface of degree 2 with a single Du Val singular point of type $\mathbb{A}_2$ (see [7] for details).

6. Polarized cylinders in smooth del Pezzo surfaces

Let S be a smooth del Pezzo surface of degree d.

Remark 6.1. The Mori cone $\overline{\mathbb{NE}}(S)$ of the surface is polyhedral. Moreover, if $d \leqslant 7$, then $\overline{\mathbb{NE}}(S)$ is generated by all (-1)-curves in S. This is well-known (see, for example, [12, Theorem 8.2.23]).

Let H be an ample $\mathbb{Q}$-divisor on the surface S. Let us generalize Definition 5.1 as follows:

Definition 6.2. An H-polar cylinder in S is an Zariski open subset U of S such that

(C) $U = \mathbb{A}^1 \times Z$ for some affine curve Z, i.e., U is a cylinder in S,
(P) there is an effective $\mathbb{Q}$-divisor D on S with $D \sim_{\mathbb{Q}} H$ and $U = S \setminus \mathrm{Supp}(D)$.

This notion has been introduced and utilized by Kishimoto, Prokhorov and Zaidenberg in [19], [20] and [21]. It plays an important role in the study of the unipotent group actions on affine cones, e.g., [20, Corollary 3.2] implies

Theorem 6.3. Suppose that H is an ample Cartier divisor on S. Put

$$V := \mathrm{Spec}\Bigg(\bigoplus_{n\geqslant 0} H^0\Big(S, \mathcal{O}_S(nH)\Big)\Bigg).$$

If V is normal, then it admits an effective algebraic action of the additive group $\mathbb{C}_+$ if and only if the surface S contains an H-polar cylinder.

This theorem and Exercise 5.3 imply

Corollary 6.4. Let V be a threefold in $\mathbb{C}^3$ that is given by

$$f_3(x, y, z, w) = 0,$$

where f_3 is a homogeneous polynomial. Suppose that V has isolated singularity at the origin. Then V does is not admit an effective algebraic action of the additive group $\mathbb{C}_+$.

Example 6.5 (cf. [13, Question 2.22])**.** The threefold in $\mathbb{C}^3$ that is given by

$$x^3 + y^3 + z^3 + w^3 = 0$$

does not admit an effective algebraic action of the additive group $\mathbb{C}_+$.

Let $\mathrm{Amp}(S)$ be the ample cone of S. Denote by $\mathrm{Amp}^{cyl}(S)$ the set

$$\Big\{H \in \mathrm{Amp}(S) : \text{ there is an } H\text{-polar cylinder on } S\Big\}.$$

We will call this set *the cone of cylindrical ample divisors* of the surface S.

Exercise 6.6. Show that $\mathrm{Amp}^{cyl}(S)$ is not empty.

Exercise 6.7. Suppose that $d \geqslant 8$. Show that $\mathrm{Amp}^{cyl}(S) = \mathrm{Amp}(S)$.

By Exercises 5.3 and 5.4, we know that

$$-K_S \in \mathrm{Amp}^{cyl}(S) \iff d \geqslant 4.$$

To study $\mathrm{Amp}^{cyl}(S)$ more systematically, let us recall the invariant of the pair (S, H) defined by Hasset, Tanimoto and Tschinkel in [17, Definition 2.2]. This number was implicitly introduced by Fujita in [14]. It plays an essential role in Manin's conjecture (see, for example, [17]).

Definition 6.8. The Fujita invariant of the pair (S, H) is the positive rational number

$$\mu_H := \inf\Big\{\lambda \in \mathbb{Q}_{>0} \ \Big|\ \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda H \text{ is pseudo-effective}\Big\}.$$

The smallest extremal face Δ_H of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu_H H$ is called the Fujita face of H. The Fujita rank of (S, H) is defined by $r_H := \dim \Delta_H$.

Now we can generalize Exercises 3.5 and 5.2 as

Exercise 6.9. Suppose that S contains an H-polar cylinder U. Denote by $C_1, \ldots, C_n$ the irreducible curves in S such that $S \setminus U = \sum_{i=1}^n C_i$. Let $\lambda_1, \ldots, \lambda_n$ be some non-negative rational numbers such that

$$\sum_{i=1}^{n} \lambda_i C_i \sim_{\mathbb{Q}} H.$$

Show that the singularities of the log pair $(S, \mu_H \sum_{i=1}^n \lambda_i C_i)$ are not log canonical. Conclude that $\alpha(S, H) < \frac{1}{\mu_H}$.

Let $\phi_H \colon S \to Z$ be the contraction given by the Fujita face Δ_H of the divisor H. Then either ϕ_H is a birational morphism, or ϕ_H is a conic bundle with $Z \cong \mathbb{P}^1$. In the former case, the ample $\mathbb{Q}$-divisor H is said to be of type $B(r_H)$, and in the latter case it is said to be of type $C(r_H)$.

Remark 6.10. If H is of type $B(0)$, then ϕ_H is an isomorphism and

$$H \sim_{\mathbb{Q}} -\lambda K_S$$

for some positive rational number λ. In this case, every H-polar cylinder is an anticanonical cylinder.

The Fujita invariants can be used to describe the ample cone $\mathrm{Amp}(S)$ explicitly.

Exercise 6.11. Suppose that H is of type $B(r_H)$ and $r_H > 0$. Show that Z is a del Pezzo surface of degree $d + r_H$, and the Fujita face Δ_H is generated by r_H disjoint (-1)-curves on S contracted by ϕ_H, where $r_H \leqslant 9 - d$. Denote these (-1)-curves by $E_1, \ldots, E_{r_H}$. Show that

$$K_S + \mu_H H \sim_{\mathbb{Q}} \sum_{i=1}^{r_H} a_i E_i$$

for some positive rational numbers $a_1, \ldots, a_{r_H}$ such that $a_i < 1$ for every i. Vice versa, for every positive rational numbers $\epsilon_1, \ldots, \epsilon_{r_H}$ that are less than 1, show that the divisor $-K_S + \sum_{i=1}^{r_H} \epsilon_i E_i$ is ample.

Let us denote the set of all ample $\mathbb{Q}$-divisors of type $B(r_H)$ on S by $\mathrm{Amp}^B_{r_H}(S)$. It follows from Remark 6.10 that $\mathrm{Amp}^B_0(S)$ is the ray generated by the anticanonical divisor $-K_S$.

Exercise 6.12. Suppose that H is of type $C(r_H)$. Show that $r_H = 9 - d > 0$, the Fujita face Δ_H is generated by the (-1)-curves in the $8 - d$ reducible fibers of ϕ_H, and each reducible fiber consists of two (-1)-curves that intersect transversally at one point. Denote by B the general fiber of ϕ_H. Show that there are $(8 - d)$ disjoint (-1)-curves $E_1, E_2, \ldots, E_{8-d}$, each of which is contained in a distinct fiber of ϕ_H, such that

$$K_S + \mu_H H \sim_{\mathbb{Q}} aB + \sum_{i=1}^{8-d} a_i E_i$$

for some positive rational number a and non-negative rational numbers $a_1, \ldots, a_{8-d}$ such that $a_i < 1$ for every i. Conclude that the curves B and $E_1, E_2, \ldots, E_{8-d}$ generate the Fujita face Δ_H.

Vice versa, show that for every positive rational number ϵ and non-negative rational numbers $\epsilon_1, \ldots, \epsilon_{8-d}$ such that $\epsilon_i < 1$ for each i, the divisor $-K_S + \epsilon B + \sum_{i=1}^{8-d} \epsilon_i E_i$ is ample.

In the case when H is of type $C(r_H)$, we put

$$\ell_H = \big|\{a_i | a_i \neq 0\}\big|$$

and say that H is to be of length ℓ_H. The set of all ample $\mathbb{Q}$-divisors of type $C(r_H)$ with length ℓ_H on S is denoted by $\mathrm{Amp}^C_{\ell_H}(S)$. It is clear that

$$\mathrm{Amp}(S) = \bigcup_{\ell=0}^{8-d} \mathrm{Amp}^C_\ell(S) \cup \bigcup_{r=0}^{9-d} \mathrm{Amp}^B_r(S).$$

Exercise 6.13. Suppose that $d \geqslant 4$. Show that $\mathrm{Amp}^{cyl}(S) = \mathrm{Amp}(S)$.

Exercise 6.14. Suppose that $d = 3$. Show that $\mathrm{Amp}^{cyl}(S) = \mathrm{Amp}(S) \setminus \mathrm{Amp}^B_0(S)$.

Thus, we have a complete description of the set $\mathrm{Amp}^{cyl}(S)$ for $d \geqslant 3$. Unfortunately, we do not have such description for $d \leqslant 2$. But we know a lot about $\mathrm{Amp}^{cyl}(S)$ in the case $d = 2$, and we know something about $\mathrm{Amp}^{cyl}(S)$ in the case $d = 1$.

Exercise 6.15. If $d = 2$, show that $\mathrm{Amp}^{cyl}(S)$ is disjoined from $\mathrm{Amp}^B_0(S)$ and $\mathrm{Amp}^B_1(S)$.

Exercise 6.16. Suppose that $d = 2$. Show that

$$\left(\bigcup_{\ell=3}^{6} \mathrm{Amp}^C_\ell(S)\right) \bigcup \left(\bigcup_{r=3}^{7} \mathrm{Amp}^B_r(S)\right) \subset \mathrm{Amp}^{cyl}(S),$$

show that the sets

$$\mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^B_2(S), \mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^C_2(S), \mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^C_1(S)$$

are not empty. Furthermore, if there is a tacnodal curve in $|-K_S|$, show that $\mathrm{Amp}^{cyl}(S)$ also contains $\mathrm{Amp}^C_0(S)$ and $\mathrm{Amp}^C_1(S)$.

Exercise 6.17. If $d = 1$, show that $\mathrm{Amp}^{cyl}(S)$ is disjoined from the sets $\mathrm{Amp}^B_0(S)$, $\mathrm{Amp}^B_1(S)$, and $\mathrm{Amp}^B_2(S)$

Exercise 6.18. Suppose that $d = 1$. Show that

$$\mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^B_r(S) \neq \emptyset$$

for each $3 \leqslant r \leqslant 8$, and show that

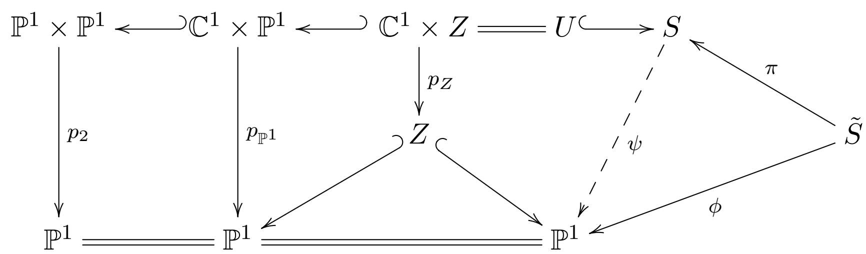
$$\mathrm{Amp}^{cyl}(S) \bigcap \mathrm{Amp}^C_\ell(S) \neq \emptyset$$

for each $2 \leqslant \ell \leqslant 7$.

Solutions to selected exercises

Exercise 1.2. Applying Minimal Model Program to S, we obtain either a birational morphism $S\to\mathbb{F}_n$ or a birational morphism $S\to\mathbb{P}^2$. Considering appropriate cylinders in $\mathbb{F}_n$ and $\mathbb{P}^2$, we obtain the required assertion. $\square$

Exercise 1.3. Suppose S contains a cylinder U. Then U be a Zariski open subset in S such that $U=\mathbb{C}^1\times Z$ for some affine curve Z. Consider the commutative diagram



such that p_Z and $p_{\mathbb{P}^1}$ are natural projections, p_2 is the projection to the second factor, ψ is a rational map, π is a birational morphism, $\tilde{S}$ is a smooth surface, and ϕ is a morphism. By construction, general fiber of ϕ is $\mathbb{P}^1$. Let $C_1,\ldots,C_n$ be irreducible curves in S such that

$$S\setminus U=\bigcup_{i=1}^{n}C_i.$$

Let $E_1,\ldots,E_r$ be the π-exceptional curves of π (if π is an isomorphism, we simply put $r=0$), and let Γ be the section of p_2 that is a complement of $\mathbb{C}^1\times\mathbb{P}^1$ in $\mathbb{P}^1\times\mathbb{P}^1$. Denote by $\tilde{C}_1,\ldots,\tilde{C}_n$ and $\tilde{\Gamma}$ the proper transforms of the curves $C_1,\ldots,C_n$ and Γ on the surface $\tilde{S}$, respectively. Then $\tilde{\Gamma}$ is a section of ϕ. Moreover, the curve $\tilde{\Gamma}$ is one of the curves $\tilde{C}_1,\ldots,\tilde{C}_n$ and $E_1,\ldots,E_r$. Furthermore, all other curves among $\tilde{C}_1,\ldots,\tilde{C}_n$ and $E_1,\ldots,E_r$ are irreducible components of some fibers of ϕ. Thus, we may assume that either $\tilde{\Gamma}=\tilde{C}_1$, or $\tilde{\Gamma}=E_r$. On the other hand, we have

$$K_{\tilde{S}}+\sum_{i=1}^{r}\mu_iE_i\sim_{\mathbb{Q}}\pi^*(K_S)$$

for some rational numbers $\mu_1,\ldots,\mu_r$. Since S has quotient singularities, all these numbers are less than 1 (see [16]). Let $\tilde{F}$ be a general fiber of ϕ. Then $K_{\tilde{S}}\cdot\tilde{F}=-2$ by the adjunction formula. Put $F=\pi(\tilde{F})$. Then $K_S\cdot F\geqslant 0$, because $\tilde{F}$ is a general fiber of ϕ. On the other hand, if $\tilde{\Gamma}=E_r$, then

$$\begin{aligned}-1>-2+\mu_r=-2+\mu_rE_r\cdot\tilde{F}=-2+\sum_{i=1}^{r}\mu_iE_i\cdot\tilde{F}=\\ =\Big(K_{\tilde{S}}+\sum_{i=1}^{r}\mu_iE_i\Big)\cdot\tilde{F}=\pi^*(K_S)\cdot\tilde{F}=K_S\cdot F\geqslant 0,\end{aligned}$$

which is absurd. Similarly, if $\tilde{\Gamma}=C_1$, then

$$-2=-2+\sum_{i=1}^{r}\mu_iE_i\cdot\tilde{F}=(K_{\tilde{S}}+\sum_{i=1}^{r}\mu_iE_i)\cdot\tilde{F}=\pi^*(K_S)\cdot\tilde{F}=K_S\cdot F\geqslant 0,$$

which is absurd as well. $\square$

Exercise 1.4. By construction, the divisor $6K_S$ is linearly trivial. Since there is no non-zero regular 1-form on $E\times E$ invariant by $\mathrm{diag}(-\tau,-\tau)$, we obtain $h^1(S,\mathcal{O}_S)=0$. Using Castelnuovo rationality criterion and rationality of quotient singularities, we conclude that the surface S is a rational surface. For details, see the proof of [1, Proposition 5.1]. □

Exercise 1.5. See the proof of [22, Theorem 39]. □

Exercise 2.2. Use $D=\frac{1}{1+\epsilon_0}D_{\epsilon_0}+\frac{\epsilon_0}{1+\epsilon_0}R$ and Definition 2.1. □

Exercise 2.3. Use Definition 2.1. □

Exercise 2.4. Use Exercise 2.3 and induction on n (see [16, Exercise 6.18]). □

Exercise 2.5. Use Exercises 2.4 and 2.3. □

Exercise 2.6. The required assertion is well-known (see, for example, [3, Theorem 7]). Put $m=\mathrm{mult}(\Delta)$. If $m>1$, then we are done, since

$$\mathrm{mult}_P(C_1\cdot\Delta)\geqslant m.$$

In particular, we may assume that the log pair (S,D) is log canonical in a punctured neighborhood of the point P. Since the log pair (S,D) is not log canonical at P, there exists a birational morphism $h\colon\hat{S}\to S$ that is a composition of $r\geqslant 1$ blow ups of smooth points dominating P, and there exists an h-exceptional divisor, say E_r, such that $e_r>1$, where e_r is a rational number determined by

$$K_{\hat{S}}+a_1\hat{C}_1+\hat{\Delta}+\sum_{i=1}^{r}e_iE_i\sim_{\mathbb{Q}}h^*(K_S+D),$$

where each e_i is a rational number, each E_i is an h-exceptional divisor, $\hat{\Delta}$ is a proper transform on $\hat{S}$ of the divisor Δ, and $\hat{C}_1$ is a proper transform on $\hat{S}$ of the curve C_1.

Let $f\colon\bar{S}\to S$ be the blow up of the point P, let $\bar{\Delta}$ be the proper transform of the divisor Δ on the surface $\bar{S}$, let E be the f-exceptional curve, and let $\bar{C}_1$ be the proper transform of the curve C_1 on the surface $\bar{S}$. Then the log pair $(\bar{S},a_1\bar{C}_1+(a_1+m-1)E+\bar{\Delta})$ is not log canonical at some point $Q\in E$ by Exercise 2.3.

Let us prove the inequality $\mathrm{mult}_P(C_1\cdot\Delta)>1$ by induction on r. If $r=1$, then

$$a_1+m-1>1,$$

which implies that $m>2-a_1\geqslant 1$. This implies that $\mathrm{mult}_P(C_1\cdot\Delta)>1$ in the case when $r=1$. Thus, we may assume that $r\geqslant 2$. Since

$$\mathrm{mult}_P\Big(C_1\cdot\Delta\Big)\geqslant m+\mathrm{mult}_Q\Big(\bar{C}_1\cdot\bar{\Delta}\Big),$$

it is enough to prove that $m+\mathrm{mult}_Q(\bar{C}_1\cdot\bar{\Delta})>1$. Moreover, we may assume that $m\leqslant 1$, since $\mathrm{mult}_P(C_1\cdot\Delta)\geqslant m$. Then the log pair

$$\Big(\bar{S},a_1\bar{C}_1+(a_1+m-1)E+\bar{\Delta}\Big)$$

is log canonical at a punctured neighborhood of the point $Q\in E$, since $a_1+m-1\leqslant 2$.

If $Q\notin\bar{C}_1$, then the log pair $(\bar{S},(a_1+m-1)E+\bar{\Delta})$ is not log canonical at the point Q, which implies that

$$m=\bar{\Delta}\cdot E\geqslant\mathrm{mult}_Q\Big(\bar{\Delta}\cdot E\Big)>1$$

by induction. The latter implies that $Q=\bar{C}_1\cap E$, since $m\leqslant 1$. Then

$$a_1+m-1+\mathrm{mult}_Q\Big(\bar{C}_1\cdot\bar{\Delta}\Big)=\mathrm{mult}_Q\Bigg(\Big((a_1+m-1)E+\bar{\Delta}\Big)\cdot\bar{C}_1\Bigg)>1$$

by induction. This implies that $\mathrm{mult}_Q(\bar{C}\cdot\bar{\Delta})>2-a_1-m$. Then

$$m+\mathrm{mult}_Q\Big(\bar{C}_1\cdot\bar{\Delta}\Big)>2-a_1\geqslant 1$$

as required. □

Exercise 2.7. If $\mathrm{mult}_P(D)\leqslant 2$ and $(\bar{S},\bar{D}+(\lambda\mathrm{mult}_P(D)-1)E)$ is not log canonical at two distinct points P_1 and $\tilde{P}_1$, then

$$2\geqslant\mathrm{mult}_P(D)=\bar{D}\cdot E\geqslant\mathrm{mult}_{P_1}\Big(\bar{D}\cdot E\Big)+\mathrm{mult}_{\tilde{P}_1}\Big(\bar{D}\cdot E\Big)>2$$

by Exercise 2.6. Now use Exercise 2.3. □

Exercise 2.8. The required assertion is [4, Lemma 2.5]. We have

$$K_{\hat{S}}+\hat{D}+\sum_{i=1}^{4}a_iE_i\sim_{\mathbb{Q}}g^*\big(K_S+D\big).$$

This implies that the log pair (S,D) is not log canonical at P if and only if the log pair $(\hat{S},\hat{D}+\sum_{i=1}^{4}a_iE_i)$ is not log canonical at some point in $E_1\cup E_2\cup E_3\cup E_4$. This follow from Definition 2.1. This, if $a_4>1$, then (S,D) is not log canonical.

To complete the solution, we may assume that $a_4\leqslant 1$. We must show that (S,D) is log canonical. Then the log pair $(\hat{S},\hat{D}+\sum_{i=1}^{4}a_iE_i)$ is not log canonical at some point $Q\in E_1\cup E_2\cup E_3\cup E_4$. Without loss of generality, we may assume that $Q\in E_1\cup E_2$.

We have $\hat{D}\cdot E_1\geqslant 0$, $\hat{D}\cdot E_2\geqslant 0$, $\hat{D}\cdot E_3\geqslant 0$ and $\hat{D}\cdot E_4\geqslant 0$. This gives the system of equations

$$\begin{cases}2a_1-a_4\geqslant 0,\\ 2a_2-a_4\geqslant 0,\\ 2a_3-a_4\geqslant 0,\\ 2a_4-a_1-a_2-a_3\geqslant 0,\\ a_4\leqslant 1,\end{cases}$$

which implies that $a_1\leqslant 1$, $a_2\leqslant 1$, and $a_3\leqslant 1$.

Suppose that $Q\not\in E_4$. Then the log pair $(\hat{S},\hat{D}+a_1E_1)$ is not log canonical at Q. By Exercise 2.6, we get

$$2a_1-a_4=\hat{D}\cdot E_1>1,$$

which contradicts the system of equations above. This shows that $Q\in E_4$.

We have $Q=E_1\cap E_4$. Then $(\hat{S},\hat{D}+a_1E_1+a_4E_4)$is not log canonical at Q. Applying Exercise 2.6 to this pair and the curve E_1, we get

$$2a_1-a_4=\hat{D}\cdot E_1>1-a_4,$$

which give $a_1>\frac{1}{2}$. Applying Exercise 2.6 to the same pair and the curve E_4, we get

$$2a_4-a_1-a_2-a_3=\hat{D}\cdot E_4>1-a_1,$$

which give $2a_4 > 1 + a_2 + a_3$. Thus, we have

$$\begin{cases} a_1 > \dfrac{1}{2}, \\ 2a_2 - a_4 \geqslant 0, \\ 2a_3 - a_4 \geqslant 0, \\ 2a_4 > 1 + a_2 + a_3, \\ a_4 \leqslant 1. \end{cases}$$

This system of equations is inconsistent. □

Exercise 2.9. The required assertion is [3, Theorem 13]. We may assume that $a_1 \leqslant 1$ and $a_2 \leqslant 1$. Put $m = \mathrm{mult}_P(\Delta)$. Since $m \leqslant 1$, the log pair $(S, a_1C_1 + a_2C_2 + \Delta)$ is log canonical in a punctured neighborhood of the point P. Thus, there exists a birational morphism $h\colon \hat{S} \to S$ that is a composition of $r \geqslant 1$ blow ups of smooth points dominating P, and there exists an h-exceptional divisor, say E_r, such that $e_r > 1$, where e_r is a rational number determined by

$$K_{\hat{S}} + a_1\hat{C}_1 + a_2\hat{C}_2 + \hat{\Delta} + \sum_{i=1}^{r} e_iE_i \sim_{\mathbb{Q}} h^*\big(K_S + a_1C_1 + a_2C_2 + \Delta\big),$$

where e_i is a rational number, each E_i is an h-exceptional divisor, $\hat{\Delta}$ is a proper transform on $\hat{S}$ of the divisor Δ, $\hat{C}_1$ and $\hat{C}_2$, are proper transforms on $\hat{S}$ of the curves C_1 and C_2, respectively.

Let $f\colon \bar{S} \to S$ be the blow up of the point P, let $\bar{\Delta}$ be the proper transform of the divisor Δ on the surface $\bar{S}$, let E be the f-exceptional curve, let $\bar{C}_1$ and $\bar{C}_2$ be the proper transforms of the curves C_1 and C_2 on the surface $\bar{S}$, respectively. Then

$$\Big(\bar{S}, a_1\bar{C}_1 + a_2\bar{C}_2 + (a_1 + a_2 + m - 1)E + \bar{\Delta}\Big)$$

is not log canonical at some point $Q \in E$ by Exercise 2.3.

If $r = 1$, then $a_1 + a_2 + m - 1 > 1$, which implies that $m > 2 - a_1 - a_2$. On the other hand, if $m > 2 - a_1 - a_2$, then either $m > 2(1 - a_1)$ or $m > 2(1 - a_2)$, because otherwise we would have

$$2m \leqslant 4 - 2(a_1 + a_2),$$

which contradicts to $m > 2 - a_1 - a_2$. Thus, if $r = 1$, then $\mathrm{mult}_P(\Delta \cdot C_1) > 2(1 - a_2)$ or $\mathrm{mult}_P(\Delta \cdot C_2) > 2(1 - a_1)$ as desired.

Let us prove the required assertion by induction on r. The case $r = 1$ is already done. Thus, we may assume that $r \geqslant 2$. If $Q \neq E \cap \bar{C}_1$ and $Q \neq E \cap \bar{C}_2$, then it follows from Exercise 2.6 that

$$m = \bar{\Delta} \cdot E > 1,$$

which is impossible, since $m \leqslant 1$ by assumption. Thus, either $Q = E \cap \bar{C}_1$ or $Q = E \cap \bar{C}_2$. Without loss of generality, we may assume that $Q = E \cap \bar{C}_1$.

By induction, we can apply the required assertion to the log pair

$$\Big(\bar{S}, a_1\bar{C}_1 + (a_1 + a_2 + m - 1)E + \bar{\Delta}\Big)$$

at the point Q. This implies that either

$$\mathrm{mult}_Q\Big(\bar{\Delta} \cdot \bar{C}_1\Big) > 2\Big(1 - (a_1 + a_2 + m - 1)\Big) = 4 - 2a_1 - 2a_2 - 2m$$

or $\mathrm{mult}_Q(\bar{\Delta} \cdot E) > 2(1 - a_1)$. In the latter case, we have

$$\mathrm{mult}_P\Big(\Delta \cdot C_2\Big) \geqslant m > 2(1 - a_1),$$

since $m = \mathrm{mult}_Q(\bar{\Delta}\cdot E) > 2(1-a_1)$, which is exactly what we want. Thus, to complete the proof, we may assume that $\mathrm{mult}_Q(\bar{\Delta}\cdot\bar{C}_1) > 4-2a_1-2a_2-2m$.

If $\mathrm{mult}_P(\Delta\cdot C_2) > 2(1-a_1)$, then we are done. Thus, we assume $\mathrm{mult}_P(\Delta\cdot C_2) \leqslant 2(1-a_1)$. This gives $m \leqslant 2(1-a_1)$, because $\mathrm{mult}_P(\Delta\cdot C_2) \geqslant m$. Then

$$\mathrm{mult}_P\Big(\Delta\cdot C_1\Big) \geqslant m + \mathrm{mult}_Q\Big(\bar{\Delta}\cdot\bar{C}_1\Big) > m + 4 - 2a_1 - 2a_2 - 2m = \\ = 4 - 2a_1 - 2a_2 - m > 2(1-a_2),$$

because $m \leqslant 2(1-a_1)$. □

Exercise 2.10. The required assertion is [4, Theorem 1.28]. Suppose that $\mathrm{mult}_P(\Delta\cdot C_1) \leqslant M + Aa_1 - a_2$ and $\mathrm{mult}_P(\Delta\cdot C_2) \leqslant N + Ba_2 - a_1$. Let us seek for a contradiction.

First we observe that $A+M \geqslant 1$, $B > 1$,

$$\alpha(B+1-MB-N)+\beta(A+1-AN-M) \geqslant AB-1,$$

$\beta(1-N)+B\alpha \geqslant B$, $\alpha(2-M)B+\beta(1-N)(A+1) \geqslant B(A+1)$,

$$\frac{\alpha(2-M)}{A+1}+\frac{\beta(2-N)}{B+1} \geqslant 1,$$

$a_1 > \frac{1-M}{A}$, $a_2 > \frac{1-N}{B}$, $a_1 < 1$ and $a_2 < 1$.

Put $m_0 = \mathrm{mult}_P(\Delta)$. Then $m_0 \leqslant M + Aa_1 - a_2$ and $m_0 \leqslant N + Ba_2 - a_1$. Then the above inequalities imply that $m_0 + a_1 + a_2 \leqslant 2$.

Let $\pi_1\colon S_1 \to S$ be the blow up of the point P. Denote by F_1 the π_1-exceptional curve, and denote by Δ^1, C_1^1 and C_2^1 the proper transforms of Δ, C_1, C_2 on the surface S_1, respectively. Then the log pair

$$\Big(S_1,\ \Delta^1 + a_1C_1^1 + a_2C_2^1 + (m_0+a_1+a_2-1)F_1\Big)$$

is not log canonical at some point $P_1 \in F_1$ by Exercise 2.3, and this point is unique by Exercise 2.7. Note that $m_0+a_1+a_2-1 \geqslant 0$ by Exercise 2.4.

We claim that either $P_1 = F_1\cap C_1^1$ or $P_1 = F_1 \cap C_2^1$. Indeed, suppose that $P_1 \notin C_1^1 \cup C_2^1$. Then the log pair $(S_1, \Delta^1 + (m_0+a_1+a_2-1)F_1)$ is not log canonical at P_1. Then

$$m_0 = \Delta^1\cdot F_1 > 1$$

by Exercise 2.6. Thus, we have

$$m_0\Bigg(\frac{\beta+B\alpha}{AB-1}+\frac{\alpha+A\beta}{AB-1}\Bigg) \leqslant (M+Aa_1-a_2)\frac{\beta+B\alpha}{AB-1}+(N+Ba_2-a_1)\frac{\alpha+A\beta}{AB-1},$$

because $m_0 \leqslant M + Aa_1 - a_2$ and $m_0 \leqslant N + Ba_2 - a_1$. On the other hand, we have

$$(M+Aa_1-a_2)\frac{\beta+B\alpha}{AB-1}+(N+Ba_2-a_1)\frac{\alpha+A\beta}{AB-1} \leqslant 1+\frac{M\beta+MB\alpha+N\alpha+AN\beta}{AB-1},$$

because $\alpha a_1 + \beta a_2 \leqslant 1$ and $AB-1>0$. But we already proved that $m_0 > 1$. Thus, we see that

$$\beta+B\alpha+\alpha+A\beta \leqslant AB-1+M\beta+MB\alpha+N\alpha+AN\beta,$$

which is impossible. This shows that either $P_1 = F_1 \cap C_1^1$ or $P_1 = F_1\cap C_2^1$.

Now we claim that $P_1 \neq F_1\cap C_1^1$. Indeed, suppose that this is not the case. Then the log pair $(S_1, \Delta^1 + a_1C_1^1 + (m_0+a_1+a_2-1)F_1)$ is not log canonical at the point P_1. Applying Exercise 2.6 to this pair and the curve C_1^1, we get

$$M + Aa_1 - a_2 - m_0 = \Delta^1\cdot C_1^1 > 1 - \big(m_0+a_1+a_2-1\big).$$

This gives $a_1 > \frac{2-M}{A+1}$. Then

$$\frac{2-M\alpha}{A+1}+\frac{\beta(1-N)}{B}<\alpha a_1+\beta a_2\leqslant 1,$$

because $a_2 > \frac{1-N}{B}$. Thus, we see that $\frac{2-M\alpha}{A+1}+\frac{\beta(1-N)}{B}<1$ which is impossible. This shows that $P_1\neq F_1\cap C_1^1$.

Since $P_1=F_1\cap C_2^1$, the log pair $(S_1,\Delta^1+a_1C_1^1+a_2C_2^1+(m_0+a_1+a_2-1)F_1)$ is not log canonical at the point P_1.

For any positive integer n, we consider a sequence of blow ups

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S$$

such that $\pi_{i+1}\colon S_{i+1}\to S_i$ is a blow up of the point $F_i\cap C_2^i$ for every $i<n$, where we denote by F_i the exceptional curve of the morphism π_i, and we denote by C_2^i the proper transform of the curve C_2 on the surface S_i. For every positive $k\leqslant n$ and $i\leqslant k$, denote by Δ^k, C_1^k and F_i^k the the proper transforms on S_k of the divisors Δ, C_1 and F_i, respectively. Put $m_i=\mathrm{mult}_{P_i}(\Delta^i)$ for every $i\leqslant n$. For every $k\leqslant n$, put $P_k=F_k\cap C_2^k$ The log pair

$$\left(S_n,\Delta^n+a_1C_1^n+a_2C_2^n+\sum_{i=1}^{n}\left(a_1+ia_2-i+\sum_{j=0}^{i-1}m_j\right)F_i^n\right)$$

is the log pull back of the log pair (S,D) on the surface S_n. By Exercise 2.3, it is not log canonical at some point of the set $F_1^n\cup F_2^n\cup\cdots\cup F_n^n$. We claim that this log pair is log canonical at every point of this set except the point P_n, and

$$1\geqslant a_1+ia_2-i+\sum_{j=0}^{i-1}m_j\geqslant 0,$$

for every $i\leqslant n$. If we prove this claim for every $n\geqslant 1$, we immediately obtain a contradiction, because the fact that (S,D) is not log canonical at P implies that

$$a_1+na_2-n+\sum_{j=0}^{n-1}m_j>1$$

for some $n\geqslant 1$. Let us prove this claim by induction on n. The case $n=1$ is already done. Thus, we may assume that $n\geqslant 2$.

For every $k<n$, the log pair

$$\left(S_k,\Delta^k+a_1C_1^k+a_2C_2^k+\sum_{i=1}^{k}\left(a_1+ka_2-k+\sum_{j=0}^{i-1}m_j\right)F_i^k\right)$$

is the log pull is the log pull back of the log pair (S,D) on the surface S_k. By induction, it is not log canonical at P_k and is log canonical at every point of the set $F_1^k\cup F_2^k\cup\cdots\cup F_k^k$ that is different from P_k. Thus, it is not log canonical at P_k by Exercise 2.3. Similarly, we have $1\geqslant a_1+ka_2-k+\sum_{j=0}^{k-1}m_j\geqslant 0$ for every $k<n$. We must show that the same assertions hold for $k=n$.

By induction, the log pair

$$\left(S_{n-1},\Delta^{n-1}+a_2C_2^k+\left(a_1+(n-1)a_2-(n-1)+\sum_{j=0}^{n-2}m_j\right)F_{n-1}^n\right)$$

is not log canonical at the point P_{n-1}. By Exercise 2.4, we have $a_1+na_2-n+\sum_{j=0}^{n-1}m_j\geqslant 0$. Moreover, applying Exercise 2.6 to this log pair and the curve C_2^{n-2}, we obtain

$$N-Ba_2-a_1-\sum_{j=0}^{n-2}m_j=\Delta^{n-1}\cdot C_2^{n-1}>1-\Bigg(a_1+(n-1)a_2-(n-1)+\sum_{j=0}^{n-2}m_j\Bigg),$$

which implies that $a_2>\frac{n-N}{B+n-1}$.

Now let us prove that $a_1+na_2-n+\sum_{j=0}^{n-1}m_j\leqslant 1$. Suppose that this is not true, i.e., we have $a_1+na_2-n+\sum_{j=0}^{n-1}m_j>1$. We have $m_0+a_2\leqslant M+Aa_1$. Then

$$a_1+nM+nAa_1-n\geqslant a_1+na_2-n+nm_0\geqslant a_1+na_2-n+\sum_{j=0}^{n-1}m_j>1,$$

which immediately implies $a_1>\frac{n+1-Mn}{nA+1}$. One the other hand, we just proved that $a_2>\frac{n-N}{B+n-1}$. Therefore, we see that

$$\Bigg(\frac{\alpha-M}{A}+\beta\Bigg)+\alpha\frac{A-1+M}{A(An+1)}+\beta\frac{1-B-N}{B+n-1}=$$
$$=\alpha\frac{n+1-Mn}{nA+1}+\beta\frac{n-N}{B+n-1}<\alpha a_1+\beta a_2\leqslant 1,$$

where $\alpha\frac{1-M}{A}+\beta\geqslant 1$. Therefore, one has

$$\alpha\frac{A+M-1}{A(An+1)}<\beta\frac{B+N-1}{B+n-1},$$

where $n\geqslant 2$. But $A+M>1$ and $B>1$. Thus, we see that

$$\frac{A(An+1)}{\alpha(A+M-1)}>\frac{B+n-1}{\beta(B+N-1)},$$

while $A^2(B+N-1)\beta\leqslant\alpha(A+M-1)$ by assumption. Then

$$\frac{A}{\alpha(A+M-1)}-\frac{B-1}{\beta(B+N-1)}\geqslant$$
$$\geqslant\Bigg(\frac{A^2}{\alpha(A+M-1)}-\frac{1}{\beta(B+M-1)}\Bigg)n+\frac{A}{\alpha(A+M-1)}-\frac{B-1}{\beta(B+N-1)}>0,$$

which implies that $\beta A(B+N-1)>\alpha(B-1)(A+M-1)$. Then

$$\frac{\alpha(A+M-1)}{A}\geqslant\beta A\big(B+N-1\big)>\alpha\big(B-1\big)\big(A+M-1\big),$$

because $A^2(B+N-1)\beta\leqslant\alpha(A+M-1)$ by assumption. Then $\alpha\neq 0$ and $A(B-1)<1$, which is impossible, because $A(B-1)\geqslant 1$ by assumption. This shows that $a_1+na_2-n+\sum_{j=0}^{n-1}m_j\leqslant 1$.

Now let us show that the log pull back of the log pair (S,D) on the surface S_n is log canonical in every point of F_n that is different $F_n\cap F_{n-1}^n$ and $F_n\cap C_2^n$. Suppose that this is not true, so that this log pair is not log canonical at some point $Q\in F_n$ that is different from $F_n\cap F_{n-1}^n$ and $F_n\cap C_2^n$. Then the log pair

$$\Bigg(S_n,\Delta^n+\Bigg(a_1+na_2-n+\sum_{j=0}^{n-1}m_j\Bigg)F_n\Bigg)$$

is also not log canonical at the point Q. Thus, $m_0 \geqslant m_{n-1} = \Delta^n \cdot F_n > 1$ by Exercise 2.6. Then

$$m_0\left(\frac{\beta + B\alpha}{AB-1} + \frac{\alpha + A\beta}{AB-1}\right) \leqslant (M + Aa_1 - a_2)\frac{\beta + B\alpha}{AB-1} + (N + Ba_2 - a_1)\frac{\alpha + A\beta}{AB-1},$$

because $m_0 \leqslant M + Aa_1 - a_2$ and $m_0 \leqslant N + Ba_2 - a_1$. Thus, we have

$$(M + Aa_1 - a_2)\frac{\beta + B\alpha}{AB-1} + (N + Ba_2 - a_1)\frac{\alpha + A\beta}{AB-1} \leqslant 1 + \frac{M\beta + MB\alpha + N\alpha + AN\beta}{AB-1},$$

because $\alpha a_1 + \beta a_2 \leqslant 1$ and $AB - 1 > 0$. But $m_0 > 1$. This gives

$$\beta + B\alpha + \alpha + A\beta \leqslant AB - 1 + M\beta + MB\alpha + N\alpha + AN\beta,$$

which contradicts one of our initial assumptions.

To finish the proof of the claim (and complete the solution of the exercise), we must prove the log pull back of the log pair (S, D) on the surface S_n is log canonical at the point $F_n \cap F^n_{n-1}$. Suppose that this is not the case. Then the log pair

$$\left(S_n, \Delta^n + \left(a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j\right)F^n_{n-1} + \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j\right)F_n\right)$$

is also not log canonical at the point $F_n \cap F^n_{n-1}$. Then

$$m_{n-2} - m_{n-1} = \Delta^n \cdot F_{n-2} > 1 - \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j\right)$$

by Exercise 2.6. Since

$$M + Aa_1 - a_2 - m_0 \geqslant \mathrm{mult}_P\Big(\Delta \cdot C_1\Big) - m_0 \geqslant \Delta \cdot C_1 - m_0 = \Delta^1 \cdot C^1_1 \geqslant 0,$$

we have $m_0 + a_2 \leqslant Aa_1 + M$. Then

$$nM + nAa_1 - na_2 \geqslant nm_0 \geqslant (n+1)m_0 - m_{n-1} \geqslant m_{n-2} - m_{n-1} + \sum_{j=0}^{n-1} m_j > n + 1 - a_1 - na_2,$$

which gives $a_1 > \frac{n+1-nM}{An+1}$. Arguing as in the proof of the inequality $a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \leqslant 1$, we immediately obtain a contradiction. This completes the solution of the exercise. □

Exercise 2.11. Let X be a cone over the curve C_i whose vertex is a general enough point in $\mathbb{P}^3$. Then

$$X \cap S = C_i + \hat{C}_i,$$

where $\hat{C}_i$ is an irreducible curve of degree $(\deg(S) - 1)\deg(C_i)$. Moreover, $\hat{C}_i$ is not contained in the support of the divisor D, and the intersection $C_i \cap \hat{C}_i$ consists of exactly $\deg(\hat{C}_i)$ singular points. Thus, we have

$$\deg(\hat{C}_i) = D \cdot \hat{C}_i \geqslant a_i C_i \cdot \hat{C}_i \geqslant a_i \deg(\hat{C}_i),$$

which implies that $a_i \leqslant 1$. This solution is due to Pukhlikov. For an alternative solution, see the proof of [16, Lemma 5.36]. □

Exercise 2.12. Suppose that $a_1 > 1$. Let us seek for a contradiction. We may write $D = a_1 C_1 + \Omega$, where $\Omega = \sum_{i=2}^{r} a_i C_i$. Since

$$2 = -K_S \cdot D = -K_S \cdot (a_1 C_1 + \Omega) = -a_1 K_S \cdot C_1 - K_S \cdot \Omega \geqslant -a_1 K_S \cdot C_1 > -K_S \cdot C_1,$$

we have $-K_S \cdot C_1 = 1$. Then $\tau(C_1)$ is a line in $\mathbb{P}^2$. Thus, there exists an irreducible reduced curve C'_1 on S such that $C_1 + C'_1 \sim -K_S$ and $\tau(C_1) = \pi(C'_1)$. Note that $C_1 = C'_1$ if and only if the line $\pi(C_1)$ is an irreducible component of the branch curve C. Since C is irreducible, this is not the case. Thus, we have $C_1 \neq C_1'$.

Note that $C_1^2 = (C_1')^2$ because C_1 and C_1' are interchanged by the biregular involution of S induced by the double cover τ. Thus, we have

$$2 = (-K_S)^2 = \Big(C_1 + C_1'\Big)^2 = 2C_1^2 + 2C_1 \cdot C_1',$$

which implies that $C_1 \cdot C_1' = 1 - C_1^2$. Since C_1 and C_1' are smooth rational curves, we can easily obtain

$$C_1^2 = (C_1')^2 = -1 + \frac{k}{2},$$

where k is the number of singular points of S that lie on C_1. Now we write $D = a_1C_1 + a_1'C_1' + \Gamma$, where a_1' is a non-negative rational number and Γ is an effective $\mathbb{Q}$-divisor whose support contains neither C_1 nor C_1'. Then

$$1 = C_1 \cdot (a_1C_1 + a_1'C_1' + \Gamma) = \\ = a_1C_1^2 + a_1'C_1 \cdot C_1' + C_1 \cdot \Gamma \geqslant \\ \geqslant a_1C_1^2 + a_1'C_1 \cdot C_1' = a_1C_1^2 + a_1'(1 - C_1^2),$$

and hence $1 \geqslant a_1C_1^2 + a_1'(1 - C_1^2)$. Similarly, from $C_1' \cdot D = 1$, we obtain

$$1 \geqslant a_1'C_1^2 + a_1(1 - C_1^2).$$

The obtained two inequalities imply that $a_1 \leqslant 1$ and $a_1' \leqslant 1$ since $C_1^2 = -1 + \frac{k}{2}$, $k = 0, 1, 2$. Since $a_1 > 1$ by our assumption, this is a contradiction.

We see that $a_1 \leqslant 0$. Similarly, we see that $a_i \leqslant 1$ for every i. Now we suppose that (S, D) is not log canonical at P. Let us show that $\tau(P) \in C$. Suppose that this is not the case, i.e., $\tau(P) \notin C$.

Let H be a general curve in $|-K_S|$ that passes through the point P. Since $\tau(P) \notin C$, the surface S is smooth at the point P. Then

$$2 = H \cdot D \geqslant \mathrm{mult}_P(H)\mathrm{mult}_P(D) \geqslant \mathrm{mult}_P(D),$$

and hence $\mathrm{mult}_P(D) \leqslant 2$.

Let $f\colon \bar{S} \to S$ be the blow up of the surface S at P. Denote by $\bar{D}$ the proper transform of the divisor D on $\bar{D}$, and denote by E the exceptional curve of the blow up f. Then it follows from Exercise 2.3 that the log pair

$$\Big(\bar{S}, \bar{D} + (\mathrm{mult}_P(D) - 1)\Big)$$

is not log canonical at some point $Q \in E$. Moreover, this point is unique by Exercise 2.7. Applying Exercise 2.4 to this log pair, we get

$$\mathrm{mult}_P(D) + \mathrm{mult}_Q(\bar{D}) > 2.$$

Since $\tau(P) \notin C$, there exists a unique reduced but possibly reducible curve $R \in |-K_S|$ such that R passes through P and its proper transform on $\bar{S}$ passes through the point Q. Note that R is smooth at P. This enables us to assume that the support of D does not contain at least one irreducible component of R by Exercise 2.2. Denote by $\bar{R}$ the proper transform of R on the surface $\bar{R}$. If the curve R is irreducible, then

$$2 - \mathrm{mult}_P(D) = 2 - \mathrm{mult}_P(C)\mathrm{mult}_P(D) = \bar{R} \cdot \bar{D} \geqslant \mathrm{mult}_Q(\bar{R})\mathrm{mult}_Q(\bar{D}) = \mathrm{mult}_Q(\bar{D}),$$

which is impossible, since we already proved that $\mathrm{mult}_P(D) + \mathrm{mult}_Q(\bar{D}) > 2$. Thus, the curve R must be reducible.

We may write $R = R_1 + R_2$, where R_1 and R_2 are irreducible smooth curves. Without loss of generality we may assume that the curve R_1 is not contained in the support of D. Then the point P must belong to R_2, because otherwise we would have

$$1 = D \cdot R_1 \geqslant \mathrm{mult}_P(D) > 1,$$

since $\mathrm{mult}_P(D) > 1$ by Exercise 2.4. Thus, we put $D = aR_2 + \Omega$, where a is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve R_2. Then

$$1 = R_1 \cdot D = \Big(2 - \frac{1}{2}l\Big)a + R_1 \cdot \Omega \geqslant \Big(2 - \frac{1}{2}l\Big)a,$$

where l is the number of singular points of S contained in the curve R_1. Denote by $\bar{R}_2$ the proper transform of the curve R_2 on the surface $\bar{S}$, and denote by $\bar{\Omega}$ the proper transform of the divisor Ω on the surface $\bar{S}$. Then the log pair

$$\Big(\bar{S}, a\bar{R}_2 + \bar{\Omega} + (\mathrm{mult}_P(D) - 1)E\Big)$$

is not log canonical at Q. Note that we already proved that $a \leqslant 1$. Thus, it follows from Exercise 2.6 that

$$\Big(2 - \frac{1}{2}l\Big)a = \bar{R}_2 \cdot \Big(\bar{\Omega} + (\mathrm{mult}_P(D) - 1)E\Big) > 1.$$

This is a contradiction. □

Exercise 3.2. Use the fact that $K_S^2 > 0$ and $-K_S \cdot C > 0$ for every curve C on S (see [12]). □

Exercise 3.3. Let $f\colon \tilde{S} \to S$ be the minimal resolution of singularities. Then $K_{\tilde{S}} \sim f^*(K_S)$, so that $-K_{\tilde{S}}$ is big and nef, i.e., $K_{\tilde{S}}^2 > 0$ and $-K_S \cdot C \geqslant 0$ for every curve C on $\tilde{S}$. Use this to show that either S is a quadric in $\mathbb{P}^3$ and $d = 8$, or $\tilde{S}$ is a blow up of $\mathbb{P}^2$ in $9 - d$ points such that no four of them lie on a one line, and no seven of them lie on a one conic. See [11] for details. □

Exercise 3.4. If $d = 3$, then each a_i does not exceed 1 by Exercise 2.11. If $d = 3$, then $a_i \leqslant 1$ for each i by Exercise 2.12. If $d = 1$, we have

$$1 = d = K_S^2 = D \cdot (-K_S) = \sum_{i=1}^{r} a_i C_i \cdot (-K_S) \geqslant a_i C_i \cdot (-K_S),$$

which immediately implies that $a_i \leqslant 1$ for each i.

Suppose now that (S, D) is not log canonical at some point $P \in S$. Let us show that there exists a unique divisor $T \in |-K_S|$ such that T is singular at P, the log pair (S, T) is not log canonical at P, and all irreducible components of T is contained in $\mathrm{Supp}(D)$. We consider the cases $d = 1$, $d = 2$ and $d = 3$ separately. For an alternative prove in the case $d = 3$, see [6, Theorem 1.12].

Suppose that $d = 1$. Let C be a curve in $|-K_S|$ that passes through P. Then C is irreducible. If C is not contained in the support of D, we have

$$1 = d \geqslant K_S^2 = D \cdot C \geqslant \mathrm{mult}_P(D) > 1,$$

by Exercise 2.4. This shows that C is contained in the support of D. If (S, C) is not log canonical at P, then we can put $T = C$ and we are done. Thus, we may assume that (S, C) is log canonical at P. Then Exercise 2.2 implies the existence of an effective $\mathbb{Q}$-divisor D' such that $D' \sim_{\mathbb{Q}} -K_S$, the curve C is not contained in the support of D', and (S, D') is not log canonical at P. Now Exercise 2.4 implies that

$$1 = d \geqslant K_S^2 = D' \cdot C \geqslant \mathrm{mult}_P(D') > 1,$$

which is absurd.

Now we consider the case $d = 2$. In this case there exists a double cover $\tau\colon S \to \mathbb{P}^2$ branched over a smooth quartic curve C. Moreover, we have

$$D \sim_{\mathbb{Q}} -K_S \sim \tau^*(L),$$

where L is a line in $\mathbb{P}^2$. By Exercise 2.12, we have $\tau(P) \in C$. Now we may assume that L is tangent to the curve C. Denote by R the curve in $|-K_S|$ that is mapped to L by τ. Then R is singular at P by construction. If R is irreducible and is not contained in the support of D, then Exercise 2.4 gives

$$2 = d \geqslant K_S^2 = D \cdot R \geqslant \mathrm{mult}_P(D)\mathrm{mult}_P(R) \geqslant 2\mathrm{mult}_P(D) > 2,$$

which is absurd. Note that either R is irreducible or R consists of two (-1)-curves that both pass through P. Thus, if one component of R is not contained in the support of D, then we obtain a contradiction in a similar way by intersecting D with this irreducible component of D. Thus, we may assume that all irreducible component of R are contained in the support of D. Now we can use Exercise 2.2 as in the case $d = 1$ to conclude that (S, R) is not log canonical at P. Hence, we can put $T = R$ and we are done again.

Finally, let us consider the case $d = 3$. In this case, S is a smooth cubic surface in $\mathbb{P}^3$. Denote by T_P the intersection of S with the hyperplane in $\mathbb{P}^3$ that is tangent to S at the point P. By Exercise 2.11, T_P is a reduced cubic curve that is singular at P. If (S, T_P) is not log canonical at P and all irreducible components of T_P are contained in $\mathrm{Supp}(D)$, we can put $T = T_P$ and we are done. Thus, we may assume that this is not the case. Now using Exercise 2.2, we may assume that at lest one irreducible components of T_P is not contained the support of the divisor D. To complete the solution, we must obtain a contradiction.

If L_P is a line in S that passes through P, then L_P is contained in in $\mathrm{Supp}(D)$, because otherwise we would get

$$1 = d \geqslant D \cdot L_P \geqslant \mathrm{mult}_P(D)\mathrm{mult}_P(L_P) \geqslant \mathrm{mult}_P(D) > 1$$

by Exercise 2.4. Thus, we see that $\mathrm{mult}_P(T_P) = 2$.

Let $f\colon \tilde{S} \to S$ be the blow up of the point P. Denote by $\tilde{D}$ the proper transform of the divisor D on the surface $\tilde{S}$, denote by $\tilde{T}_P$ the proper transform of the curve T_P on the surface $\tilde{S}$, and denote by E the f-exceptional curve. Then $\mathrm{mult}_P(D) > 1$ by Exercise 2.4, and the log pair

$$\Big(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E\Big)$$

is not log canonical at some point $Q \in E$ by Exercise 2.3. Moreover, there exists a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\quad g \quad} & \bar{S} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ S & \overset{\psi}{\dashrightarrow} & \mathbb{P}^2, \end{array}$$

where ψ is a projection from P, the morphism g is a contraction of the proper transforms of all lines in S that pass through P, and h is a double cover branched over a quartic curve. This quartic curve has at most two ordinary double points, because $\mathrm{mult}_P(T_P) \neq 3$. Now applying Exercise 2.12, we see $Q \in E \cap \tilde{T}_P$.

Note that T_P is one of the following curves: an irreducible cubic curve, a union of a conic and a line, a union of three lines. Let us consider this cases separately.

Suppose that T_P splits as a union of a conic and a line. Then $T_P = L_P + C_P$, where L_P is a line, and C_P is an irreducible conic. We already proved that L_P is contained in the support of D. Hence, C_P is not contained in the support of D. Thus, we write $D = aL_P + \Omega$, where a is a positive rational number, and Ω is an effective $\mathbb{Q}$-divisor on S whose support contains none of the curves L_P and C_P. Put $m = \mathrm{mult}_P(\Omega)$. Then $\mathrm{mult}_P(D) = m + a$ and

$$2 - 2a = \Omega \cdot C_P \geqslant m,$$

which gives $m + 2a \leqslant 2$. Similarly, we have

$$1 + a = L_P \cdot D = \Omega \cdot L_P \geqslant m,$$

which gives $1 + a \geqslant m$. Denote by $\tilde{C}_P$ the proper transform of the conic C_P on the surface $\tilde{S}$, denote by $\tilde{L}_P$ be the proper transform of the line L_P on the surface $\tilde{S}$, and denote by $\tilde{\Omega}$ be the proper transform of the divisor Ω on the surface $\tilde{S}$. Put $\tilde{m} = \mathrm{mult}_Q(\tilde{\Omega})$. Then the log pair

$$\Big(\tilde{S}, a\tilde{L}_P + \tilde{\Omega} + (m + a - 1)E\Big)$$

is not log canonical at P. Applying Exercise 2.4 to this log pair, we obtain $2a + m + \tilde{m} > 2$. One the other hand, if $Q \in \tilde{C}_P$, then

$$2 - 2a - m = \tilde{\Omega} \cdot \tilde{C}_P \geqslant \tilde{m},$$

which implies that $Q \notin \tilde{C}_P$. Since we already proved that $Q \in \tilde{T}_P$, we see that $Q \in \tilde{L}_P$. Now we can apply Exercise 2.10 to the log pair $(\tilde{S}, a\tilde{L}_P + \tilde{\Omega} + (m + a - 1)E)$ at the point Q. Put $C_1 = E$, $C_2 = \tilde{L}_P$, $M = 1$, $A = 1$, $N = 0$, $B = 2$, and $\alpha = \beta = 1$. One can easily check that all hypotheses of Exercise 2.10 are satisfied. Thus, Exercise 2.10 gives

$$m = \mathrm{mult}_Q(\tilde{\Omega} \cdot E) > 1 + (n + m - 1) - n = m$$

or

$$1 + n - m = \mathrm{mult}_Q(\tilde{\Omega} \cdot \tilde{L}) > 2n - (n + m - 1) = 1 + n - m,$$

which is absurd. Note that we can obtain a contradiction in the case also by using Exercise 2.9 instead of Exercise 2.10.

We see that T_P a union of three lines. Denote these lines by L_1, L_2 and L_3. Without loss of generality, we may assume that $P = L_1 \cap L_2$, and $P \notin L_3$. We proved earlier that L_1 and L_2 are contained in the support of D. Thus, we write $D = a_1 L_1 + a_2 L_2 + \Delta$, where a_1 and a_2 are positive rational numbers, and Δ is an effective $\mathbb{Q}$-divisor whose support does not contain the lines L_1 and L_2. Note that the support of Δ does not contain the curve L_3 by assumption. Put $n = \mathrm{mult}_P(\Delta)$. Then

$$n \leqslant \Delta \cdot L_1 = \Big(H - a_1 L_1 - a_2 L_2\Big) \cdot L_1 = 1 + a_1 - a_2,$$

because $L_1 \cdot L_2 = 1$ and $L_1^2 = -1$ on the surface S. Similarly, we see that

$$n \leqslant \Delta \cdot L_2 = \Big(H - a_1 L_1 - a_2 L_2\Big) \cdot L_2 = 1 - a_1 + a_2,$$

because $L_2^2 = -2$ on the surface S. Adding these inequalities, we get $n \leqslant 1$. Thus, applying Exercise 2.9, we get

$$1 + a_1 - a_2 = \Delta \cdot L_1 > 2(1 - a_2)$$

or

$$1 - a_1 + a_2 = \Delta \cdot L_2 > 2(1 - a_1).$$

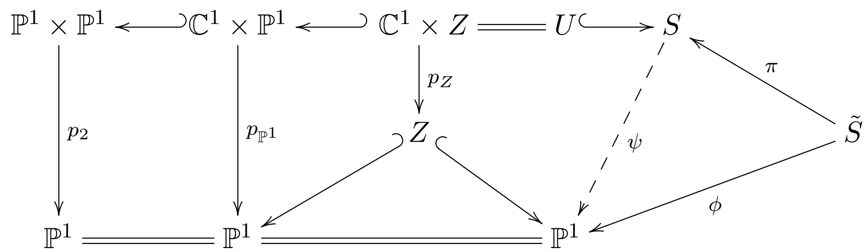
Thus, we get $a_1 + a_2 > 1$. On the other hand, we have

$$0 \leqslant \Delta \cdot L_3 = \Big(H - a_1 L_1 - a_2 L_2\Big) \cdot L_3 = 1 - a_1 - a_2,$$

which implies that $a_1 + a_2 \leqslant 1$. The obtained contradiction completes the solution. $\square$

Exercise 3.5. Since $\mathrm{Cl}(U)$ is trivial, the curves $C_1, \cdots, C_n$ generate the group $\mathrm{Cl}(S)$. Since S has at most quotient singularities, the group $\mathrm{Cl}(S) \otimes \mathbb{Q}$ coincides with the group $\mathrm{Pic}(S) \otimes \mathbb{Q}$. Thus, the curves $C_1, \cdots, C_n$ generate the vector space $\mathrm{Pic}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$, which implies that their number n is at least the dimension of this space.

Since U is a cylinder, $U = \mathbb{C}^1 \times Z$ for some affine curve Z. Consider the commutative diagram



such that p_Z and $p_{\mathbb{P}^1}$ are natural projections, p_2 is the projection to the second factor, ψ is a rational map, π is a birational morphism, $\tilde{S}$ is a smooth surface, and ϕ is a morphism. By construction, general fiber of ϕ is $\mathbb{P}^1$. Let $E_1, \ldots, E_r$ be the π-exceptional curves of π (if π is an isomorphism, we simply put $r = 0$), and let Γ be the section of p_2 that is a complement of $\mathbb{C}^1 \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\tilde{C}_1, \ldots, \tilde{C}_n$ and $\tilde{\Gamma}$ the proper transforms of the curves $C_1, \ldots, C_n$ and Γ on the surface $\tilde{S}$, respectively. Then $\tilde{\Gamma}$ is a section of ϕ. Moreover, the curve $\tilde{\Gamma}$ is one of the curves $\tilde{C}_1, \ldots, \tilde{C}_n$ and $E_1, \ldots, E_r$. Furthermore, all other curves among $\tilde{C}_1, \ldots, \tilde{C}_n$ and $E_1, \ldots, E_r$ are irreducible components of some fibers of ϕ. Thus, we may assume that

- either $\tilde{\Gamma} = \tilde{C}_1$,
- or $\tilde{\Gamma} = E_r$.

If $\tilde{\Gamma} = \tilde{C}_1$, then ψ is a morphism, so that we may assume that π is an isomorphism. If $\tilde{\Gamma} = E_r$, then we may assume that π is a composition of r blow ups of centers the discrete valuation ν_Γ associated to the curve Γ, so that $\tilde{\Gamma}$ is the exceptional curve of the last blow up. Then

$$K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i \sim_{\mathbb{Q}} \pi^*\Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big) \sim_{\mathbb{Q}} 0.$$

for some rational numbers $\mu_1, \ldots, \mu_r$. Let $\tilde{F}$ be a general fiber of ϕ. Then $K_{\tilde{S}} \cdot \tilde{F} = -2$ by the adjunction formula. Put $F = \pi(\tilde{F})$. If $\tilde{\Gamma} = E_r$, then

$$\begin{aligned} -2 + \mu_r &= -2 + \mu_r E_r \cdot \tilde{F} = -2 + \sum_{i=1}^{n} \lambda_i \tilde{C}_i \cdot \tilde{F} + \sum_{i=1}^{r} \mu_i E_i \cdot \tilde{F} = \\ &= \Big(K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i\Big) \cdot \tilde{F} = \bigg(\pi^*\Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big)\bigg) \cdot \tilde{F} = \Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big) \cdot F = 0 \end{aligned}$$

Similarly, if $\tilde{\Gamma} = C_1$, then

$$\begin{aligned} -2 + \lambda_1 &= -2 + \lambda_1 \tilde{C}_1 \cdot \tilde{F} = -2 + \sum_{i=1}^{n} \lambda_i \tilde{C}_i \cdot \tilde{F} + \sum_{i=1}^{r} \mu_i E_i \cdot \tilde{F} = \\ &= \Big(K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i\Big) \cdot \tilde{F} = \bigg(\pi^*\Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big)\bigg) \cdot \tilde{F} = \Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big) \cdot F = 0 \end{aligned}$$

Therefore, we see that

- either $\lambda_1 = 2$ (in the case when $\tilde{\Gamma} = \tilde{C}_1$),
- or $\mu_r = 2$ (in the case when $\tilde{\Gamma} = E_r$).

In particular, the singularities of the log pair $(S, \sum_{i=1}^{n} \lambda_i C_i)$ are not log canonical. □

Exercise 3.6. The existence of desired surfaces is well-known. See, for example, [15]. The absence of cylinders on S follows from [7, Theorem 1.5]. Indeed, suppose that S contains a cylinder U. Denote by $C_1, \dots, C_n$ the irreducible curves in S such that $S \setminus U = \sum_{i=1}^{n} C_i$. Note that the rank of the Picard group of S is 1. Hence, there is a positive rational number λ such that

$$\lambda \sum_{i=1}^{n} C_i \sim_{\mathbb{Q}} -K_S.$$

Put $D = \lambda \sum_{i=1}^{n} C_i$. Then the singularities of the log pair (S, D) are not log canonical at some point $P \in S$ by Exercise 3.5.

Let C be a curve in $|-K_S|$ that passes through P. Then C is an irreducible curve. Note that C contains at most one singular point of S. This implies that $C \neq D$, because $S \setminus U$ is smooth, and $S \setminus C$ is not smooth. Thus, there exists a positive rational number $\mu > 0$ such that the support of the $\mathbb{Q}$-divisor

$$(1+\mu)C - \mu C$$

does not contain at least one irreducible component of P. Applying Exercise 3.5 to this divisor, we see that the log pair $(S, (1+\mu)D - \mu C)$ is not log canonical at P. Replacing D by $(1+\mu)D - \mu C$, we may assume that (S, D) is not log canonical at P, and C is not contained in the support of the divisor D. Let us show that this leads to a contradiction.

If S is smooth at P, then

$$1 = K_S^2 = C \cdot D \geqslant \mathrm{mult}_P(C)\mathrm{mult}_P(D) \geqslant \mathrm{mult}_P(D) > 1$$

by Exercise 2.4. Hence, P is a singular point of S.

Let $f\colon \tilde{S} \to S$ be the minimal resolution of the singular point P. Denote by $E_1, \dots, E_r$ the f-exceptional curves, denote by $\tilde{D}$ the proper transform of the divisor D on the surface $\tilde{S}$, and denote by $\tilde{C}$ the proper transform of the curve C on the surface $\tilde{S}$. Then there are non-negative rational numbers $a_1, \dots, a_r$ such that

$$K_{\tilde{S}} + \tilde{D} + \sum_{i=1}^{r} a_i E_i \sim_{\mathbb{Q}} f^*(K_S + D) \sim_{\mathbb{Q}} 0.$$

We can immediately see how the proper transform $\tilde{C}$ of the effective anticanonical divisor C intersects the exceptional divisors E_i.

Suppose that P is a singular point of type $\mathbb{D}_4$. Then $r = 4$ and we may assume that the exceptional divisor E_4 is the (-2)-curve that intersects all the other three (-2)-curves. We see that, $\tilde{C} \cdot E_4 = 1$ and $\tilde{C} \cdot E_1 = \tilde{C} \cdot E_2 = \tilde{C} \cdot E_3 = 0$. We then obtain

$$1 - a_4 = \Big(f^*(-K_S) - \sum_{i=1}^{r} a_i E_i\Big) \cdot \tilde{C} = \tilde{D} \cdot \tilde{C} \geqslant 0.$$

Thus, the log pair (S, D) is log canonical at P by Exercise 2.8, which is a contradiction. We see that P is not a singular point of type $\mathbb{D}_4$.

Suppose that P is a singular point of type $\mathbb{A}_r$, where $r \leqslant 3$ by assumption. If $r > 1$, then we assume that E_1 and E_r are the tail curves, i.e., the (-2)-curves intersecting only one (-2)-curve, respectively. In this case the curve $\tilde{C}$ intersects E_1 and E_r, respectively, at one point transversally, and it does not intersect the other (-2)-curve in the case when $r = 3$. If $r = 1$, then $\tilde{C} \cdot E_1 = 2$. Therefore, we have

$$1 - a_1 - a_r = \Big(f^*(-K_S) - \sum_{i=1}^{r} a_i E_i\Big) \cdot \tilde{C} = \tilde{D} \cdot \tilde{C} \geqslant 0,$$

and hence $a_1 + a_r \leqslant 1$ (if $r = 1$, then $a_1 \leqslant \frac{1}{2}$).

Consider the case $r = 1$. Since $\tilde{D} \cdot E_1 = 2a_1 \leqslant 1$, the log pair $(\tilde{S}, \tilde{D} + a_1E_1)$ is log canonical along the exceptional curve E_1 by Exercise 2.6. Therefore, the log pair (S, D) is log canonical at P.

Next we consider the case $r = 2$. We then have $a_1 + a_2 \leqslant 1$. Moreover, we obtain $2a_1 \geqslant a_2$ from the inequality

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geqslant 0.$$

Similarly, $2a_2 \geqslant a_1$. Since $a_1 + a_2 \leqslant 1$, we may assume that $a_1 \leqslant \frac{1}{2}$. We obtain

$$\Big(\tilde{D} + a_2E_2\Big) \cdot E_1 = 2a_1 \leqslant 1,$$

and hence the log pair $(\tilde{S}, \tilde{D} + a_1E_1 + a_2E_2)$ is log canonical along the curve E_1 by Exercise 2.6. Furthermore, the inequality

$$\tilde{D} \cdot E_2 = 2a_2 - a_1 \leqslant 2a_1 + (a_2 - a_1) = a_1 + a_2 \leqslant 1$$

and Exercise 2.6 imply that the log pair $(\tilde{S}, \tilde{D} + a_1E_1 + a_2E_2)$ is log canonical along the curve E_2. Consequently, the log pair (S, D) is log canonical at P.

Finally we consider the case $r = 3$. We have $a_1 + a_3 \leqslant 1$. Moreover, we may obtain $2a_1 \geqslant a_2$, $2a_2 \geqslant a_1 + a_3$ and $2a_3 \geqslant a_2$ from

$$\begin{cases} 2a_1 - a_2 = \tilde{D} \cdot E_1 \geqslant 0, \\ 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geqslant 0, \\ 2a_3 - a_2 = \tilde{D} \cdot E_3 \geqslant 0. \end{cases}$$

We may assume that $a_1 \leqslant \frac{1}{2}$, since $a_1 + a_3 \leqslant 1$. Since

$$\Big(\tilde{D} + a_2E_2 + a_3E_3\Big) \cdot E_1 = 2a_1 \leqslant 1,$$

the log pair $(\tilde{S}, \tilde{D} + a_1E_1 + a_2E_2 + a_3E_3)$ is log canonical along the curve E_1 by Exercise 2.6. By Exercise 2.6, the log pair $(\tilde{S}, \tilde{D} + a_1E_1 + a_2E_2 + a_3E_3)$ is log canonical at every point of $E_2 \cup E_3$ that is different from $E_3 \cap E_2$, since

$$\begin{cases} \tilde{D} \cdot E_3 = 2a_3 - a_2 \leqslant (2a_2 - a_1) + a_3 - a_2 \leqslant a_1 + a_3 \leqslant 1, \\ \tilde{D} \cdot E_2 = 2a_2 - a_1 - a_3 \leqslant 2(a_1 + a_3) - (a_1 + a_3) = a_1 + a_3 \leqslant 1. \end{cases}$$

Let Q be the intersection point of E_2 and E_3. We have

$$\begin{cases} \tilde{D} \cdot E_2 = 2a_2 - a_1 - a_3 \leqslant (4a_1 - a_1 + a_3) - 2a_3 = 2a_1 + (a_1 + a_3) - 2a_3 \leqslant 2(1 - a_3), \\ \tilde{D} \cdot E_3 = 2a_3 - a_2 = 2a_3 + a_2 - 2a_2 \leqslant 2a_3 + 2a_1 - 2a_2 \leqslant 2(1 - a_2), \end{cases}$$

and $\mathrm{mult}_Q(\tilde{D}) \leqslant E_3 \cdot \tilde{D} = 2a_3 - a_2 \leqslant 1$. Thus, the log pair $(\tilde{S}, \tilde{D} + a_1E_1 + a_2E_2 + a_3E_3)$ is log canonical at Q by Exercise 2.9.

We proved that the log pair $(\tilde{S}, \tilde{D} + a_1E_1 + a_2E_2 + a_3E_3)$ is log canonical along the three exceptional curves, and hence (S, D) is log canonical at P, which is a contradiction. □

Exercise 3.7. The required assertion follows from Exercise 3.5. □

Exercise 4.1. Use Exercises 2.2 and 2.4. □

Exercise 4.2. It follows from Exercises 3.4 and 2.2 that $\alpha(S,-K_S)=\alpha_1(S,-K_S)$. The number $\alpha_1(S,-K_S)$ is easy to compute. Thus, if $d=1$, then

$$\alpha(S,-K_S)=\begin{cases} 1 \text{ if } |-K_S| \text{ contains a cuspidal curve,}\\ \dfrac{5}{6} \text{ if } |-K_S| \text{ does not contain cuspidal curves.}\end{cases}$$

Similarly, if $d=2$, one has

$$\alpha(S,-K_S)=\begin{cases} \dfrac{3}{4} \text{ if } |-K_S| \text{ contains a tacknodal curve,}\\ \dfrac{5}{6} \text{ if } |-K_S| \text{ does not contain tacknodal curves.}\end{cases}$$

Finally, if $d=3$, we have

$$\alpha(S,-K_S)=\begin{cases} \dfrac{2}{3} \text{ if } S \text{ contains an Eckardt point,}\\ \dfrac{3}{4} \text{ if } S \text{ does not contain Eckardt points.}\end{cases}$$

For an alternative solution, see the proof of [2, Theorem 1.7]. □

Exercise 4.3. This follows from the definitions of $\alpha(X,L)$ and $\alpha_n(X,L)$. □

Exercise 4.4. By Exercises 4.1 and 4.2, we may assume that $7\geqslant K_S^2\geqslant 4$. Then

$$\alpha(S,-K_S)\leqslant\alpha_1(S,-K_S)\leqslant\frac{2}{3}.$$

Suppose that $\alpha(S,-K_S<\alpha_1(S,-K_S)$. Then there exists an effective $\mathbb{Q}$-divisor D such that $D\sim_{\mathbb{Q}}-K_S$ and $(S,\lambda D)$ is not log canonical for some $\lambda<\alpha_1(S,-K_S)$. One can easily see that the log pair $(S,\lambda D)$ is log canonical outside of finitely many points.

Let P be one of these points at which $(S,\lambda D)$ is not log canonical. Then there exists a birational morphism $f\colon S\to\mathbb{P}^2$ such that f contracts $9-K_S^2$ disjoint (-1)-curves and f is an isomorphism in a neighborhood of the point P. Then

$$\Big(\mathbb{P}^2,\lambda f(D)\Big)$$

is not log canonical at the point $f(P)$ and is log canonical outside of finitely many points. Note that $f(D)\sim_{\mathbb{Q}}-K_{\mathbb{P}^2}$. This easily leads to a contradiction (see the proof of [2, Theorem 1.7]). □

Exercise 4.5. If $d\leqslant 2$, then the required assertion follows from Exercise 4.1. If $d=3$, then the required assertion follows from Exercise 4.2. Thus, we assume that $d=4$. Suppose that $\alpha(S,H)<\alpha_1(S,H)$. Then there exists an effective $\mathbb{Q}$-divisor D such that $D\sim_{\mathbb{Q}}H$ and $(S,\lambda D)$ is not log canonical for some $\lambda<\alpha_1(S,H)$. By Exercise 2.11, the log pair $(S,\lambda D)$ is log canonical outside of finitely many points.

Let P be one of these points at which $(S,\lambda D)$ is not log canonical. Then the support of D contains all lines in S that passes through P. Indeed, if L is such a line and L is not contained in the support of D, then

$$1=L\cdot H=L\cdot D\geqslant\mathrm{mult}_P(L)\mathrm{mult}_P(D)=m>\frac{1}{\lambda}>1$$

by Exercise 2.4.

Consider the quartic curve T_P that is cut out on S by the hyperplane in $\mathbb{P}^3$ that is tangent to S at the point P. By Exercise 2.11, T_P is a reduced plane quartic curve. Note that T_P is singular at the point P. By Exercise 2.5, one has

$$\mathrm{lct}_P(S, T_P) \leqslant \frac{2}{\mathrm{mult}_P(T_P)}.$$

In particular, if $\mathrm{mult}_P(T_P) \geqslant 3$, then $\lambda < \frac{2}{3}$. Use parameter count to show that $\alpha_1(S, H) \leqslant \frac{3}{4}$, so that $\lambda < \frac{3}{4}$.

By Exercise 2.2, we may assume that the support of the divisor D does not contain at least one irreducible component of the plane quartic curve T_P. In particular, we see that $\mathrm{mult}_P(T_P) < 4$, since the support of D contains all lines in S that passes through P. Similarly, if C is an irreducible quartic curve and $\mathrm{mult}_P(T_P) = 3$, then

$$4 = H \cdot C = D \cdot C \geqslant \mathrm{mult}_P(C)\mathrm{mult}_P(D) \geqslant 3\mathrm{mult}_P(D) > \frac{3}{\lambda},$$

which is impossible, since $\lambda < \frac{3}{4}$. In all other cases, we can obtain a contradiction in a similar way using Exercises 2.6 and 2.9, the fact that $(S, \lambda D)$ is not log canonical at P, the inequalities $\lambda < \frac{3}{4}$ and $\lambda < \frac{2}{\mathrm{mult}_P(T_P)}$, and the assumption that the support of the divisor D does not contain at least one irreducible component of the plane quartic curve T_P. Let us consider just the case when T_P consists of a (possibly reducible) conic and two lines, and the two lines intersect at P and P does not lie on the conic.

We suppose that T_P consists of two lines L_1 and L_2, and a possibly reducible conic C_1, where P is the intersection point of the lines L_1 and L_2, and P is not contained in the conic C_1. We denote by $C_\star$ the irreducible component of the curve T_P that is not contained in the support of the divisor D. We already know that both lines L_1 and L_2 are contained in the support of the divisor D. In particular, $C_\star \neq L_1$ and $C_\star \neq L_2$. Write $D = \Omega + a_1L_1 + a_2L_2$, where a_1 and a_2 are positive rational numbers, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the lines L_1 and L_2. Note that the support of Ω does not contain the curve $C_\star$ by assumption. Put $n = \mathrm{mult}_P(\Omega)$. Then

$$n \leqslant \Omega \cdot L_1 = \Big(H - a_1L_1 - a_2L_2\Big) \cdot L_1 = 1 + 2a_1 - a_2,$$

because $L_1 \cdot L_2 = 1$ and $L_1^2 = -2$ on the surface S. Similarly, we see that

$$n \leqslant \Omega \cdot L_2 = \Big(H - a_1L_1 - a_2L_2\Big) \cdot L_2 = 1 - a_1 + 2a_2,$$

because $L_2^2 = -2$ on the surface S. Finally, we have

$$0 \leqslant \Omega \cdot C_\star = \Big(H - a_1L_1 - a_2L_2\Big) \cdot C_\star = \deg(C_\star)(1 - a_1 - a_2),$$

which implies that $a_1 + a_2 \leqslant 1$. Adding these three inequalities together, we get $n \leqslant \frac{3}{2}$.

Let $f\colon \widetilde{S} \to S$ be a blow up of the surface S at the point P. Denote by E the f-exceptional curve, and denote by $\widetilde{\Omega}$ the proper transform of the divisor Ω on the surface $\widetilde{\Omega}$. Similarly, denote by $\widetilde{L}_1$ and $\widetilde{L}_2$ the proper transform of the lines L_1 and L_2 on the surface $\widetilde{\Omega}$, respectively. Then the log pair

$$\Big(\widetilde{S}, \lambda a_1\widetilde{L}_1 + \lambda a_2\widetilde{L}_2 + \lambda\widetilde{\Omega} + (\lambda(a_1 + a_2 + n) - 1)E\Big).$$

is not log canonical at some point $Q \in E$ by Exercise 2.3. On the other hand, $n + a_1 + a_1 \leqslant \frac{5}{2}$, because $a_1 + a_2 \leqslant 1$ and $n \leqslant \frac{3}{2}$. Thus, the latter log pair is log canonical at every point of the curve E that is different from Q.

Put $\widetilde{n}=\mathrm{mult}_Q(\widetilde{\Omega})$. Then $\widetilde{n}\leqslant n$.

Suppose $Q\in\widetilde{L}_1$. Then $Q\not\in\widetilde{L}_2$ and

$$\widetilde{n}\leqslant\widetilde{\Omega}\cdot\widetilde{L}_1=\Omega\cdot L_1-n=1+2a_1-a_2-n.$$

This gives $2\widetilde{n}\leqslant\widetilde{n}+n\leqslant 1+2a_1-a_2$, because $\widetilde{n}\leqslant n$. Since, we already know that $n\leqslant 1-a_1+2a_2$, we get

$$3\widetilde{n}\leqslant 2\widetilde{n}+n\leqslant 2+a_1+a_2\leqslant 3,$$

because $a_1+a_2\leqslant 1$. Thus, we see that $\widetilde{n}\leqslant 1$. On the other hand, the log pair

$$\Big(\widetilde{S},\lambda a_1\widetilde{L}_1+\lambda\widetilde{\Omega}+(\lambda(a_1+a_2+n)-1)E\Big).$$

is not log canonical at Q. Thus, we can apply Exercise 2.9 to this log pair. This gives

$$\lambda\Big(1+2a_1-a_2-n\Big)=\lambda\Big(\Omega\cdot L_1-n\Big)=\lambda\widetilde{\Omega}\cdot\widetilde{L}_1>2\Big(1-(\lambda(a_1+a_2+n)-1\Big)$$

or $\lambda n=\lambda\widetilde{\Omega}\cdot E>2(1-\lambda a_1)$. Since $\lambda\leqslant\frac{3}{4}$, the former inequality gives $n+4a_1+a_2>\frac{13}{3}$, and the later inequality gives $n+2a_1>\frac{8}{3}$. Since we already proved that $n\leqslant 1+2a_2-a_1$ and $a_1+a_2\leqslant 1$, the inequality $n+4a_1+a_2>\frac{13}{3}$ leads to a contradiction, and the inequality $n+2a_1>\frac{8}{3}$ gives $a_2>\frac{2}{3}$. Hence, we have $a_2>\frac{2}{3}$. Now applying Exercise 2.6, we obtain

$$\begin{aligned}\lambda+3\lambda a_1-1&=\lambda\big(1+2a_1-a_2\big)+\lambda a_1+\lambda a_2-1=\\&=\lambda\Big(H-a_1L_1-a_2L_2\Big)\cdot L_1+\lambda a_1+\lambda a_2-1=\lambda\Omega\cdot L_1+\lambda a_1+\lambda a_2-1=\\&=\lambda(\Omega\cdot L_1-n)+\lambda a_1+\lambda a_2+\lambda n-1=\lambda\widetilde{\Omega}\cdot\widetilde{L}_1+\lambda a_1+\lambda a_2+\lambda n-1=\\&=\Big(\lambda\widetilde{\Omega}+(\lambda(a_1+a_2+n)-1)E\Big)\cdot\widetilde{L}_1>1,\end{aligned}$$

which results in $a_1>\frac{5}{9}$. On the other hand, we have $a_1+a_2\leqslant 1$ and $a_2>\frac{2}{3}$, which is absurd.

We see that $Q\not\in\widetilde{L}_1$. Similarly, we see that $Q\not\in\widetilde{L}_2$.

Let $g\colon\overline{S}\to\widetilde{S}$ be the blow up of the surface $\widetilde{S}$ at the point Q, and let F be the exceptional curve of g. Denote by $\overline{E}$ the proper transforms of E on the surface $\overline{S}$, and denote by $\overline{\Omega}$ the proper transform of the divisor Ω on the surface $\overline{S}$. Since $Q\not\in\widetilde{L}_1\cup\widetilde{L}_2$, the log pair

$$\Big(\overline{S},\lambda\overline{\Omega}+(\lambda(a_1+a_2+n)-1)\overline{E}+(\lambda(a_1+a_2+n+\widetilde{n})-2)F\Big)$$

is not log canonical at some point $O\in F$. Since $a_1+a_2\leqslant 1$ and $\widetilde{n}\leqslant n\leqslant\frac{3}{2}$, we have

$$a_1+a_2+n+\widetilde{n}\leqslant a_1+a_2+2n\leqslant 4<\frac{3}{\lambda},$$

because $\lambda<\frac{3}{4}$. Thus, it follows from Exercise 2.7 the latter log pair is log canonical at every point of F that is different from O. If $O=F\cap\overline{E}$, then

$$\begin{aligned}\lambda\big(a_1+a_2+2n\big)-2&=\lambda\big(n-\widetilde{n}\big)+\lambda\big(a_1+a_2+n+\widetilde{n}\big)-2=\\&=\lambda\overline{\Omega}\cdot\overline{E}+\lambda\big(a_1+a_2+n+\widetilde{n}\big)-2=\Big(\lambda\overline{\Omega}+\big(\lambda(a_1+a_2+n+\widetilde{n})-2\big)F\Big)\cdot\overline{E}>1\end{aligned}$$

by Exercise 2.6, which implies that $a_1+a_2+2n>\frac{3}{\lambda}>4$, because $\lambda<\frac{3}{4}$. Since we already proved that $a_1+a_2\leqslant 1$ and $n\leqslant\frac{3}{2}$, we see that $O\neq F\cap\overline{E}$. Applying Exercise 2.4, we get

$$a_1+a_2+n+\widetilde{n}+\mathrm{mult}_O\big(\overline{\Omega}\big)>\frac{3}{\lambda}>4.$$

Consider the linear system

$$\overline{\mathcal{M}} := \big|(f\circ g)^*(H) - 2F - \overline{E}\big|.$$

It is a free pencil, because $Q \not\in \widetilde{L}_1 \cup \widetilde{L}_2$. Thus, $\overline{\mathcal{M}}$ contains a unique curve that passes through the point O. Denote this curve by $\overline{M}$, and denote its proper transform on S by M. Then M is a hyperplane section of the surface S and $P \in M$. In particular, M is reduced by Exercise 2.11. Moreover, $M \neq T_P$ by construction. Thus, M is smooth at P, which implies that $\overline{M}$ is the proper transform of the curve M on the surface $\overline{S}$.

Since M is smooth at P, the log pair $(S, \lambda M)$ is log canonical at P. Thus, it follows from Exercise 2.2 that there exists an effective $\mathbb{Q}$-divisor D' such that $D' \sim_{\mathbb{Q}} H$, the log pair $(S, \lambda D')$ is not log canonical at P, the support of the divisor D' is contained in the support of the divisor D, and the support of the divisor D' does not contain at least one irreducible component of the curve M. Replacing D by D', we may assume that D enjoys all these properties.

Denote by $M_\star$ the irreducible component of the curve M that is not contained in the support of D. Similarly, denote by $\overline{M}'$ the irreducible component of the curve $\overline{M}$ that contain O, and denote its image on S by M'. If $M_\star = M'$, then

$$\mathrm{mult}_O(\overline{\Omega}) \leqslant \overline{M}' \cdot \overline{\Omega} \leqslant \deg(M') - a_1 - a_2 - n - \tilde{n} \leqslant 4 - -a_1 - a_2 - n - \tilde{n},$$

which contradicts the inequality we obtained earlier. Thus, we see that $M_\star \neq M'$. In particular, the curve M is not irreducible.

Since M is smooth at P and $P \in M'$, we have $P \not\in M_\star$. Since $Q \not\in \widetilde{L}_1 \cup \widetilde{L}_2$, the curve M' is not a line. Hence, either M' is a conic or M' is a cubic curve. Put $\Omega = aM' + \Delta$, where a is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain M'. Then $a \leqslant 1$ by Exercise 2.11. In fact, we can say more. Indeed, we have

$$\deg(M_\star) = H \cdot M_\star = D \cdot M_\star \geqslant aM' \cdot M_\star.$$

Since $M' \cdot M_\star = \deg(M')\deg(M_\star)$ on the surface S, we have

$$a \leqslant \frac{\deg(M_\star)}{\deg(M')\deg(M_\star)},$$

which implies that $a \leqslant \frac{1}{2}$.

Denote by $\widetilde{\Delta}$ the proper transform of the divisor Δ on the surface $\widetilde{S}$. Put $m = \mathrm{mult}_P(\Delta)$ and $\widetilde{m} = \mathrm{mult}_Q(\widetilde{\Delta})$. Since $O \neq \overline{E} \cap F$ and $Q \not\in \widetilde{L}_1 \cup \widetilde{L}_2$, the log pair

$$\Big(\overline{S}, \lambda a\overline{M}' + \lambda\overline{\Delta} + \big(\lambda m + \lambda a_1 + \lambda a_2 + \lambda\widetilde{m} + 2\lambda a - 2\big)F\Big).$$

is not log canonical at the point the point O. Applying Exercise 2.6 to this log pair, we obtain

$$\overline{M}' \cdot \overline{\Delta} + \big(\lambda n + \lambda a_1 + \lambda a_2 + \lambda\widetilde{m} + 2\lambda a - 2\big) = \overline{M}' \cdot \Big(\lambda\overline{\Delta} + \big(\lambda m + \lambda a_1 + \lambda a_2 + \lambda\widetilde{m} + 2\lambda a - 2\big)F\Big) > 1.$$

This gives

$$\overline{M}' \cdot \overline{\Delta} + m + a_1 + a_2 + \widetilde{m} + 2a > \frac{3}{\lambda}.$$

On the other hand, we have

$$\overline{M}' \cdot \overline{\Delta} = M' \cdot \Delta - m - \widetilde{m} = M' \cdot (H - a_1 L_1 - a_2 L_2 - aM') - m - \widetilde{m} \leqslant \deg(M') - a_1 - a_2 - a(M')^2 - m - \widetilde{m}.$$

Therefore, we obtain

$$\deg(M') - a(M')^2 > \frac{3}{\lambda} - 2a > 4 - 2a,$$

because $\lambda > \frac{3}{4}$. Thus, we have

$$a\Big(2-(M')^2\Big) > 4-\deg(M').$$

This gives $a > \frac{1}{2}$, which is impossible, because $a \leqslant \frac{1}{2}$. $\square$

Exercise 4.6. We must show that $\alpha(S,H) < \alpha_1(S,H)$. First, we use parameter count to show that $\alpha_1(S,H) = \frac{3}{4}$. To show that $\alpha(S,H) < \frac{3}{4}$, pick a point P in S, and consider a blow up $f\colon \widetilde{S} \to S$ of the surface S at the point P. Denote by E the f-exceptional curve, and denote by H the class of a hyperplane section of the surface S. Fix a rational number m such that $\frac{8}{3} < m < \sqrt{d}$. Put $\mathcal{D} = f^*(H) - mE$, so that

$$\mathcal{D}^2 = d - m^2 > 0,$$

because $\frac{8}{3} < m < \sqrt{d}$. Let n be a sufficiently large integer such that mn is an integer. By the Riemann-Roch formula for surfaces, we get

$$h^0(n\mathcal{D}) + h^2(n\mathcal{D}) \geqslant h^0(n\mathcal{D}) - h^1(n\mathcal{D}) + h^2(n\mathcal{D}) = \chi(\mathcal{O}_{\widetilde{S}}) + \frac{1}{2}\left(n^2\mathcal{D}^2 - n\mathcal{D}\cdot K_{\widetilde{S}}\right).$$

By Serre duality, we have

$$h^2(n\mathcal{D}) = h^0(K_{\widetilde{S}} - n\mathcal{D}) = h^0\big((d-4-n)f^*(H) + (mn+1)E\big),$$

which vanishes for $n > d-4$. Hence, it follows from positivity of $\mathcal{D}^2$ that $h^0(n\mathcal{D}) > 0$ for large enough n. Fix such n. Pick $\widetilde{M} \in |n\mathcal{D}|$. Then

$$\widetilde{M} \sim n\widetilde{H} - nmE.$$

Denote by M the proper transform of the divisor $\widetilde{M}$ on the surface S. Put $D = \frac{1}{n}M$. Then $D \sim_{\mathbb{Q}} H$ and $\mathrm{mult}_P(D) \geqslant m$. By Exercise 2.3, the log pair $(S, \frac{3}{4}D)$ is not log canonical, and hence $\alpha(S,H) < \alpha_1(S,H) = \frac{3}{4}$. $\square$

Exercise 5.2. The required assertion follow from Exercise 3.5. $\square$

Exercise 5.3. If $d \leqslant 2$, the required assertion is [21, Proposition 5.1]. If $d = 3$, the required assertion is [6, Theorem 1.7]. In all cases, the required assertion follow from Exercises 3.4 and 5.2. Indeed, suppose that S contains an anticanonical cylinder U. Denote by $C_1, \ldots, C_n$ the irreducible curves in S such that $S \setminus U = \sum_{i=1}^n C_i$. Then there are positive rational numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\sum_{i=1}^n \lambda_i C_i \sim_{\mathbb{Q}} -K_S.$$

Put $D = \sum_{i=1}^n \lambda_i C_i$. Then the singularities of the log pair (S, D) are not log canonical at some point $P \in S$ by Exercise 5.2. Hence, by Exercise 3.4, there exists a unique divisor $T \in |-K_S|$ such that T is singular at P, the log pair (S,T) is not log canonical at P, and all irreducible components of T is contained in $\mathrm{Supp}(D)$. Note that $D \neq T$, because $n > 3$ by Exercise 5.2, and T does not have more than $d \leqslant 3$ irreducible components. Thus, there exists a positive rational number $\mu > 0$ such that the support of the $\mathbb{Q}$-divisor

$$(1+\mu)D - \mu T$$

does not contain at least one irreducible component of P. Applying Exercise 5.2 to this divisor, we see that the log pair

$$\big(S, (1+\mu)D - \mu T\big)$$

is not log canonical at P. This contradicts to Exercise 3.4, because $(1+\mu)D - \mu T \sim_{\mathbb{Q}} -K_S$. $\square$

Exercise 5.4. The required assertion is [19, Theorem 3.19]. Let us consider the case $d = 4$. Then there exists a birational morphism $f\colon S \to \mathbb{P}^2$ such that f is the blow up of $\mathbb{P}^2$ at five points that lie on a unique irreducible conic. Denote this conic by C. Let $\tilde{C}$ be the proper transform of the conic C on the surface S and let $E_1, \ldots, E_5$ be the exceptional divisors of the morphism f. Let L be a sufficiently general line in $\mathbb{P}^2$ that is tangent to C. Denote by $\tilde{L}$ its proper transform on S. Then

$$-K_S \sim_{\mathbb{Q}} (1+\epsilon)\tilde{C} + (1-2\epsilon)\tilde{L} + \sum_{i=1}^{5} \epsilon E_i,$$

where ϵ is any positive rational number such that $\epsilon < \frac{1}{2}$. On the other hand, we have

$$\tilde{S} \setminus \Big(\tilde{C} \cup \tilde{L} \cup E_1 \cup E_2 \cup E_3 \cup E_3 \cup E_5\Big) \cong \mathbb{P}^2 \setminus \Big(C \cup L\Big)$$

is a cylinder. □

Exercise 5.5. The required assertion follows from [7, Theorem 1.5]. In fact, its proof is almost identical to the solution to Exercise 3.6. □

Exercise 5.6. The required assertion follows from [7, Theorem 1.5]. Its proof is mixture of solutions to Exercises 3.6 and 5.3, see also Exercise 2.12. For details, see the proof of [7, Theorem 1.5]. □

Exercise 5.7. Use projection from a singular point. For details, see [7]. □

Exercise 5.8. Use Exercise 5.7 (see [7]). □

Exercise 5.9. By Exercises 5.3, 5.5, 5.6, 5.7 and 5.8, we may assume that either $d = 2$ and S has a singular point that is not an ordinary double point, or $d = 1$ and S has a singular point that is not of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, or $\mathbb{D}_4$. Then we have many possibilities for S. In all of them we can construct an anticanonical cylinder on S similar to Example 5.10. These constructions can be used to prove that all such del Pezzo surfaces admit anticanonical cylinders. This is done in [7]. Namely, for a given singular del Pezzo surface S we find an effective $\mathbb{Q}$-divisor D_S such that $D_S \sim_{\mathbb{Q}} -K_S$ and the complement of the support of D_S is a cylinder. To this end, instead of the singular surface S, we can consider its minimal resolution $f\colon \tilde{S} \to S$. Since we only allow du Val singularities on the surface S, the surface $\tilde{S}$ is a smooth weak del Pezzo surface, i.e., a smooth surface with nef and big anticanonical class $-K_{\tilde{S}}$ (see Exercise 3.3). On this smooth weak del Pezzo surface, it is enough to find an effective $\mathbb{Q}$-divisor $D_{\tilde{S}}$ such that $D_{\tilde{S}} \sim_{\mathbb{Q}} -K_{\tilde{S}}$, its support contains all the (-2)-curves on $\tilde{S}$, and the complement of the support of $D_{\tilde{S}}$ is a cylinder. Then we can take the divisor D_S as $f(D_{\tilde{S}})$. In order to find such a divisor $D_{\tilde{S}}$, we start with the projective plane $\mathbb{P}^2$ and one of the following effective $\mathbb{Q}$-divisors $D_{\mathbb{P}^2}$ on it:

- a triple line $3L$;
- $a_1L_1 + a_2L_2$, where $a_1 + a_2 = 3$ and L_1, L_2 are distinct lines;
- $aL + bC$, where $a + 2b = 3$, C is an irreducible conic and L is a line tangent to the conic C;
- $a_1L_1 + a_2L_2 + a_3L_3$, where $a_1 + a_2 + a_3 = 3$ and L_1, L_2, L_3 are three distinct lines meeting at a single point.

In all these cases, $D_{\mathbb{P}^2} \sim_{\mathbb{Q}} -K_{\mathbb{P}^2}$, and the complement $\mathbb{P}^2 \setminus \mathrm{Supp}(D_{\mathbb{P}^2})$ is a cylinder.

Let S be a given del Pezzo surface with du Val singularities and $\tilde{S}$ be its minimal resolution. Starting from $\mathbb{P}^2$ with one of the divisors $D_{\mathbb{P}^2}$ we will present the composition of a sequence of blow ups $h\colon \check{S} \to \mathbb{P}^2$ and a contraction $g\colon \check{S} \to \tilde{S}$ with the following properties. We write

$$K_{\check{S}} \sim h^*(K_{\mathbb{P}^2}) + \sum a_i E_i,$$

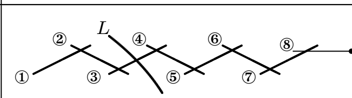
where E_i's are h-exceptional curves. Then we consider an effective $\mathbb{Q}$-divisor $D_{\check{S}}$ on $\check{S}$ such that $(\check{S}, D_{\check{S}})$ is the log pull back of $(\mathbb{P}^2, D_{\mathbb{P}^2})$ (see Exercise 2.3). Then $D_{\check{S}} \sim_{\mathbb{Q}} -K_{\check{S}}$, since $D_{\mathbb{P}^2} \sim_{\mathbb{Q}} -K_{\mathbb{P}^2}$. In all cases, we will see that $D_{\check{S}}$ is effective, its support contains all the exceptional curves of h, and its support contains all the curves contracted by g. We put $D_{\tilde{S}} = g(D_{\check{S}})$ and $D_S = f(D_{\tilde{S}})$. Existence of such birational morphisms h and g shows that the given surface S admits an anticanonical cylinder.

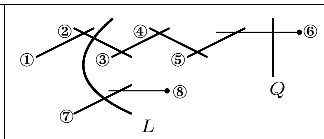
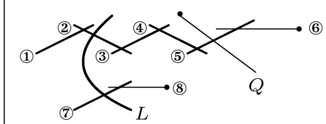
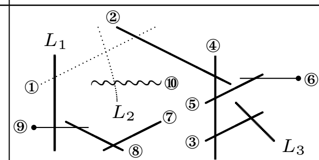
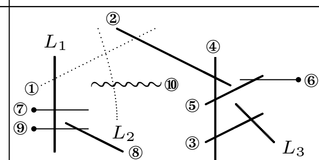
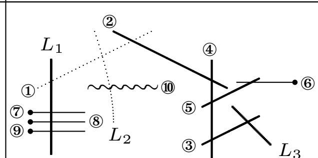
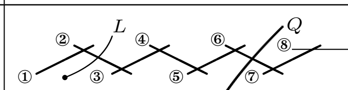
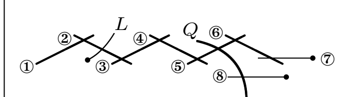
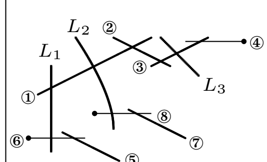
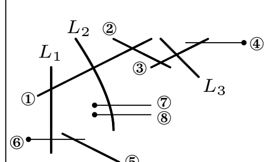
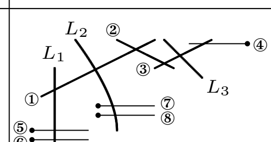
For a given del Pezzo surface S that satisfies the above restrictions, we provide the divisor $D_{\mathbb{P}^2}$ and the birational morphisms h and g. They are described in Tables 1 and 2, below. We read these tables in the following way. In the first column the singularity types are given in normal size letters. The singularity types in small letters in Table 1 are those for del Pezzo surfaces of degree 2. These singularity types in small letters will be explained later. The birational morphism h is obtained by successive blow ups with exceptional curves $E_{①}, \ldots, E_{⑬}$ in this order. The configuration of these exceptional curves given in the third column shows how to take these blow ups. The exceptional curves $E_{①}, \ldots, E_{⑬}$ are labelled by ①, ... , ⑬, respectively, in the third column. The configuration in the third column also shows $D_{\mathbb{P}^2}$. We denote the proper transforms of lines from $\mathbb{P}^2$ by L_i (or L). We denote the proper transforms of an irreducible conic from $\mathbb{P}^2$ by Q. In the second column, the sum of the first divisor and the second divisor (if any) is the divisor $D_{\check{S}}$. If we have the second divisor in the second column, the birational morphism g is obtained by contracting curves drawn by dotted curves in the third column. The second divisor in the second column is contracted by g. Indeed, each component of the second divisor is depicted by a dotted curve in the third column. If we do not have the second divisor in the second column, then $\check{S} = \tilde{S}$ and the morphism g is the identity. The fat curves in the third column are the curves to be (-2)-curves on $\tilde{S}$. The wiggly lines are the curves to be non-negative curves on $\tilde{S}$. The thin lines with dots at one of the ends are the curves to be (-1)-curves on $\tilde{S}$. The curves without superscripts are (-2)-curves on $\check{S}$. The curves superscripted by black-circled numbers are the smooth rational curves on $\check{S}$ with self-intersection numbers of the negatives of the black-circled numbers. The curves superscripted by the circled numbers are the smooth rational curves on $\check{S}$ with self-intersection numbers of the circled numbers.

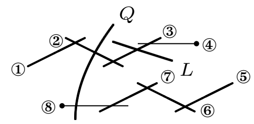
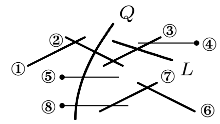
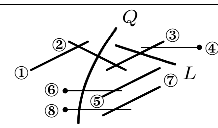
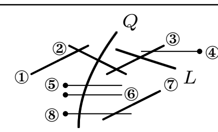
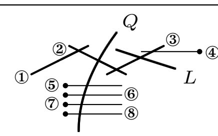
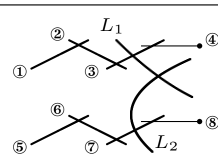
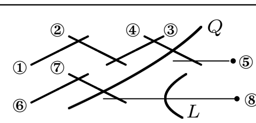
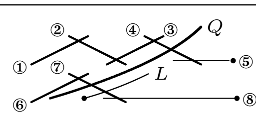
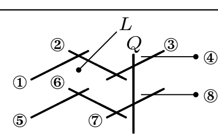
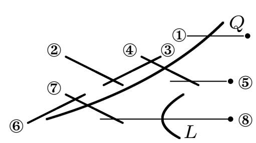
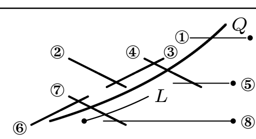
For a del Pezzo surface of degree 2 with a singularity type written in small letters in Table 1 the divisor $D_{\mathbb{P}^2}$ and the birational morphisms h and g can be easily obtained by contracting one of the (-1)-curves (thin lines with dots at one of the ends) in the third column. Only for singularity types $\mathbb{D}_4$, $\mathbb{A}_3$ and $\mathbb{A}_2$ they cannot be obtained in this way. For these three types, we provide the divisor $D_{\mathbb{P}^2}$ and the birational morphisms h and g separately.

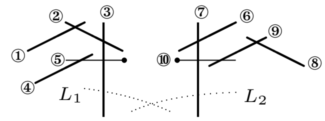
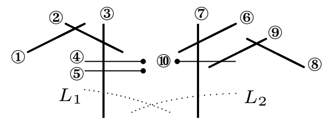
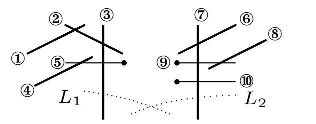
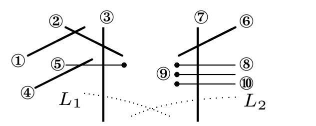
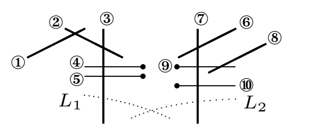
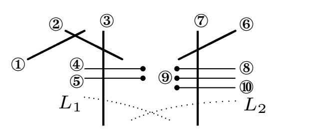
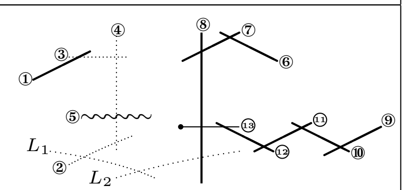
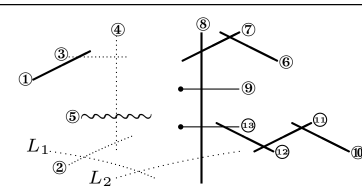
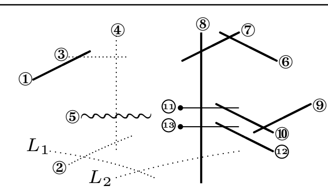
The methods are given according to the singularity types of singular del Pezzo surfaces. Even though they show how to construct the birational morphisms h and g for *a seemingly single* del Pezzo surface S of a given singularity type, they indeed demonstrate how to obtain the birational morphisms h and g for *every* del Pezzo surface S of a given singularity type (see [7] for details).

Table 1: Degree 1

Singularity Type	Tiger/ Divisor contracted (if any)	Construction
$\mathbb{E}_8$ $\mathbb{E}_7$	$2E_{①}+4E_{②}+6E_{③}+5E_{④}+4E_{⑤}+3E_{⑥}+2E_{⑦}+E_{⑧}^{❶}+3L$	

$\mathbb{E}_7+\mathbb{A}_1$ $\mathbb{E}_6$, $\mathbb{D}_6+\mathbb{A}_1$	$\frac{5}{3}E_{①}+\frac{10}{3}E_{②}+\frac{8}{3}E_{③}+2E_{④}+\frac{4}{3}E_{⑤}+\frac{2}{3}E_{⑥}^{❶}+\frac{4}{3}E_{⑦}+\frac{1}{3}E_{⑧}^{❶}+\frac{1}{3}Q+\frac{7}{3}L$	
$\mathbb{E}_7$ $\mathbb{E}_6$, $\mathbb{D}_6$	$\frac{5}{3}E_{①}+\frac{10}{3}E_{②}+\frac{8}{3}E_{③}+2E_{④}+\frac{4}{3}E_{⑤}+\frac{1}{3}E_{⑥}^{❶}+\frac{4}{3}E_{⑦}+\frac{1}{3}E_{⑧}^{❶}+\frac{1}{3}Q^{❶}+\frac{7}{3}L$	
$\mathbb{E}_6+\mathbb{A}_2$ $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_2$	$\frac{12}{7}E_{②}^{❸}+\frac{10}{7}E_{③}+\frac{15}{7}E_{④}+\frac{8}{7}E_{⑤}+\frac{1}{7}E_{⑥}^{❶}+\frac{1}{7}E_{⑦}+\frac{2}{7}E_{⑧}+\frac{3}{7}E_{⑨}^{❶}+\frac{1}{7}E_{⑩}^{❶}+\frac{8}{7}L_1^{❸}+\frac{5}{7}L_3$ $\frac{8}{7}L_2^{❶}+2E_1$	
$\mathbb{E}_6+\mathbb{A}_1$ $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{D}_5$, $(\mathbb{A}_5+\mathbb{A}_1)'$	$\frac{12}{7}E_{②}^{❸}+\frac{10}{7}E_{③}+\frac{15}{7}E_{④}+\frac{8}{7}E_{⑤}+\frac{1}{7}E_{⑥}^{❶}+\frac{1}{7}E_{⑦}^{❶}+\frac{1}{7}E_{⑧}+\frac{2}{7}E_{⑨}^{❶}+\frac{1}{7}E_{⑩}^{❶}+\frac{8}{7}L_1^{❸}+\frac{5}{7}L_3$ $\frac{8}{7}L_2^{❶}+2E_1$	
$\mathbb{E}_6$ $\mathbb{D}_5$, $(\mathbb{A}_5)'$	$\frac{12}{7}E_{②}^{❸}+\frac{10}{7}E_{③}+\frac{15}{7}E_{④}+\frac{8}{7}E_{⑤}+\frac{1}{7}E_{⑥}^{❶}+\frac{1}{7}E_{⑦}^{❶}+\frac{1}{7}E_{⑧}^{❶}+\frac{1}{7}E_{⑨}^{❶}+\frac{1}{7}E_{⑩}^{❶}+\frac{8}{7}L_1^{❸}+\frac{5}{7}L_3$ $\frac{8}{7}L_2^{❶}+2E_1$	
$\mathbb{D}_8$ $\mathbb{D}_6+\mathbb{A}_1$, $\mathbb{A}_7$	$\frac{3}{4}E_{①}+\frac{3}{2}E_{②}+\frac{7}{4}E_{③}+2E_{④}+\frac{9}{4}E_{⑤}+\frac{5}{2}E_{⑥}+\frac{3}{2}E_{⑦}+\frac{1}{2}E_{⑧}^{❶}+\frac{1}{2}L^{❶}+\frac{5}{4}Q$	
$\mathbb{D}_7$ $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{A}_6$	$\frac{3}{4}E_{①}+\frac{3}{2}E_{②}+\frac{7}{4}E_{③}+2E_{④}+\frac{9}{4}E_{⑤}+\frac{5}{2}E_{⑥}+\frac{1}{4}E_{⑦}^{❶}+\frac{1}{4}E_{⑧}^{❶}+\frac{1}{2}L+\frac{5}{4}Q$	
$\mathbb{D}_6+2\mathbb{A}_1$ $\mathbb{D}_4+3\mathbb{A}_1$, $(\mathbb{A}_5+\mathbb{A}_1)''$	$2E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}+\frac{2}{5}E_{⑥}^{❶}+\frac{1}{5}E_{⑦}+\frac{2}{5}E_{⑧}^{❶}+\frac{6}{5}L_1+\frac{6}{5}L_2+\frac{3}{5}L_3$	
$\mathbb{D}_6+\mathbb{A}_1$ $\mathbb{D}_4+2\mathbb{A}_1$, $(\mathbb{A}_5)''$, $(\mathbb{A}_5+\mathbb{A}_1)''$	$2E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}+\frac{2}{5}E_{⑥}^{❶}+\frac{1}{5}E_{⑦}^{❶}+\frac{1}{5}E_{⑧}^{❶}+\frac{6}{5}L_1+\frac{6}{5}L_2+\frac{3}{5}L_3$	
$\mathbb{D}_6$ $\mathbb{D}_4+\mathbb{A}_1$, $(\mathbb{A}_5)''$	$2E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}^{❶}+\frac{1}{5}E_{⑥}^{❶}+\frac{1}{5}E_{⑦}^{❶}+\frac{1}{5}E_{⑧}^{❶}+\frac{6}{5}L_1+\frac{6}{5}L_2+\frac{3}{5}L_3$	

$\mathbb{D}_5+\mathbb{A}_3$ $2\mathbb{A}_3+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$	$\frac{4}{5}E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}+\frac{2}{5}E_{⑥}+\frac{3}{5}E_{⑦}+\frac{4}{5}E_{⑧}^{❶}+\frac{3}{5}L+\frac{6}{5}Q$	
$\mathbb{D}_5+\mathbb{A}_2$ $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_1$	$\frac{4}{5}E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}^{❶}+\frac{1}{5}E_{⑥}+\frac{2}{5}E_{⑦}+\frac{3}{5}E_{⑧}^{❶}+\frac{3}{5}L+\frac{6}{5}Q$	
$\mathbb{D}_5+2\mathbb{A}_1$ $\mathbb{A}_3+3\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_1$	$\frac{4}{5}E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}+\frac{2}{5}E_{⑥}^{❶}+\frac{1}{5}E_{⑦}+\frac{2}{5}E_{⑧}^{❶}+\frac{3}{5}L+\frac{6}{5}Q$	
$\mathbb{D}_5+\mathbb{A}_1$ $(\mathbb{A}_3+2\mathbb{A}_1)'$, $\mathbb{A}_4+\mathbb{A}_1$, $\mathbb{A}_4$	$\frac{4}{5}E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}^{❶}+\frac{1}{5}E_{⑥}^{❶}+\frac{1}{5}E_{⑦}+\frac{2}{5}E_{⑧}^{❶}+\frac{3}{5}L+\frac{6}{5}Q$	
$\mathbb{D}_5$ $(\mathbb{A}_3+\mathbb{A}_1)'$, $\mathbb{A}_4$	$\frac{4}{5}E_{①}+\frac{8}{5}E_{②}+\frac{6}{5}E_{③}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}^{❶}+\frac{1}{5}E_{⑥}^{❶}+\frac{1}{5}E_{⑦}^{❶}+\frac{1}{5}E_{⑧}^{❶}+\frac{3}{5}L+\frac{6}{5}Q$	
$\mathbb{A}_8$ $\mathbb{A}_5+\mathbb{A}_2$	$\frac{1}{2}E_{①}+E_{②}+\frac{3}{2}E_{③}+\frac{1}{2}E_{④}^{❶}+\frac{1}{2}E_{⑤}+E_{⑥}+\frac{3}{2}E_{⑦}+\frac{1}{2}E_{⑧}^{❶}+\frac{3}{2}L_1+\frac{3}{2}L_2$	
$\mathbb{A}_7+\mathbb{A}_1$ $2\mathbb{A}_3+\mathbb{A}_1$, $(\mathbb{A}_5+\mathbb{A}_1)''$	$\frac{1}{3}E_{①}+\frac{2}{3}E_{②}+E_{③}+\frac{4}{3}E_{④}+\frac{1}{3}E_{⑤}^{❶}+\frac{2}{3}E_{⑥}+\frac{4}{3}E_{⑦}+\frac{2}{3}E_{⑧}^{❶}+\frac{1}{3}L+\frac{4}{3}Q$	
$(\mathbb{A}_7)'$ $2\mathbb{A}_3$, $(\mathbb{A}_5+\mathbb{A}_1)''$	$\frac{1}{3}E_{①}+\frac{2}{3}E_{②}+E_{③}+\frac{4}{3}E_{④}+\frac{1}{3}E_{⑤}^{❶}+\frac{2}{3}E_{⑥}+\frac{4}{3}E_{⑦}+\frac{1}{3}E_{⑧}^{❶}+\frac{1}{3}L^{❶}+\frac{4}{3}Q$	
$(\mathbb{A}_7)''$ $\mathbb{A}_4+\mathbb{A}_2$, $(\mathbb{A}_5+\mathbb{A}_1)'$	$\frac{7}{12}E_{①}+\frac{7}{6}E_{②}+\frac{19}{12}E_{③}+\frac{7}{12}E_{④}^{❶}+\frac{5}{12}E_{⑤}+\frac{5}{6}E_{⑥}+\frac{5}{4}E_{⑦}+\frac{1}{4}E_{⑧}^{❶}+\frac{1}{6}L^{❶}+\frac{17}{12}Q$	
$\mathbb{A}_6+\mathbb{A}_1$ $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_1$	$\frac{5}{12}E_{①}^{❶}+\frac{5}{12}E_{②}+\frac{5}{6}E_{③}+\frac{5}{4}E_{④}+\frac{1}{4}E_{⑤}^{❶}+\frac{7}{12}E_{⑥}+\frac{7}{6}E_{⑦}+\frac{1}{3}E_{⑧}^{❶}+\frac{1}{6}L+\frac{17}{12}Q$	
$\mathbb{A}_6$ $\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_1$	$\frac{5}{12}E_{①}^{❶}+\frac{5}{12}E_{②}+\frac{5}{6}E_{③}+\frac{5}{4}E_{④}+\frac{1}{4}E_{⑤}^{❶}+\frac{7}{12}E_{⑥}+\frac{7}{6}E_{⑦}+\frac{1}{6}E_{⑧}^{❶}+\frac{1}{6}L^{❶}+\frac{17}{12}Q$	

$\mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1$ $3\mathbb{A}_2$, $\mathbb{A}_3+3\mathbb{A}_1$	$\frac{2}{5}E_{①}+\frac{4}{5}E_{②}+\frac{6}{5}E_{③}^{❸}+\frac{1}{5}E_{④}+\frac{2}{5}E_{⑤}^{❶}+\frac{3}{5}E_{⑥}+\frac{6}{5}E_{⑦}^{❹}+\frac{1}{5}E_{⑧}+\frac{2}{5}E_{⑨}+\frac{3}{5}E_{⑩}^{❶}$ $\frac{7}{5}L_1+\frac{8}{5}L_2^{❶}$	
$\mathbb{A}_5+\mathbb{A}_2$ $3\mathbb{A}_2$, $\mathbb{A}_3+3\mathbb{A}_1$	$\frac{2}{5}E_{①}+\frac{4}{5}E_{②}+\frac{6}{5}E_{③}^{❸}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}^{❶}+\frac{3}{5}E_{⑥}+\frac{6}{5}E_{⑦}^{❹}+\frac{1}{5}E_{⑧}+\frac{2}{5}E_{⑨}+\frac{3}{5}E_{⑩}^{❶}$ $\frac{7}{5}L_1+\frac{8}{5}L_2^{❶}$	
$\mathbb{A}_5+2\mathbb{A}_1$ $2\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $(\mathbb{A}_3+2\mathbb{A}_1)''$	$\frac{2}{5}E_{①}+\frac{4}{5}E_{②}+\frac{6}{5}E_{③}^{❸}+\frac{1}{5}E_{④}+\frac{2}{5}E_{⑤}^{❶}+\frac{3}{5}E_{⑥}+\frac{6}{5}E_{⑦}^{❹}+\frac{1}{5}E_{⑧}+\frac{2}{5}E_{⑨}^{❶}+\frac{1}{5}E_{⑩}^{❶}$ $\frac{7}{5}L_1+\frac{8}{5}L_2^{❶}$	
$(\mathbb{A}_5+\mathbb{A}_1)'$ $(\mathbb{A}_3+2\mathbb{A}_1)''$, $2\mathbb{A}_2$	$\frac{2}{5}E_{①}+\frac{4}{5}E_{②}+\frac{6}{5}E_{③}^{❸}+\frac{1}{5}E_{④}+\frac{2}{5}E_{⑤}^{❶}+\frac{3}{5}E_{⑥}+\frac{6}{5}E_{⑦}^{❹}+\frac{1}{5}E_{⑧}^{❶}+\frac{1}{5}E_{⑨}^{❶}+\frac{1}{5}E_{⑩}^{❶}$ $\frac{7}{5}L_1+\frac{8}{5}L_2^{❶}$	
$(\mathbb{A}_5+\mathbb{A}_1)''$ $(\mathbb{A}_3+\mathbb{A}_1)''$, $2\mathbb{A}_2+\mathbb{A}_1$	$\frac{2}{5}E_{①}+\frac{4}{5}E_{②}+\frac{6}{5}E_{③}^{❸}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}^{❶}+\frac{3}{5}E_{⑥}+\frac{6}{5}E_{⑦}^{❹}+\frac{1}{5}E_{⑧}+\frac{2}{5}E_{⑨}^{❶}+\frac{1}{5}E_{⑩}^{❶}$ $\frac{7}{5}L_1+\frac{8}{5}L_2^{❶}$	
$\mathbb{A}_5$ $(\mathbb{A}_3+\mathbb{A}_1)''$, $2\mathbb{A}_2$	$\frac{2}{5}E_{①}+\frac{4}{5}E_{②}+\frac{6}{5}E_{③}^{❸}+\frac{1}{5}E_{④}^{❶}+\frac{1}{5}E_{⑤}^{❶}+\frac{3}{5}E_{⑥}+\frac{6}{5}E_{⑦}^{❹}+\frac{1}{5}E_{⑧}^{❶}+\frac{1}{5}E_{⑨}^{❶}+\frac{1}{5}E_{⑩}^{❶}$ $\frac{7}{5}L_1+\frac{8}{5}L_2^{❶}$	
$2\mathbb{A}_4$ $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$	$\frac{19}{30}E_{①}^{❸}+\frac{1}{6}E_{⑤}^{❶}+\frac{11}{30}E_{⑥}+\frac{11}{15}E_{⑦}+\frac{11}{10}E_{⑧}^{❻}+\frac{1}{10}E_{⑨}+\frac{1}{5}E_{⑩}+\frac{3}{10}E_{⑪}+\frac{2}{5}E_{⑫}+\frac{1}{2}E_{⑬}^{❶}$ $\frac{19}{15}E_{②}^{❸}+\frac{9}{10}E_{③}+\frac{7}{6}E_{④}+\frac{49}{30}L_1^{❶}+\frac{41}{30}L_2$	
$\mathbb{A}_4+\mathbb{A}_3$ $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+\mathbb{A}_1$	$\frac{19}{30}E_{①}^{❸}+\frac{1}{6}E_{⑤}^{❶}+\frac{11}{30}E_{⑥}+\frac{11}{15}E_{⑦}+\frac{11}{10}E_{⑧}^{❻}+\frac{1}{10}E_{⑨}^{❶}+\frac{1}{10}E_{⑩}+\frac{1}{5}E_{⑪}+\frac{3}{10}E_{⑫}+\frac{2}{5}E_{⑬}^{❶}$ $\frac{19}{15}E_{②}^{❸}+\frac{9}{10}E_{③}+\frac{7}{6}E_{④}+\frac{49}{30}L_1^{❶}+\frac{41}{30}L_2$	
$\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$ $\mathbb{A}_2+3\mathbb{A}_1$, $2\mathbb{A}_2+\mathbb{A}_1$	$\frac{19}{30}E_{①}^{❸}+\frac{1}{6}E_{⑤}^{❶}+\frac{11}{30}E_{⑥}+\frac{11}{15}E_{⑦}+\frac{11}{10}E_{⑧}^{❻}+\frac{1}{10}E_{⑨}+\frac{1}{5}E_{⑩}+\frac{3}{10}E_{⑪}^{❶}+\frac{1}{10}E_{⑫}+\frac{1}{5}E_{⑬}^{❶}$ $\frac{19}{15}E_{②}^{❸}+\frac{9}{10}E_{③}+\frac{7}{6}E_{④}+\frac{49}{30}L_1^{❶}+\frac{41}{30}L_2$	

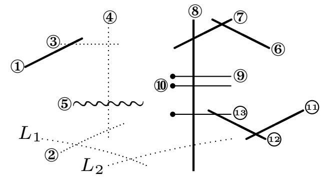
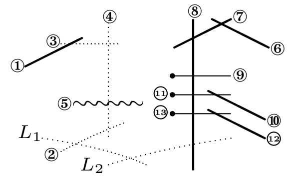
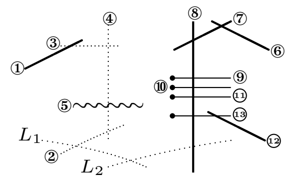
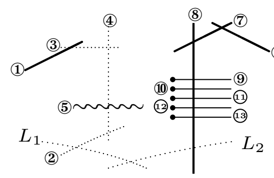
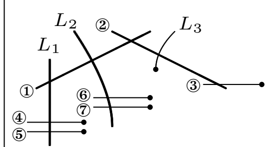
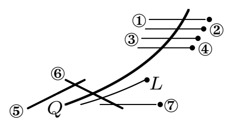
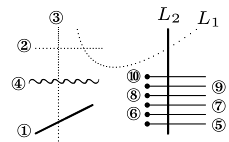
$\mathbb{A}_4+\mathbb{A}_2$ $2\mathbb{A}_2+\mathbb{A}_1,\ \mathbb{A}_2+2\mathbb{A}_1$	$\frac{19}{30}E_{①}^{❸}+\frac{1}{6}E_{⑤}^{❶}+\frac{11}{30}E_{⑥}+\frac{11}{15}E_{⑦}+\frac{11}{10}E_{⑧}^{❻}+\frac{1}{10}E_{⑨}^{❶}+\frac{1}{10}E_{⑩}^{❶}+\frac{1}{10}E_{⑪}+\frac{1}{5}E_{⑫}+\frac{3}{10}E_{⑬}^{❶}$ $\frac{19}{15}E_{②}^{❸}+\frac{9}{10}E_{③}+\frac{7}{6}E_{④}+\frac{49}{30}L_1^{❶}+\frac{41}{30}L_2$	
$\mathbb{A}_4+2\mathbb{A}_1$ $\mathbb{A}_2+3\mathbb{A}_1,\ \mathbb{A}_2+2\mathbb{A}_1$	$\frac{19}{30}E_{①}^{❸}+\frac{1}{6}E_{⑤}^{❶}+\frac{11}{30}E_{⑥}+\frac{11}{15}E_{⑦}+\frac{11}{10}E_{⑧}^{❻}+\frac{1}{10}E_{⑨}^{❶}+\frac{1}{10}E_{⑩}+\frac{1}{5}E_{⑪}^{❶}+\frac{1}{10}E_{⑫}+\frac{1}{5}E_{⑬}^{❶}$ $\frac{19}{15}E_{②}^{❸}+\frac{9}{10}E_{③}+\frac{7}{6}E_{④}+\frac{49}{30}L_1^{❶}+\frac{41}{30}L_2$	
$\mathbb{A}_4+\mathbb{A}_1$ $\mathbb{A}_2+2\mathbb{A}_1,\ \mathbb{A}_2+\mathbb{A}_1$	$\frac{19}{30}E_{①}^{❸}+\frac{1}{6}E_{⑤}^{❶}+\frac{11}{30}E_{⑥}+\frac{11}{15}E_{⑦}+\frac{11}{10}E_{⑧}^{❻}+\frac{1}{10}E_{⑨}^{❶}+\frac{1}{10}E_{⑩}^{❶}+\frac{1}{10}E_{⑪}^{❶}+\frac{1}{10}E_{⑫}+\frac{1}{5}E_{⑬}^{❶}$ $\frac{19}{15}E_{②}^{❸}+\frac{9}{10}E_{③}+\frac{7}{6}E_{④}+\frac{49}{30}L_1^{❶}+\frac{41}{30}L_2$	
$\mathbb{A}_4$ $\mathbb{A}_2+\mathbb{A}_1$	$\frac{19}{30}E_{①}^{❸}+\frac{1}{6}E_{⑤}^{❶}+\frac{11}{30}E_{⑥}+\frac{11}{15}E_{⑦}+\frac{11}{10}E_{⑧}^{❻}+\frac{1}{10}E_{⑨}^{❶}+\frac{1}{10}E_{⑩}^{❶}+\frac{1}{10}E_{⑪}^{❶}+\frac{1}{10}E_{⑫}^{❶}+\frac{1}{10}E_{⑬}^{❶}$ $\frac{19}{15}E_{②}^{❸}+\frac{9}{10}E_{③}+\frac{7}{6}E_{④}+\frac{49}{30}L_1^{❶}+\frac{41}{30}L_2$	

Table 2: Degree 2

Singularity Type	Tiger/ Divisor contracted (if any)	Construction
$\mathbb{D}_4$	$2E_{①}+\frac{4}{3}E_{②}+\frac{1}{3}E^{❶}_{③}+\frac{1}{3}E^{❶}_{④}+\frac{1}{3}E^{❶}_{⑤}+\frac{1}{3}E^{❶}_{⑥}+\frac{1}{3}E^{❶}_{⑦}+\frac{4}{3}L_1+\frac{4}{3}L_2+\frac{1}{3}L^{❶}_3$	
$\mathbb{A}_3$	$\frac{1}{4}E^{❶}_{①}+\frac{1}{4}E^{❶}_{②}+\frac{1}{4}E^{❶}_{③}+\frac{1}{4}E^{❶}_{④}+\frac{3}{4}E_{⑤}+\frac{3}{2}E_{⑥}+\frac{1}{2}E^{❶}_{⑦}+\frac{1}{2}L^{❶}+\frac{5}{4}Q$	
$\mathbb{A}_2$	$\frac{3}{4}E^{❸}_{①}+\frac{1}{4}E^{❶}_{④}+\frac{1}{4}E^{❶}_{⑤}+\frac{1}{4}E^{❶}_{⑥}+\frac{1}{4}E^{❶}_{⑦}+\frac{1}{4}E^{❶}_{⑧}+\frac{1}{4}E^{❶}_{⑨}+\frac{1}{4}E^{❶}_{⑩}+\frac{5}{4}L^{❺}_2$ $\frac{6}{4}E_{②}+\frac{5}{4}E_{③}+\frac{7}{4}L^{❶}_1$	

□

Exercise 6.6. See the solution of Exercise 1.2. For details, see [19, Proposition 3.13]. □

Exercise 6.7. Straightforward (cf. Exercise 4.1). □

Exercise 6.9. The required assertion follows from the solution to Exercise 3.5. □

Exercise 6.11. Straightforward. □

Exercise 6.12. Straightforward. □

Exercise 6.13. The required assertion was proved by Perepechko in [23]. By Exercise 6.7, we my assume that $d \leqslant 7$. Now we can use Exercises 6.11 and 6.12 together with the solution to Exercise 5.4. Namely, let H be an arbitrary ample divisor on S, let μ be the Fujita invariant of H, let Δ be the Fujita face of H and let r be the Fujita rank of H. Let $\phi\colon S \to Z$ be the contraction given by Δ. By Exercise 5.4, we may assume that Δ is not an origin.

Suppose that H is of type $B(r)$ and $Z \ncong \mathbb{P}^1 \times \mathbb{P}^1$. Let us show that S contains an H-polar cylinder. Let $E_1, \ldots, E_r$ be the (-1)-curves that generate the face Δ. Then

$$K_S + \mu H \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i$$

for some positive rational numbers $a_1, \ldots, a_r$. Note that $r \leqslant 9-d$ and $E_1, \ldots, E_r$ are disjoint. The surface Z is a smooth del Pezzo surface of degree $d+r$. Since $Z \ncong \mathbb{P}^1 \times \mathbb{P}^1$, either $Z = \mathbb{P}^2$ or Z is a blow up of $\mathbb{P}^2$ in $9-d-r$ distinct points in general position. Let $\psi\colon Z \to \mathbb{P}^2$ be such a blow up. Put $k = 9-d$ and $\sigma = \psi \circ \phi$. If $k > r$, denote the proper transforms of these ψ-exceptional curves on S by $E_{r+1}, \ldots, E_k$. Put $P_i = \sigma(E_i)$. Let C be an irreducible conic in $\mathbb{P}^2$ passing through the points $P_2, \ldots, P_k$. Such a conic exists because $k \leqslant 6$. Let L be a line in $\mathbb{P}^2$ passing through the point P_1 and tangent to the conic C. For a positive rational number ε we have $-K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1+\varepsilon)C + (1-2\varepsilon)L$. Hence,

$$-K_S \sim \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{k} E_i \sim_{\mathbb{Q}} (1+\varepsilon)\tilde{C} + (1-2\varepsilon)\tilde{L} - 2\varepsilon E_1 + \varepsilon \sum_{i=2}^{k} E_i,$$

where $\tilde{C}$ and $\tilde{L}$ are the proper transforms in S of C and L, respectively. Thus, we have

$$H \sim_{\mathbb{Q}} \frac{1}{\mu}\left((1+\varepsilon)\tilde{C} + (1-2\varepsilon)\tilde{L} + (a_1 - 2\varepsilon)E_1 + \sum_{i=2}^{r}(a_i+\varepsilon)E_i + \varepsilon\sum_{i=r+1}^{k} E_i\right).$$

For $0 < \varepsilon < \frac{a_1}{2}$, this is an H-polar cylinder.

Suppose that H is of type $B(8-d)$ and $Z \not\cong \mathbb{P}^1 \times \mathbb{P}^1$. Let us show that S contains an H-polar cylinder. Let $E_1, \ldots, E_r$ be the (-1)-curves that generate the face Δ. Note that $r = 8-d$. Then

$$K_S + \mu H \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i$$

for some positive rational numbers $a_1, \ldots, a_r$. The (-1)-curves $E_1, \ldots, E_r$ are disjoint. Put $P_i = \sigma(E_i)$. Since $r \leqslant 5$, there is an irreducible curve C of type $(2,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the points $P_1, \cdots, P_r$. Let L be a curve of type $(0,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ that is tangent to the curve C. Let P be the intersection point of C and L. Then there is a unique curve M of type $(1,0)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the point P. For a positive rational number ε we have $-K_{\mathbb{P}^1\times\mathbb{P}^1} \sim_{\mathbb{Q}} (1-\varepsilon)C + (1+\varepsilon)L + 2\varepsilon M$. Hence,

$$-K_S \sim \phi^*(-K_{\mathbb{P}^1\times\mathbb{P}^1}) - \sum_{i=1}^{r} E_i \sim_{\mathbb{Q}} (1-\varepsilon)\tilde{C} + (1+\varepsilon)\tilde{L} + 2\varepsilon\tilde{M} - \varepsilon\sum_{i=1}^{r} E_i,$$

where $\tilde{C}$, $\tilde{L}$, and $\tilde{M}$ are the proper transforms in S of C, L, and M, respectively. Thus, we have

$$H \sim_{\mathbb{Q}} \frac{1}{\mu}\left((1-\varepsilon)\tilde{C} + (1+\varepsilon)\tilde{L} + 2\varepsilon\tilde{M} + \sum_{i=1}^{r}(a_i-\varepsilon)E_i\right).$$

Furthermore, we see

$$S \setminus (\tilde{C} \cup \tilde{L} \cup \tilde{M} \cup E_1 \cup \cdots \cup E_r) \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus (C \cup L \cup M).$$

By taking $0 < \varepsilon < \min\{a_1, \ldots, a_r\}$ we obtain an H-polar cylinder on S

To complete the solution, we may assume that the contraction ϕ is a conic bundle. Let us show that S contains an H-polar cylinder. If the contraction ϕ is a conic bundle, then, we may write

$$K_S + \mu H \sim_{\mathbb{Q}} aB + \sum_{i=1}^{r} a_i E_i$$

where B is an irreducible fiber of ϕ, a is a positive rational number, a_i's are non-negative rational numbers, and $r = 8-d$. We may assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$. Let $\phi_1 : S \to Z$ be the birational morphism obtained by contracting the disjoint (-1)-curves $E_1, \ldots, E_r$.

Case 1. $a_r \neq 0$ and $Z \cong \mathbb{F}_1$.

There is a (-1)-curve E on S whose image by ϕ_1 is the unique (-1)-curve on Z. Let $\psi : Z \to \mathbb{P}^2$ be the birational morphism given by contracting E. Put $\sigma = \phi_1 \circ \phi$. Denote the points $\sigma(E_i)$ by P_i, $i = 1, \cdots, r$, the point $\sigma(E)$ by P, and the line $\sigma(B)$ by M. Note that the line M passes through the point P.

Let C be the conic passing the five points $P_1, \cdots, P_r$. Such a conic exists because $r \leqslant 5$. Let L be a line that passes through the point P and that is tangent to the conic C. We may assume that the line L is different from M.

For any rational number ε we have $-K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1-\varepsilon)C + (1+2\varepsilon+a)L - aM$. Hence,

$$-K_S \sim \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{r} E_i - E$$
$$\sim_{\mathbb{Q}} (1-\varepsilon)\tilde{C} + (1+2\varepsilon+a)\tilde{L} + 2\varepsilon E - aB - \varepsilon \sum_{i=1}^{r} E_i,$$

where $\tilde{C}$ and $\tilde{L}$ are the proper transforms of C and L on S, respectively. Thus, we have

$$H \sim_{\mathbb{Q}} \frac{1}{\mu}\left((1-\varepsilon)\tilde{C} + (1+2\varepsilon+a)\tilde{L} + 2\varepsilon E + \sum_{i=1}^{r}(a_i-\varepsilon)E_i\right).$$

By taking a sufficiently small positive rational number ε we obtain an H-polar cylinder on S.

Case 2. $a_r = 0$ or $a_r \neq 0$, $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\bar{E}_r$ be the other (-1)-curve in the fiber of ϕ contained the (-1)-curve E_r.

In case where $a_r = 0$, by contracting $\bar{E}_r$ instead of E_r, we may assume that $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let E be the (-1)-curve, in the fiber of ϕ containing E_r, that is contracted by ϕ_1. The curve E is either E_r or $\bar{E}_r$.

Denote the points $\phi_1(E_i)$ by P_i, $i = 1, \cdots, r-1$, the point $\phi_1(E)$ by P, and the curve $\phi_1(B)$ by M. The curve M is a curve of type $(0,1)$ or $(1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We may assume that it is of type $(0,1)$.

There is a unique curve C of type $(1,2)$ passing through the points $P, P_1, \cdots, P_{r-1}$. There is a curve L of type $(1,0)$ that is tangent to C. Let Q be the point at which L meets C and let N be the curve of type $(0,1)$ passing through the point Q.

For an arbitrary rational number ε we have $-K_{\mathbb{P}^1\times\mathbb{P}^1} \sim_{\mathbb{Q}} (1+\varepsilon)C + (1-\varepsilon)L + (a-2\varepsilon)N - aM$. Hence,

$$-K_S \sim \phi_1^*(-K_{\mathbb{P}^1\times\mathbb{P}^1}) - E - \sum_{i=1}^{r-1} E_i$$
$$\sim_{\mathbb{Q}} (1+\varepsilon)\tilde{C} + (1-\varepsilon)\tilde{L} + (a-2\varepsilon)\tilde{N} - aB + \varepsilon E + \sum_{i=1}^{r-1} \varepsilon E_i,$$

where $\tilde{C}$, $\tilde{L}$, and $\tilde{N}$ are the proper transforms of C, L, N on S, respectively. Thus, we have

$$H \sim_{\mathbb{Q}} \frac{1}{\mu}\left((1+\varepsilon)\tilde{C} + (1-\varepsilon)\tilde{L} + (a-2\varepsilon)\tilde{N} + \varepsilon E + \sum_{i=1}^{r-1}(a_i+\varepsilon)E_i,\right).$$

By taking a sufficiently small positive rational number ε we obtain an H-polar cylinder on S ☐

Exercise 6.14. This follows from the solution to Exercise 6.13. We just need to use Exercise 5.3 instead of Exercise 5.4. ☐

Exercise 6.15. Use Exercise 5.3 and its proof. The set $\mathrm{Amp}^{cyl}(S)$ is disjoined from $\mathrm{Amp}_0^B(S)$ Exercise 5.3. Let us show that $\mathrm{Amp}^{cyl}(S)$ is disjoint from $\mathrm{Amp}_1^B(S)$. To do this, let E be a (-1)-curve on S. For a positive rational number a the surface S does not contain any $(-K_S+aE)$-polar cylinder. Suppose that there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} -K_S + aE$ and $S \setminus \mathrm{Supp}(D)$ is a cylinder.

Let $f\colon S \to \bar{S}$ be the contraction of the curve E. Put $\bar{D} = f(D)$. Then $\bar{S}$ is a smooth del Pezzo surface of degree $d+1 \leqslant 3$. Moreover, we have $\bar{D} \sim_{\mathbb{Q}} -K_{\bar{S}}$. This implies that $E \not\subset \operatorname{Supp}(D)$. Indeed, if $E \subset \operatorname{Supp}(D)$, then

$$\bar{S} \setminus \operatorname{Supp}(\bar{D}) \cong S \setminus \operatorname{Supp}(D) \cong Z \times \mathbb{A}^1,$$

which implies that $\bar{S} \setminus \operatorname{Supp}(\bar{D})$ is a $(-K_{\bar{S}})$-polar cylinder on $\bar{S}$. This contradicts Theorem **??**.

Since

$$1 - a = (-K_S + aE) \cdot E = D \cdot E \geqslant 0,$$

we see that $a \leqslant 1$. Note that the divisor D is nef and big.

Put $D = \sum_{i=1}^{n} a_i D_i$, where $D_1, \ldots, D_n$ are irreducible curves, and $a_1, \ldots, a_n$ are positive rational numbers. None of the the curves $D_1, \ldots, D_n$ is contracted by the morphism f and

$$\sum_{i=1}^{n} a_i f(D_i) = \bar{D} \sim_{\mathbb{Q}} -K_{\bar{S}}.$$

Therefore, we have $a_i \leqslant 1$ for each $i = 1, \ldots, n$ by Exercise 3.4, and hence it follows from Exercise 6.9 that there exists a point P on S such that for every effective $\mathbb{Q}$-divisor B on the surface S such that $K_S + B$ is pseudo-effective and $\operatorname{Supp}(B) \subset \operatorname{Supp}(D)$, the log pair (S, B) is not log canonical at P . In particular, we see that (S, D) is not log canonical at the point P.

The inequality

$$1 \geqslant 1 - a = (-K_S + aE) \cdot E = D \cdot E \geqslant \operatorname{mult}_P(D)\operatorname{mult}_P(E)$$

and Exercise 2.4 show that P lies outside of E. Therefore, $(\bar{S}, \bar{D})$ is not log canonical at the point $f(P)$.

Let $\bar{T}$ be the unique divisor in $|-K_{\bar{S}}|$ that is singular at $f(P)$. Denote by T its proper transform on the surface S. Since $\bar{D} \sim_{\mathbb{Q}} -K_{\bar{S}}$ and $(\bar{S}, \bar{D})$ is not log canonical at the point $f(P)$, it follows from Exercise 3.4 that $(\bar{S}, \bar{T})$ is not log canonical at $f(P)$ and $\operatorname{Supp}(\bar{T}) \subset \operatorname{Supp}(\bar{D})$. Hence, $\operatorname{Supp}(T) \subset \operatorname{Supp}(D)$.

For every non-negative rational number μ, put $D_\mu = (1+\mu)D - \mu T$ and $\bar{D}_\mu = (1+\mu)\bar{D} - \mu\bar{T}$. Since $-K_{\bar{S}} \cdot \bar{T} = K_{\bar{S}}^2 \leqslant 3$, the divisor T consists of at most 3 irreducible components. Therefore, $D \neq T$ because the divisor D has at least 8 component by Exercise 3.5. Put

$$\nu = \sup\Big\{\mu \in \mathbb{R}_{\geqslant 0} \ \Big|\ D_\mu \text{ is effective}\Big\}.$$

Then $\operatorname{Supp}(T) \not\subset \operatorname{Supp}(D_\nu)$ and $\operatorname{Supp}(\bar{T}) \not\subset \operatorname{Supp}(\bar{D}_\nu)$. In particular, we have $\nu > 0$ since $\operatorname{Supp}(T) \subset \operatorname{Supp}(D)$.

We have $\bar{D}_\mu \sim_{\mathbb{Q}} \bar{D} \sim_{\mathbb{Q}} \bar{T} \sim_{\mathbb{Q}} -K_{\bar{S}}$ for each rational number μ. This implies that

$$D_\mu \sim_{\mathbb{Q}} -K_S + a_\mu E$$

for some rational number a_μ. Note that a_μ is either linear or constant in μ.

Suppose that $a_\nu \geqslant 0$. Then $K_S + D_\nu$ is pseudo-effective. Therefore, the log pair (S, D_ν) is not log canonical at the point P by Exercise 6.9. Then $(\bar{S}, \bar{D}_\nu)$ is not log canonical at $f(P)$. The latter contradicts Exercise 5.9 because $\operatorname{Supp}(\bar{T}) \not\subset \operatorname{Supp}(\bar{D}_\nu)$ by the choice of ν.

Suppose that $a_\nu < 0$. Since $a_0 = a > 0$, there exists a positive rational number $\lambda \in (0, \nu)$ such that $a_\lambda = 0$. It follows from $\lambda < \nu$ that $\operatorname{Supp}(T) \subset \operatorname{Supp}(D_\lambda)$ and $\operatorname{Supp}(D_\lambda) = \operatorname{Supp}(D)$. Therefore,

$$S \setminus \operatorname{Supp}(D_\lambda) \cong S \setminus \operatorname{Supp}(D) \cong Z \times \mathbb{A}^1$$

is a cylinder. However, this contradicts Exercise 5.3, because $a_\lambda = 0$, i.e., $D_\lambda \sim_{\mathbb{Q}} -K_S$. □

Exercise 6.16. Use the solution to Exercise 6.13. □

Exercise 6.17. Use Exercise 6.15 together with Exercise 5.3 and its proof. The set $\mathrm{Amp}^{cyl}(S)$ is disjoined from $\mathrm{Amp}_0^B(S)$ Exercise 5.3. The solution of Exercise 6.15 implies that $\mathrm{Amp}^{cyl}(S)$ is disjoint from $\mathrm{Amp}_1^B(S)$. Let us show that $\mathrm{Amp}^{cyl}(S)$ is disjoint from $\mathrm{Amp}_2^B(S)$. To do this, let E and F be two disjoint (-1)-curves on S. We must show that the surface S contains no $(-K_S+aE+bF)$-polar cylinder for any positive rational numbers a and b. Suppose that there exists an effective $\mathbb{Q}$-divisor D such that $D\sim_{\mathbb{Q}}-K_S+aE+bF$ and such that $S\setminus\mathrm{Supp}(D)$ is isomorphic a cylinder. Let us seek for a contradiction.

Let $g\colon S\to\hat{S}$ be the contraction of the curve E. Put $\hat{D}=g(D)$ and $\hat{F}=g(F)$. Then $\hat{S}$ is a smooth del Pezzo surface of degree 2, $\hat{F}$ is a (-1)-curve and $\hat{D}\sim_{\mathbb{Q}}-K_{\hat{S}}+b\hat{F}$. This implies that $E\not\subset\mathrm{Supp}(D)$. Indeed, if $E\subset\mathrm{Supp}(D)$, then

$$\hat{S}\setminus\mathrm{Supp}(\hat{D})\cong S\setminus\mathrm{Supp}(D)\cong Z\times\mathbb{A}^1$$

is a $\hat{D}$-polar cylinder on $\hat{S}$, which is impossible by Exercise 6.15. Since $E\not\subset\mathrm{Supp}(D)$, we have

$$1-a=(-K_S+aE+bF)\cdot E=D\cdot E\geqslant 0,$$

which implies that $a\leqslant 1$. Similarly, we see that $F\not\subset\mathrm{Supp}(D)$ and $b\leqslant 1$.

Write $D=\sum_{i=1}^n a_iD_i$, where $D_1,\ldots,D_n$ are irreducible curves and $a_1,\ldots,a_n$ are positive rational numbers.

Let $f\colon S\to\bar{S}$ be the contraction of the curves E and F. Put $\bar{D}=f(D)$. Then $\bar{S}$ is a smooth cubic surface and $\bar{D}\sim_{\mathbb{Q}}-K_{\bar{S}}$. None of the the curves $D_1,\ldots,D_n$ is contracted by the morphism f and

$$\sum_{i=1}^n a_if(D_i)=\bar{D}\sim_{\mathbb{Q}}-K_{\bar{S}}.$$

Therefore, we have $a_i\leqslant 1$ for each $i=1,\ldots,n$ by Exercise 3.4, and hence it follows from Exercise 6.9 that there exists a point P on S such that for every effective $\mathbb{Q}$-divisor B on the surface S such that K_S+B is pseudo-effective and $\mathrm{Supp}(B)\subset\mathrm{Supp}(D)$, the log pair (S,B) is not log canonical at P . In particular, we see that (S,D) is not log canonical at the point P.

We claim that P belongs to neither E nor F. Indeed, if $P\in E$, then

$$1\geqslant 1-a=(-K_S+aE)\cdot E=D\cdot E\geqslant\mathrm{mult}_P(D)>1$$

by Exercise 2.4. This shows that $P\notin E$. Similarly, we see that $P\notin F$. Therefore, the birational morphism f is an isomorphism in a neighborhood of the point P, and hence the log pair $(\bar{S},\bar{D})$ is not log canonical at the point $f(P)$.

Let $\bar{T}$ be the unique divisor in $|-K_{\bar{S}}|$ that is singular at $f(P)$. Denote by T its proper transform on the surface S. Since $\bar{D}\sim_{\mathbb{Q}}-K_{\bar{S}}$ and $(\bar{S},\bar{D})$ is not log canonical at the point $f(P)$, it follows from Exercise 3.4 that $(\bar{S},\bar{T})$ is not log canonical at $f(P)$ and $\mathrm{Supp}(\bar{T})\subset\mathrm{Supp}(\bar{D})$. Hence, $\mathrm{Supp}(T)\subset\mathrm{Supp}(D)$.

For every non-negative rational number μ, put $D_\mu=(1+\mu)D-\mu T$ and $\bar{D}_\mu=(1+\mu)\bar{D}-\mu\bar{T}$. Since $-K_{\bar{S}}\cdot\bar{T}=K_{\bar{S}}^2=3$, the divisor T consists of at most 3 irreducible components. Therefore, $D\neq T$ because the divisor D has at least 9 component by Exercise 3.5. Put

$$\nu=\sup\Big\{\mu\in\mathbb{R}_{\geqslant 0}\ \Big|\ D_\mu\text{ is effective}\Big\}.$$

Then $\mathrm{Supp}(T)\not\subset\mathrm{Supp}(D_\nu)$ and $\mathrm{Supp}(\bar{T})\not\subset\mathrm{Supp}(\bar{D}_\nu)$. In particular, we have $\nu>0$ since $\mathrm{Supp}(T)\subset\mathrm{Supp}(D)$.

We have $\bar{D}_\mu\sim_{\mathbb{Q}}\bar{D}\sim_{\mathbb{Q}}\bar{T}\sim_{\mathbb{Q}}-K_{\bar{S}}$ for each rational number μ. This implies that

$$D_\mu\sim_{\mathbb{Q}}-K_S+a_\mu E+b_\mu F$$

for some rational numbers a_μ and b_μ. From $-K_S+E+F \sim_{\mathbb{Q}} f^*(\bar{D}_\mu)=(1+\mu)f^*(\bar{D})-\mu f^*(\bar{T})$ and $a_0=a$, $b_0=b$ we obtain

$$\begin{cases} a_\mu = \Big(\mathrm{mult}_{f(E)}(\bar{T})-\mathrm{mult}_{f(E)}(\bar{D})\Big)\mu+a \\ b_\mu = \Big(\mathrm{mult}_{f(F)}(\bar{T})-\mathrm{mult}_{f(F)}(\bar{D})\Big)\mu+b. \end{cases}$$

Because $\mathrm{mult}_{f(E)}(\bar{T}) \geqslant \mathrm{mult}_{f(E)}(\bar{D})$ and $\mathrm{mult}_{f(F)}(\bar{T}) \geqslant \mathrm{mult}_{f(F)}(\bar{D})$, we have $a_\nu>0$ and $b_\nu>0$. Then K_S+D_ν is pseudo-effective, and hence the log pair (S, D_ν) is not log canonical at the point P by Exercise 6.9. Then $(\bar{S}, \bar{D}_\nu)$ is not log canonical at $f(P)$. Since $\mathrm{Supp}(\bar{T}) \not\subset \mathrm{Supp}(\bar{D}_\nu)$, this contradicts Exercise 5.9. □

Exercise 6.18. Use the solution to Exercise 6.13. □

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