Analytic constructions of p-adic L-functions and Eisenstein series

Alexei PANCHISHKIN
Institut Fourier, Université Grenoble-1
B.P.74, 38402 St.–Martin d'Hères, FRANCE
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Work of Ilya Piatetski-Shapiro and new ways of constructing complex and p-adic L-functions

In June 2011 Roger Howe invited me to this conference devoted to assessing the work of Ilya Piatetski-Shapiro, especially in the areas of autmorphic forms and geometry.

Many thanks to the organizers for this invitation and this occasion both to review the accomplishments of Ilya Piatetski-Shapiro and his colleagues, and to point to productive directions to take research from here.

I new Ilya since 1973-74 during our joint participation in the seminar of Manin and Kirillov on p-adic L-functions, and attending his informal lectures on GL(3) in Moscow University in April-May 1975.

After many years we met again in Jerusalem in February 1998 during the conference "p-Adic Aspects of the Theory of Automorphic Representations".

Ilya liked my construction of p-adic standard L-functions of Siegel modular forms [Pa91], and suggested to extend it to spinor L-functions, using the restriction of an Eisenstein series to the Bessel subgroup in the generalized Whittaker models (see Olga Taussky Todd memorial volume [PS3]). So we started a joint work "On p-adic L-functions for $GSp(4)$".

In 1998, a conference for Ilya Piatetski-Shapiro

was organized in the Fourier Institute (Grenoble, France), with participation of A.Andrianov, G.Henniart, H.Hida, J.-P.Labesse, J.-L.Waldspurger and others.

In IAS, we had the most intensive period of our joint work in the Fall 1999-2000.

My last meeting Ilya was on January 31, 2009 at the the Weizmann Institute home of Volodya Berkovich and his wife Lena, who is Grisha Freiman's daughter. Vera brought Ilya, Edith, and Edith's mother Ida to Berkovich's. This very pleasant gathering also included Grisha Freiman, his wife Nina, Steve and Mary Gelbart, Antoine Ducros, and my wife Marina, see also [CGS], p. 1268.

That was the last party in Ilya's life.

His favorite automorphic forms were the *Eisenstein series*, and the main subject of this talk will be my new construction of *meromorphic p-adic families of Siegel-Eisenstein series in relation to the geometry of homogeneuos spaces*, both complex and p-adic, for any prime p.

I am glad that this construction fits into the particular subject "Automorphic Forms and Related Geometry" of our conference.

p-adic Siegel-Eisenstein series and related geometry

Let us consider the *symplectic group* $\Gamma = \mathrm{Sp}_m(\mathbb{Z})$ (of $(2m \times 2m)$-matrices), and prove that the *Fourier coefficients $a_h(k)$ of the original Siegel-Eisenstein series E_k^m* admit an explicit p-adic meromorphic interpolation on k where h runs through all positive definite half integral matricies for $\det(2h)$ not divisible by p, where

$$\begin{aligned} E_k^m(z) &= \sum_{(c,d)/\sim} \det(cz+d)^{-k} = \sum_{\gamma \in P\backslash\Gamma} \det(cz+d)^{-k} \\ &= \sum_{h \in B_m} a_h \exp(\mathrm{tr}(hz)) \end{aligned}$$

on the Siegel upper half plane $\mathfrak{H}_m = \{z = {}^t z \in M_m(\mathbb{C}) | \mathrm{Im}\,(z) > 0\}$ of degree m, (c,d) runs over equivalence classes of all coprime symmetric couples,

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over equivalence classes of Γ modulo the Siegel parabolic $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

p-adic Siegel-Eisenstein series and related geometry

The homogeneuos space $X = \{(c,d)/\sim\} = P\backslash \mathrm{Sp}_m$ *and its p-adic points admit Siegel's coordinates*

$$\nu = \det(c) \text{ and } \mathfrak{R} = c^{-1}d$$

defined on the main subset given by $\det(c) \in \mathrm{GL}_1$, which is used in the construction.

I try also to present various *applications: to p-adic L-functions, to Siegel's Mass Formula, to p-adic analytic families of automorphic representations.*

Eisenstein series are basic automorphic forms, and there exist several ways to construct them via group theory, lattice theory, Galois representations, spectral theory...

For me, the Eisenstein series is the main tool of analytic constructions of complex and p-adic L-functions, in particular via the *doubling method*, see [PSR], [GRPS], [Boe85], [Shi95], [Boe-Schm],..., greatly thanks to Ilya Piatetski-Shapiro and his collaborators.

General strategy

For any Dirichlet character χ mod p^v consider Shimura's "involuted" Siegel-Eisenstein series assuming their absolute convergence (i.e. $k > m+1$):

$$E_k^*(\chi, z) = \sum_{(c,d)/\sim} \chi(\det(c)) \det(cz+d)^{-k} = \sum_{0<h\in B_m} a_h(k,\chi) q^h.$$

The two sides of the equality produce *dual approaches: geometric and algebraic.* The Fourier coefficients can be computed by Siegel's method (see

[St81], [Shi95], ...) via the singular series

$$a_h(E_k^*(\chi,z)) \tag{1}$$
$$= \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}}\Gamma_m(k)} \sum_{\mathfrak{R} \bmod 1} \chi(\nu(\mathfrak{R}))\nu(\mathfrak{R})^{-k} \det h^{k-\frac{m+1}{2}} e_m(h\mathfrak{R})$$

The *orthogonality relations* $\bmod p^v$ *produce two families of distributions* (notice that terms in the RHS are invariant under sign changes, and (3) is algebraic after multiplying by the factor in (1)):

$$\frac{1}{\varphi(p^v)} \sum_{\chi \bmod p^v} \bar{\chi}(b) \sum_{(c,d)/\sim} \frac{\chi(\det(c))}{\det(cz+d)^k} = \sum_{\substack{(c,d)/\sim \\ \det(c)\equiv b \bmod p^v}} \frac{\mathrm{sgn}(\det(c))^k}{\det(cz+d)^k} \tag{2}$$

$$\frac{1}{\varphi(p^v)} \sum_{\chi \bmod p^v} \bar{\chi}(b) \sum_{\mathfrak{R} \bmod 1} \frac{\chi(\nu(\mathfrak{R}))e_m(h\mathfrak{R})}{\nu(\mathfrak{R})^k} = \sum_{\substack{\mathfrak{R} \bmod 1 \\ \nu(\mathfrak{R})\equiv b \bmod p^v}} \frac{e_m(h\mathfrak{R})\mathrm{sgn}\nu(\mathfrak{R})^k}{\nu(\mathfrak{R})^k} \tag{3}$$

The use of Iwasawa theory and pseudomeasures

We express the integrals of Dirichlet characters θ mod p along the distributions (3) through the reciprocal of a product of L-functions, and elementary integral factors. The result turns out to be an *Iwasawa function of the variable* $t = (1+p)^k - 1$ *divided by a distinguished polynomial* provided that $\det h$ is not divisible by p.

Thus the second family (3) comes from a unique pseudomeasure μ_h^* which becomes a measure after multiplication by an explicit polynomial factor (in the sense of the convolution product).

Then we deduce that (2) determines a unique pseudomeasure with coefficients in $\mathbb{Q}[\![q^{B_m}]\!]$ whose moments are given by those of the coefficients (3) (after removing from the Fourier expansion $f(z) = \sum_{h\geq 0} a_h e_m(hz)$ all h with $\det h$ divisible by p):

$$\sum_{h>0, p\nmid \det h} a_h q^h = p^{-m(m+1)/2} \sum_{\substack{h_0 \bmod p \\ p\nmid \det h_0}} \sum_{x\in S \bmod p} e_m(-h_0x/p) f(z+(x/p)).$$

In this way a p-adic family of Siegel-Eisenstein series is geometrically produced.

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1 Complex and p-adic L-functions

Generalities about p-adic L-functions

There exist two kinds of L-functions

- Complex-analytic L-functions (Euler products)
- p-adic L-functions (Mellin transforms L_μ of p-adic measures)

Both are used in order to obtain a number (L-value) from an automorphic form. Usually such a number is algebraic (after normalization) via the embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p.$$

How to define and to compute p-adic L-functions? We use Mellin transform of a $\mathbb{Z}_p$-valued distribution μ on a profinite group

$$Y = \lim_{\substack{\leftarrow \\ i}} Y_i, \;\; \mu \in Distr(Y, \mathbb{Z}_p) = \mathbb{Z}_p[[Y]] = \lim_{\substack{\leftarrow \\ i}} \mathbb{Z}_p[Y_i] =: \Lambda_Y$$

(the *Iwasawa algebra* of Y).

$$L_\mu(x) = \int_Y x(y) d\mu, \quad x \in X_Y = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$$

(the *Mellin transform* of μ on Y).

Examples of p-adic measures and L-functions

- $Y = \mathbb{Z}_p$, $X_Y = \{\chi_t : y \mapsto (1+t)^y\}$. The Mellin transform
$$L_\mu(\chi_t) = \int_{\mathbb{Z}_p} (1+t)^y d\mu(y)$$
of any measure μ on $\mathbb{Z}_p$ is given by *the Amice transform*, which is the following power series
$$A_\mu(t) = \sum_{n\geq 0} t^n \int_{\mathbb{Z}_p} \binom{y}{n} d\mu(y) = \int_{\mathbb{Z}_p} (1+t)^y d\mu(y),$$
e.g. $A_{\delta_m} = (1+t)^m$. Thus, $Distr(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p[\![T]\!]$.
- $Y = \mathbb{Z}_p^* = \Delta \times \Gamma = \{y = \delta(1+p)^z, \delta^{p-1} = 1, z \in \mathbb{Z}_p\}$
$X_{\mathbb{Z}_p^*} = \{\theta\chi_{(t)} \mid \theta \bmod p, \chi_{(t)}(\chi_{(t)}) = (1+t)^z$, where
Δ is the subgroup of roots of unity, $\Gamma = 1 + p\mathbb{Z}_p$.
The *p-adic Mellin transform* $L_\mu(\theta\chi_{(t)}) = \int_{\mathbb{Z}_p^*} \theta(\delta)(1+t)^z \mu(y)$ of a measure μ on $\mathbb{Z}_p^*$ is given by the collection of Iwasawa series $G_{\theta,\mu}(t) = \sum_{n\geq 0} a_{n,\theta} t^n$, where
$(1+t)^z = \sum_{n\geq 0} \binom{z}{n} t^n$,
$a_{n,\theta} = \sum_{\delta \bmod p,\ n\geq 0} \theta(\delta) t^n \cdot \int_{\mathbb{Z}_p} \binom{z}{n} \mu(\delta(1+p)^z)$.
- A general idea is to construct p-adic L-functions *directly from Fourier coefficients* of modular forms (or from the Whittaker functions of automorphic forms).

2 p-adic meromorphic continuation of the Siegel-Eisenstein series

Mazur's p-adic integral

For any choice of a natural number $c \geq 1$ not divisible by p, there exists a p-adic measure μ_c on $\mathbb{Z}_p^*$, such that the special values
$$\zeta(1-k)(1-p^{k-1}) = \frac{\int_{\mathbb{Z}_p^*} y^{k-1} d\mu_c}{1-c^k} \in \mathbb{Q} \quad (k \geq 2 \text{ even }),$$
produce the *Kubota-Leopoldt p-adic zeta-function* $\zeta_p : X_p \to \mathbb{C}_p$ (where $X_p = X_{\mathbb{Z}_p^*} = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$) as the *p-adic Mellin transform*
$$\zeta_p(x) = \frac{\int_{\mathbb{Z}_p^*} x(y) d\mu_c(y)}{1 - cx(c)} = \frac{L_{\mu_c}(x)}{1 - cx(c)},$$

with a single simple pole at $x = x_p^{-1} \in X$, where $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ the Tate field, the completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$, $x \in X_p$ (a $\mathbb{C}_p$-analytic Lie group), $x_p(y) = y \in X_p$, and $x(y) = \chi(y)y^{k-1}$ as above.

Explicitly: Mazur's measure is given by

$$\mu_c(a + p^v\mathbb{Z}_p) = \frac{1}{c}\left[\frac{ca}{p^v}\right] + \frac{1-c}{2c} = \frac{1}{c}B_1(\{\frac{ca}{p^v}\}) - B_1(\frac{a}{p^v}),\ B_1(x) = x - \frac{1}{2},$$

see [LangMF], Ch.XIII.

Meromorphic p-adic continuation of $\frac{1}{\zeta(1-k)(1-p^{k-1})}$

For any odd prime p take the Iwasawa series $G_{\theta,c}(t)$ of Mazur's measure μ_c where θ is a character mod$p$, $G_{\theta,c}(t) := \int_{\mathbb{Z}_p^*} \theta(y)\chi_{(t)}(\langle y\rangle)\mu_c = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Z}_p[\![t]\!]$, and $\chi_{(t)} : (1+p)^z \mapsto (1+t)^z$, $\langle y\rangle = \dfrac{y}{\omega(y)}$, ω the Teichmüller character. Mazur's integral of the character $y^{k-1} = \omega^{k-1} \cdot \chi_{(t)}$ shows that $\theta = \omega^{k-1}, (1+t) = (1+p)^{k-1}$

$$\zeta(1-k)(1-p^{k-1}) = \frac{G_{\theta,c}((1+p)^{k-1}-1)}{1-c^k}. \tag{4}$$

By the *Weierstrass preparation theorem* we have a decomposition

$$G_{\theta,c}(t) = U_{\theta,c}(t)P_{\theta,c}(t)$$

with a *distinguished polynomial* $P_{\theta,c}(t)$ and *invertible power series* $U_{\theta,c}(t)$. The inversion of (4) for any even $k \geq 2$ gives :

$$\frac{1}{\zeta(1-k)(1-p^{k-1})} = G_{\theta,c}((1+p)^{k-1}-1)^{-1}(1-c^k).$$

The answer: for any prime $p > 2$ and even $k \geq 2$

is the following Iwasawa function on $t = t_k = (1+p)^k - 1$ divided by a distingushed polynomial:

$$\begin{aligned}\frac{1}{\zeta(1-k)(1-p^{k-1})} &= \frac{U^*_{\theta,c}((1+p)^{k-1}-1)(1-c^k)}{P_{\theta,c}((1+p)^{k-1}-1)} \\ &= \frac{U^*_{\theta,c}((1+t_k)(1+p)^{-1}-1)(1-c^k)}{P_{\theta,c}((1+t_k)(1+p)^{-1}-1)}\end{aligned} \tag{5}$$

which is meromorphic in the unit disc of the variable $t = (1+p)^k - 1$ with a finite number of poles (expressed via roots of $P_{\theta,c}$) for $\theta = \omega^{k-1}$, and

$$U^*_{\theta,c}((1+p)^{k-1}-1) := 1/U_{\theta,c}((1+p)^{k-1}-1).$$

The above formula immediately extends to all Dirichlet L-functions of characters χ mod p^v as the following Iwasawa function divided by a polynomial:

$$\frac{1}{L(1-k,\chi)(1-\chi(p)p^{k-1})} = \frac{U^*_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1)(1-\chi(c)c^k)}{P_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1)}$$

where $U^*_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1) := \dfrac{1}{U_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1)}$

Illustration: numerical values of $\zeta(1-2k)^{-1}(1-p^{2k-1})^{-1}$ for $p=37$

```
gp >zetap1(p,n)= -2*n/(bernfrac(2*n)*(1-p^(2*n-1)+O(p^5)));
gp > p=37;
gp > for(k=1,(p-1)/2, print(2*k, zetap1(p,k)))
```

$2k$	$\zeta(1-2k)^{-1}(1-p^{2k-1})^{-1}$
2	$25+24*37+24*37^2+24*37^3+24*37^4+O(37^5)$
4	$9+3*37+9*37^3+3*37^4+O(37^5)$
6	$7+30*37+36*37^2+36*37^3+36*37^4+O(37^5)$
8	$18+6*37+O(37^5)$
10	$16+33*37+36*37^2+36*37^3+36*37^4+O(37^5)$
12	$8+25*37+28*37^2+23*37^3+O(37^5)$
14	$25+36*37+36*37^2+36*37^3+36*37^4+O(37^5)$
16	$6+16*37+31*37^2+29*37^3+20*37^4+O(37^5)$
18	$3+4*37+10*37^2+32*37^3+25*37^4+O(37^5)$
20	$11+13*37+19*37^2+36*37^3+12*37^4+O(37^5)$
22	$1+26*37+15*37^2+35*37^3+9*37^4+O(37^5)$
24	$16+28*37+24*37^2+27*37^3+31*37^4+O(37^5)$
26	$4+17*37+25*37^2+25*37^3+19*37^4+O(37^5)$
28	$22+36*37+8*37^2+4*37^3+33*37^4+O(37^5)$
30	$22+5*37+35*37^2+9*37^3+5*37^4+O(37^5)$
32	$36*37^{-1}+28+3*37+19*37^2+18*37^3+O(37^4)$
34	$20+37+30*37^2+15*37^3+22*37^4+O(37^5)$
$\boxed{36}$	$36*37+29*37^2+35*37^3+5*37^4+37^5+O(37^6)$

Fourier expansion of the Siegel-Eisenstein series

has the form

$$E^m_k(z) = \sum_{\gamma\in P\backslash\Gamma} \det(cz+d)^{-k} = \sum_{h\in B_m} a_h q^h,$$

where $a_h = a_h(k) = a_h(E^m_k)$, $q^h = e^{2\pi i \operatorname{tr}(hz)}$, h runs over semi-definite half integral $m\times m$ matrices.

The *rationality of the coefficients* a_h was established in Siegel's pioneer work [Si35] in connection with a study of local densities for quadratic forms. Siegel expressed $a_h(k)$ as a product of local factors over all primes and ∞.

In a difficult later work [Si64b] Siegel proved the boundedness of their denominators, and S.Boecherer [Boe84] gave a simplified proof of a more precise result in 1984. M.Harris extended the rationality to wide classes of Eisenstein series on Shimura varieties [Ha81], [Ha84]. Their relation to the Iwasawa Main Conjecture and p-adic L-functions on the unitary groups was established in [HLiSk].

Explicit p-adic continuation of $a_h(k)$

as *Iwasawa functions on* $t = (1+p)^k - 1$ *divided by distinguished polynomials.* Let $a_h^{(p)}(k)$ denote the *p-regular part of the coefficient* $a_h(k)$ (i.e. with the Euler p-factor removed from the product). Namely, for any even k, $a_h^{(p)}(k) = a_h(E_k^m)$ times

$$\begin{cases} 1/((1-p^{k-1})(1+\psi_h(p)p^{k-\frac{m}{2}-1})\prod_{i=1}^{(m/2)-1}(1-p^{2k-2i-1})) \\ = (1-\psi_h(p)p^{k-\frac{m}{2}-1})/((1-p^{k-1})\prod_{i=1}^{m/2}(1-p^{2k-2i-1})), & m \text{ even} \\ 1/((1-p^{k-1})\prod_{i=1}^{(m-1)/2}(1-p^{2k-2i-1})), & m \text{ odd}, \end{cases}$$

where the *p-correcting factor is a p-adic unit*, and $\psi_h(n) := \left(\dfrac{\det(2h)(-1)^{m/2}}{n}\right)$.

Main Theorem 2.1 (A.P., 2012) *Let h be any positive definite half integral matrix with $\det(2h)$ not divisible by p. Then there exist explicitly given distinguished polynomials $P^E_{\theta,h}(T) \in \mathbb{Z}_p[T]$ and Iwasawa series $S^E_{\theta,h}(T) \in \mathbb{Z}_p[\![T]\!]$ such that the p-regular part $a_h^{(p)}(k)$ of the Fourier coefficient $a_h(k)$ admit the following p-adic meromorphic interpolation on all even k with $\theta = \omega^k$ fixed*

$$a_h^{(p)}(k) = \frac{S^E_{\theta,h}((1+p)^{k-1}-1)}{P^E_{\theta,h}((1+p)^{k-1}-1)}$$

with a finite number of poles expressed via the roots of $P^E_{\theta,h}(T)$ where the denominator depends only on $\det(2h) \bmod 4p$ and $k \bmod p-1$.

Computation of the Fourier coefficients

Recall that Siegel's computation of the coefficients $a_h = a_h(E_k^m)$:

$$E_k^m(z) = \sum_{\gamma \in P\backslash\Gamma} \det(cz+d)^{-k} = \sum_{h \in B_m} a_h q^h$$

is based on the Poisson summation formula giving the equality (see [Maa71], p.304):

$$\sum_{a \in S_m} \det(z+a)^{-k} = \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}}\Gamma_m(k)} \sum_{h \in C_m} \det(h)^{k-\frac{m+1}{2}} e^{2\pi i \mathrm{tr}(hz)},$$

where $\Gamma_m(k) = \pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma(s-\frac{j}{2})$, $q^h = e^{2\pi i \mathrm{tr}(hz)}$, h runs over the set C_m of positive definite half integral $m\times m$ symmetric matrices, and a runs over the set S_m of integral $m\times m$ symmetric matrices, see [Si39], p.652, [St81], p.338.

Formulas for the Fourier coefficients for $\det(2h)\neq 0$

$$\begin{aligned} a_h(E_k^m) = {} & \frac{(-2\pi i)^{mk}\Gamma_m^{-1}(k)}{\zeta(k)\prod_{i=1}^{[m/2]}\zeta(2k-2i)} \\ & \times \det(2h)^{k-\frac{m+1}{2}} M_h(k)\begin{cases} L(k-\dfrac{m}{2},\psi_h), & m \text{ even,}\\ 1, & m \text{ odd.}\end{cases}\end{aligned}$$

The *integral factor* $M_h(k) = \displaystyle\prod_{\ell\in P(h)} M_\ell(h,\ell^{-k})$ is a *finite Euler product*, extended over primes ℓ in the set $P(h)$ of prime divisors of all elementary divisors of the matrix h. The important property of the product is that for each ℓ we have that $M_\ell(h,t)\in\mathbb{Z}[t]$ is a polynomial with integral coefficients.

Notice the *L-factor* $L(k-\frac{m}{2},\psi_h)$ depends on the index h of the Fourier coefficient; this makes a difference to the case of odd m; the case of $GL(2)$ corresponds to $m=1$.

Proof: the use of the normalized Siegel-Eisenstein series

defined as in [Ike01], [PaSE] and [PaLNM1990] by

$$\mathcal{E}_k^m = E_k^m(z)2^{m/2}\zeta(1-k)\prod_{i=1}^{[m/2]}\zeta(1-2k+2i),$$

I show that it produces a nice p-adic family, namely:

Proposition 2.2

(1) For any non-degenerate matrice $h\in C_m$ the following equality holds

$$\begin{aligned} a_h(\mathcal{E}_k^m) = {} & 2^{-\frac{m}{2}}\det h^{k-\frac{m+1}{2}} M_h(k) \\ & \times\begin{cases} L(1-k+\frac{m}{2},\psi_h)C_h^{\frac{m}{2}-k+(1/2)}, & m \text{ even,}\\ 1, & m \text{ odd,}\end{cases}\end{aligned} \tag{6}$$

where C_h is the conductor of ψ_h.

(2) for any prime $p>2$, and $\det(2h)$ not divisible by p, define the p-regular part $a_h(\mathcal{E}_k^m)^{(p)}$ of the coefficient $a_h(\mathcal{E}_k^m)$ of $\mathcal{E}_k^m$ by introducing the factor $\begin{cases}(1-\psi_h(p)p^{k-\frac{m}{2}-1})C_h^{\frac{m}{2}-k+(1/2)}, & m \text{ even,}\\ 1, & m \text{ odd.}\end{cases}$
Then $a_h(\mathcal{E}_k^m)^{(p)}$ is a p-adic analytic Iwasawa function of $t=(1+p)^k-1$ for all k with ω^k fixed, and divided by the elementary factor $1-\psi_h(c_h)c_h^{k-\frac{m}{2}}$.

Proof of (1) of Proposition 2.2

Proof of (1) is deduced like at p.653 of [Ike01] from the Gauss duplication formula

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\sqrt{\pi}\Gamma(s),$$

the definition

$$\Gamma_m(k) = \pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma(s-\frac{j}{2})$$

and the functional equations

$$\begin{aligned}
\zeta(1-k) &= \frac{2(k-1)!}{(-2\pi i)^k}\zeta(k),\\
\zeta(1-2k+2i) &= \frac{2(2k-2i-1)!}{(-2\pi i)^{2k-2i}}\zeta(2k-2i),\\
L(1-k+\frac{m}{2},\psi_h) &= \frac{2(k-\frac{m}{2}-1)!}{(-2\pi i)^{k-\frac{m}{2}}}L(k-\frac{m}{2},\psi_h)C_h^{k-\frac{m}{2}-\frac{1}{2}}
\end{aligned}$$

Proof of (2) of Proposition 2.2

is then deduced easily :

Notice that for any $a \in \mathbb{Z}_p^*$, the function of $t = (1+p)^k - 1$

$$\begin{aligned}
k \mapsto a^k = \omega(a)^k\langle a\rangle^k &= \omega(a)^k(1+p)^{k\frac{\log\langle a\rangle}{\log(1+p)}} \\
&= \omega(a)^k(((1+p)^k-1)+1)^{\frac{\log\langle a\rangle}{\log(1+p)}} \\
&= \omega(a)^k\sum_{n=0}^{\infty}\binom{\frac{\log\langle a\rangle}{\log(1+p)}}{n}t^n
\end{aligned} \tag{7}$$

is a p-adic analytic Iwasawa function denoted by $\tilde{a}(t) \in \mathbb{Z}_p[\![t]\!]$, of $t = (1+p)^k - 1$ with ω^k fixed, where $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$.

Then Mazur's formula applied to $L(1-k+\frac{m}{2},\psi_h)(1-\psi_h(p)p^{k-\frac{m}{2}-1})$ shows that this function is a p-adic analytic Iwasawa function of $t = (1+p)^k - 1$ with ω^k fixed (a single simple pole may occur at $k = \frac{m}{2}$ only if $\omega^{k-\frac{m}{2}}$ is trivial).

Proof of Main Theorem 2.1

Let us use the equality

$$E_k^m = \mathcal{E}_k^m(z)\cdot\frac{2^{-m/2}}{\zeta(1-k)\prod_{i=1}^{[m/2]}\zeta(1-2k+2i)}$$

and the properties of the normalized series $\mathcal{E}_k^n(z)$ in Proposition 2.2.

First let us compute the reciprocal of the product of L-functions

$$\zeta(1-k)\prod_{i=1}^{[m/2]}\zeta(1-2k+2i)$$

using the above: for even $k \geq 2$,

$$\zeta(1-k)^{-1}(1-p^{k-1})^{-1} = \frac{U^*_{\theta_k,c}((1+p)^{k-1}-1)(1-c^k)}{P_{\theta_k,c}((1+p)^{k-1}-1)} \tag{8}$$

$$\zeta(1-2k+2i)^{-1}(1-p^{2k-2i-1})^{-1}$$
$$= \frac{U^*_{\theta_{2k-2i},c}((1+p)^{2k-2i-1}-1)(1-c^{2k-2i})}{P_{\theta_{2k-2i},c}((1+p)^{2k-2i-1}-1)} \tag{9}$$

which is meromorphic in the unit disc with a finite number of poles (expressed via roots of P_θ) for $\theta_k = \omega^{k-1}$.

Let us use again the notation $1+t = (1+p)^k$ with $k \in \mathbb{Z}_p$.

$$\frac{2^{-m/2}}{\zeta(1-k)(1-p^{k-1})\prod_{i=1}^{[m/2]}\zeta(1-2k+2i)(1-p^{2k-2i-1})}$$
$$= \frac{U^E_{\omega^k}(t)}{P^E_{\omega^k}(t)} \tag{10}$$

where the *numerator* is (an Iwasawa function) $U^E_{\omega^k}(t) =$

$$U^*_{\theta_k,c}(\frac{1+t}{1+p}-1)(1-c^k)\prod_{i=1}^{[m/2]}U^*_{\theta_{2k-2i},c}(\frac{(1+t)^2}{(1+p)^{2i+1}}-1)(1-c^{2k-2i}),$$

and

$$P^E_{\omega^k}(t) = P_{\theta,c}\left(\frac{1+t}{1+p}-1\right)\prod_{i=1}^{[m/2]}P_{\theta_{2k-2i},c}\left(\frac{(1+t)^2}{(1+p)^{2i+1}}-1\right)$$

is the *polynomial denominator* which depends only on $k \bmod p-1$.

Proof of Main Theorem 2.1: control over the conductor of ψ_h

Moreover, Mazur's formula applied to $L(1-k+\frac{m}{2},\psi_h)(1-\psi_h(p)p^{k-\frac{m}{2}-1})$ (*in the numerator*) shows that for all h with $\det(2h)$ not divisible by p,

$$L(1-k+\frac{m}{2},\psi_h)(1-\psi_h(p)^{k-\frac{m}{2}-1}) \tag{11}$$
$$= \frac{G_{\theta,h}((1+p)^{k-\frac{m}{2}-1}-1)}{1-\psi_h(c_h)c_h^{k-\frac{m}{2}}}$$

which is meromorphic in the unit disc with a possible single simple pole at $k=\frac{m}{2}$ for all k with $\theta=\omega^{k-1}$. It comes from Mazur's measure on the finite product $\prod\limits_{\ell\in P_h}\mathbb{Z}_\ell^*$ extended over primes ℓ in the set $P_h=P(h)\cup\{p\}$; recall that $P(h)$ is the set of prime divisors of all elementary divisors of the matrix h as above.

Indeed, for any choice of a natural number $c_h>1$ coprime to $\prod_{\ell\in P_h}\ell$, there exists a p-adic measure $\mu_{c_h,h}$ on $\mathbb{Z}_p^*$, such that the special values

$$\begin{aligned}L(1-k+\frac{m}{2},\psi_h)(1-\psi_h(p)p^{k-1-\frac{m}{2}}) &= \frac{\int_{\mathbb{Z}_p^*}y^{k-\frac{m}{2}-1}d\mu_{c_h,h}}{1-\psi_h(c_h)c_h^{k-\frac{m}{2}}}\\ := (1-\psi_h(c_h)c_h^{k-\frac{m}{2}})^{-1}&\int_{\prod_{\ell\in P_h}\mathbb{Z}_\ell^*}\psi_h(y)y_p^{k-\frac{m}{2}-1}d\mu_{c_h},\end{aligned}$$

where Mazur's measure μ_{c_h} extends on the product $\prod\limits_{\ell\in P_h}\mathbb{Z}_\ell^*\xrightarrow{y_p}\mathbb{Z}_p^*$ (see §3,Ch.XIII of [LangMF]):

$$\mu_{c_h}(a+(N))=\frac{1}{c_h}\left[\frac{c_h a}{N}\right]+\frac{1-c_h}{2c_h}=\frac{1}{c_h}B_1(\{\frac{c_h a}{N}\})-B_1(\frac{a}{N})$$

for any natural number N with all prime divisors in P_h.

The regularizing factor is the following Iwasawa function which depends on c_h mod $4p$ and k mod $p-1$:

$$\begin{aligned}1-\psi_h(c_h)c_h^{k-\frac{m}{2}} &= 1-(\psi_h\omega^{k-\frac{m}{2}})((\frac{(1+p)^k}{(1+p)^{\frac{m}{2}}}-1)+1)^{\frac{\log\langle c_h\rangle}{\log(1+p)}} \qquad (12)\\ &= 1-(\psi_h\omega^{k-\frac{m}{2}})(c_h)\sum_{n=0}^{\infty}\binom{\frac{\log\langle c_h\rangle}{\log(1+p)}}{n}((\frac{(1+p)^k}{(1+p)^{\frac{m}{2}}}-1)^n\\ &= 1-(\psi_h\omega^{k-\frac{m}{2}})(c_h)\sum_{n=0}^{\infty}\binom{\frac{\log\langle c_h\rangle}{\log(1+p)}}{n}(\frac{(1+t)}{(1+p)^{\frac{m}{2}}}-1)^n\in\mathbb{Z}_p[\![t]\!],\end{aligned}$$

where we write c_h in place of $i_p(c_h)$ and use the notation $1+t=(1+p)^k$. The function (12) is divisible by t or invertible in $\mathbb{Z}_p[\![t]\!]$ according as $\omega^{k-\frac{m}{2}}\psi_{c_h}$ is trivial or not because $t=0\Longleftrightarrow k=0$ and $1+t=(1+p)^k$.

Elementary factors

Notation:

$$u_{c_h}(t)=\begin{cases}(1-\psi_h(c_h)\tilde{c}_h(t))/t, & \text{if } \omega^{k-\frac{m}{2}}\psi_h \text{ is trivial},\\ 1-\psi_h(c_h)\tilde{c}_h(t), & \text{otherwise}.\end{cases}$$

By (12) we have that $u_{c_h}(t) \in \mathbb{Z}_p[\![t]\!]^*$, and we denote by $u^*_{c_h}(t)$ its inverse. Moreover, (6) gives the elementary factor

$$\mathcal{M}_h((1+p)^k - 1) = 2^{-\frac{m}{2}} \det h^{k-\frac{m+1}{2}} \prod_{\ell | P(h)} M(h, \ell^{-k}) C_h^{k-\frac{m+1}{2}}$$

which is also an *Iwasawa function* as above:

$$\mathcal{M}_h((1+p)^k - 1) = \mathcal{M}_h(t) \in \mathbb{Z}_p[\![t]\!].$$

Proof of Main Theorem 2.1: the numerator

It follows that

$$a_h^{(p)}(k) = \frac{S^E_{\theta,h}((1+p)^k - 1)}{P^E_{\theta,h}((1+p)^k - 1)} = \frac{S^E_{\theta,h}(t)}{P^E_{\theta,h}(t)},$$

where

$$\begin{aligned}
&S^E_{\theta,h} = u^*_{c_h}((1+p)^k - 1)\mathcal{M}((1+p)^k - 1)\\
&\times U_{\theta,h}((1+p)^{k-\frac{m}{2}-1} - 1)(1 - c_h^k) U^*_{\theta_k,c}((1+p)^{k-1} - 1)\\
&\times \prod_{i=1}^{[m/2]} U^*_{\theta_{2k-2i},c}((1+p)^{2k-2i-1} - 1)(1 - c_h^{2k-2i})\\
&= u^*_{c_h}(t)\mathcal{M}(t) U_{\theta,h}((1+t)(1+p)^{-\frac{m}{2}-1} - 1)\\
&\times (1 - \tilde{c}_h(t)) U^*_{\theta_k,c_h}((1+t)(1+p)^{-1} - 1)\\
&\times \prod_{i=1}^{[m/2]} U^*_{\theta_{2k-2i},c_h}((1+t)^2(1+p)^{-2i-1} - 1)(1 - \tilde{c}_h^2(t) c_h^{-2i}),
\end{aligned}$$

Proof of Main Theorem (end)

The denominator is the following distinguished polynomial

$$\begin{aligned}
&P^E_{\theta,h}((1+p)^{k-1} - 1) = (1 + ((1+p)^{k-1} - 2)\delta(\omega^{k-\frac{m}{2}}\psi_{c_h}))\\
&\times P_{\theta_k,c_h}((1+p)^{k-1} - 1) \prod_{i=1}^{[m/2]} P_{\theta_{2k-2i},c_h}((1+p)^{2k-2i-1} - 1)\\
&= (1 + (t-1)\delta(\omega^{k-\frac{m}{2}}\psi_{c_h})) P_{\theta_k,c_h}((1+t)(1+p)^{-1} - 1)\\
&\times \prod_{i=1}^{[m/2]} P_{\theta_{2k-2i},c}((1+t)^2(1+p)^{-2i-1} - 1), \text{ where}
\end{aligned}$$

$$\delta(\omega^{k-\frac{m}{2}}\psi_{c_h}) = \begin{cases} 1, & \text{if } \omega^{k-\frac{m}{2}}\psi_{c_h} \text{ is trivial,} \\ 0, & \text{otherwise,} \end{cases} \text{ so that}$$

$$1+(t-1)\delta(\omega^{k-\frac{m}{2}}\psi_{c_h}) = \begin{cases} t, & \text{if } \omega^{k-\frac{m}{2}}\psi_{c_h} \text{ is trivial,} \\ 1, & \text{otherwise.} \end{cases} .$$

It remains to notice that different choices of c_h *coprime to* $p\det(2h)$ *give the same polynomial factors* $P^E_{\theta,h}$ *(up to invertible Iwasawa function). Indeed they all give the same single simple zero.* ∎

3 Pseudomeasures and their Mellin transform

Interpretation: Mellin transform of a pseudomeasure

Pseudomeasures were introduced by J.Coates [Co] as elements of the *fraction field* $\mathcal{L}$ of the Iwasawa algebra. Such a pseudomeasure is defined by its Mellin transform which is a ring homomorphism and we can extend it by universality (the extension of the integral along measures in $\Lambda = \mathbb{Z}_p[[T]]$ to the whole fraction field $\mathcal{L}$).

The p-adic meromorphic function

$$a_h^{(p)}(k) = \frac{S^E_{\theta,h}((1+p)^k-1)}{P^E_\theta((1+p)^k-1)} = \frac{S^E_{\theta,h}(t)}{P^E_\theta(t)},$$

is attached to an explicit pseudo-measure:

$$\rho_h^E = \frac{\mu_h^E}{\nu_h^E}, \quad \frac{S^E_{\theta,h}(t)}{P^E_\theta(t)} = \frac{\int_{\mathbb{Z}_p^*}\theta\chi_{(t)}\mu_h}{\int_{\mathbb{Z}_p^*}\theta\chi_{(t)}\nu_h}$$

- $\mathcal{S}(x) = \int_{\mathbb{Z}_p^*} x\mu_h^E$ is given by the collection of Iwasawa functions $S_\theta(t) = \int_{\mathbb{Z}_p^*}\theta\chi_{(t)}\mu_h^E$ (the *numerator*),
- $\mathcal{P}(x) = \int_{\mathbb{Z}_p^*} x\nu_h^E$ is given by the collection of polynomials $P_\theta(t) = \int_{\mathbb{Z}_p^*}\theta\chi_{(t)}\nu_h^E$ (the *denominator*).

Pseudomeasure ρ as a family of distributions

A pseudomeasure ρ can be described as a certain *family of distributions, parametrized by the set* X_p *of* p*-adic characters.*

For any $x \in X_p$ we have a distribution given by the formula

$$\rho^E_{h,x}(a+(p^v)) = \frac{1}{\varphi(p^v)}\sum_{\chi \bmod p^v}{}' \chi(a)^{-1}\frac{\mathcal{S}^E(\chi x)}{\mathcal{P}^E(\chi x)}$$

where $'$ means that the terms with $\mathcal{P}(\chi x) = 0$ are omited. It follows that

$$\int_{\mathbb{Z}_p^*} \chi \rho^E_{h,x} = \begin{cases} \dfrac{\mathcal{S}^E(\chi x)}{\mathcal{P}^E(\chi x)}, & \text{if } \mathcal{P}^E(\chi x) \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned}
&\mathcal{S}^E_h(\chi x) = S^E_{(\chi x)_\Delta, h}(\chi x(1+p) - 1) = S^E_{\theta,h}((1+p)^t - 1), \\
&\theta = (\chi x)_\Delta, (\chi x)(1+p) = 1 + t, \\
&\mathcal{P}^E_h(\chi x) = P^E_{(\chi x)_\Delta, h}(\chi x(1+p) - 1) = P^E_{\theta,h}((1+p)^t - 1).
\end{aligned}$$

A geometric construction: Siegel's method and duality

For any Dirichlet character χ mod p^v consider Shimura's "involuted" Siegel-Eisenstein series assuming their absolute convergence (i.e. $k > m+1$):

$$E^*_k(\chi, z) = \sum_{(c,d)/\sim} \chi(\det(c)) \det(cz+d)^{-k} = \sum_{h \in B_m} a_h(E^*_k(\chi, z)) q^h$$

The series on the left is *geometrically defined*, and the Fourier coefficients on the right can be computed by Siegel's method (see [St81] [Shi95], ...) via the singular series

$$\begin{aligned}
&a_h(E^*_k(\chi, z)) \qquad (13) \\
&= \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}} \Gamma_m(k)} \sum_{\mathfrak{R} \bmod 1} \chi(\nu(\mathfrak{R})) \nu(\mathfrak{R})^{-k} \det h^{k - \frac{m+1}{2}} e_m(h\mathfrak{R})
\end{aligned}$$

If $\chi = \chi_0$ mod p is trivial and $p \nmid \det h$ then

$$\begin{aligned}
&a_h(E^*_k(\chi_0, z)) \qquad (14) \\
&= \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}} \Gamma_m(k)} \sum_{\mathfrak{R} \bmod 1} \chi_0(\nu(\mathfrak{R})) \nu(\mathfrak{R})^{-k} \det h^{k - \frac{m+1}{2}} e_m(h\mathfrak{R}) = \\
&a_h(E^m_k) \times \begin{cases} (1 - p^{-k})(1 + \psi_h(p) p^{-k + \frac{m}{2}}) \prod_{i=1}^{(m/2)-1} (1 - p^{-2k+2i}), & m \text{ even} \\ (1 - p^{-k}) \prod_{i=1}^{(m-1)/2} (1 - p^{-2k+2i}), & m \text{ odd.} \end{cases}
\end{aligned}$$

The formula (14) means that the series $E^*_k(\chi_0, z)$ coincides with E^m_k after removing h with $\det h$ divisible by p and normalizing by the factor in (14). Moreover, the *Gauss reciprocity law* shows that the normalizing factor depends only on $\det h \bmod 4p = \det h_0 \bmod 4p$, where $h_0 \equiv h \bmod 4p$ runs through a representative system. Let us denote this factor by $C^+(h_0, k, 4p)$: for the trivial character

$\chi = \chi_0 \bmod p$ and $\det h$ not divisible by p

$$a_h(E_k^*(\chi_0, z)) = a_h(E_k^m)C^+(h_0, k, 4p), \text{ where} \tag{15}$$

$$C^+(h_0, k, 4p) = \begin{cases} (1-p^{-k})(1+\psi_h(p)p^{-k+\frac{m}{2}}) \prod_{i=1}^{(m/2)-1}(1-p^{-2k+2i}), & m \text{ even} \\ \\ (1-p^{-k}) \prod_{i=1}^{(m-1)/2}(1-p^{-2k+2i}), & m \text{ odd.} \end{cases}$$

From the Fourier coefficients to modular forms:

If we remove in the Fourier expansion $E_k^m(z) = \sum_{h \geq 0} a_h e_m(hz)$ all terms with $\det h$ divisible by p the equality of Fourier coefficients (15) transforms to the equality of the series

$$E_k^*(\chi_0, z) = (4p)^{-m(m+1)/2} \sum_{\substack{h_0 \bmod 4p \\ p \nmid \det h_0}} C^+(h_0, k, 4p) \times \tag{16}$$
$$\sum_{x \in S \bmod 4p} e_m(-h_0 x/4p) E_k^m(z + (x/4p)).$$

A geometric construction

Let us apply the interpolation theorem (Theorem 2.1) to all the coefficients

$$a_h^{(p)}(k) = a_h(E_k^m)C^-(h_0, k, 4p), \text{ where} \tag{17}$$

$$C^-(h_0, k, 4p) = \begin{cases} \frac{1-\psi_h(p)p^{k-\frac{m}{2}-1}}{(1-p^{k-1})\prod_{i=1}^{m/2}(1-p^{2k-2i-1})}, & m \text{ even} \\ \\ \frac{1}{(1-p^{k-1})\prod_{i=1}^{(m-1)/2}(1-p^{2k-2i-1})}, & m \text{ odd,} \end{cases}$$

and (16) becomes a "geometric-algebraic equality" of two families of modular forms

$$E_k^*(\chi_0, z) = (4p)^{-m(m+1)/2} \sum_{\substack{h_0 \bmod 4p \\ p \nmid \det h_0}} C^+(h_0, k, 4p) \times \tag{18}$$
$$C^-(h_0, k, 4p) \sum_{x \in S \bmod 4p} e_m(-h_0 x/4p) E_k^m(z + (x/4p)).$$

A geometric construction (end)

We deduce by the orthogonality that

$$\sum_{x' \in S \bmod 4p} e_m(-h_0 x'/4p) E_k^*(\chi_0, z + (x'/4p)) = \tag{19}$$
$$C^+(h_0, k, 4p) \sum_{x \in S \bmod 4p} e_m(-h_0 x/4p) E_k^m(z + (x/4p)).$$

Each series $C^-(h_0,k,4p)\sum_{x\in S \bmod 4p} e_m(-h_0x/4p)E_k^m(z+(x/4p))$ in (18) determines a *unique pseudomeasure with coefficients in* $\bar{\mathbb{Q}}[\![q^{B_m}]\!]$ whose moments are given by those of the coefficients (17).The unicity means that a pseudomeasure is determined by its Mellin transform. It is also a family of distributions geometrically defined by the series

$$\frac{C^-(h_0,k,4p)}{C^+(h_0,k,4p)}\sum_{x\in S \bmod 4p} e_m(-h_0x/4p)E_k^*(\chi_0,z+(x/4p)).$$

4 Application to Minkowski-Siegel Mass constants

p-adic version of Minkowski-Siegel Mass constants.

An application of the construction is the p-adic version of Siegel's Mass formula. It expresses the Mass constant through the above product of L-values. This product can be viewed as the proportionality coefficient between two kinds of Eisenstein series in the symplectic case extending Hecke's result (1927) of the two kinds of Eisenstein series and the relation between them. However, there is no direct analogue of Hecke's computation in the symplectic case.

Thus this mass constant admits an explicit product expression through the values of the functions (5) at $t_j=(1+p)^j-1$, for $j=k$, and $j=2,4,\ldots,2k-2$.

Recall that ([ConSl98], p.409)

> unimodular latticies have the property that there are explicit formulae, the mass formulae, which give appropriately weighted sums of the theta-series of all the inequivalent latticies of a given dimension. In particular, the numbers of inequivalent latticies is given by Minkowski-Siegel Mass constants for unimodular latticies.

In the particular case of even unimodular quadratic forms of rank $m=2k\equiv 0(\mathrm{mod}8)$, this formula means that there are only finitely many such forms up to equivalence for each k and that, if we number them $Q_1,\ldots,Q_{h_k}$, then we have the relation

$$\sum_{i=1}^{h_k}\frac{1}{w_i}\Theta_{Q_i}(z)=m_kE_k$$

where w_i is the number of automorphisms of the form Θ_{Q_i} is the theta series of Q_i, E_k the normalized Eisenstein series of weight $k=m/2$ (with the constant term equal to 1),

The dimension of lattices is $2k$ and the Mass formula express an identity of a sum of *weighted theta functions* and a Siegel-Eisenstein series of weight k, multiplied by the Mass constant

$$m_k = 2^{-k}\zeta(1-k)\prod_{i=1}^{k-1}\zeta(1-2k+2i) = (-1)^k\frac{B_k}{2k}\times\prod_{j=1}^{k-1}\frac{B_{2j}}{4j}$$

which is related the above normalising coefficient.

```
gp > mass(4)
% = 1/696729600
gp > mass(8)
% = 691/277667181515243520000
```

The present result says that the p-regular part of $1/m_k$ is a product of values of the p-adic meromorphic functions (5) at $t_j = (1+p)^j - 1$, $j = k$ and $j = 2, 4, \ldots, 2k-2$.

It is known that the rational number m_k becomes very large rapidly, when k grows (using the functional equation). It means that the denominator of $1/m_k$ becomes *enormous*. The explicit formula (10) applied to the reciprocal of the product of L-functions as above shows that these are *only irregular primes* which contribute to the denominator, and this contribution can be evaluated for *all primes* knowing the *Newton polygons* of the polynomial part P_θ, which can be found directly from the Eisenstein measure. Precisely, for the distingushed polynomial $P(t) = P_\theta(t) = a_d t^d + \cdots + a_0$, $\mathrm{ord}_p a_d = 0$, and $\mathrm{ord}_p a_i > 0$ for $0 \le i \le d-1$, and $\mathrm{ord}_p(t_j) = \mathrm{ord}_p j + 1$, where $t_j = (1+p)^j - 1$ for $j = k$ and $j = 2, 4, \ldots, 2k-2$. Then

$$\mathrm{ord}_p P(t_j) = \min_{i=0,\ldots,d}\left(\mathrm{ord}_p a_{i,k} + i(\mathrm{ord}_p j + 1)\right).$$

the values $\mathrm{ord}_p a_{i,k}$ for $0 \le i \le d$ come from the Iwasawa series in the denominator in the left hand side of (10). Also, it gives an important information about the location of zeroes of the polynomial part as in (10)). However $P(t_j) \ne 0$ in our case because all the L-values in question do not vanish.

Application to Minkowski-Siegel Mass constant (numerical illustration)

```
for(k=1,10,print(2*k, factor(denominator(1/mass(2*k)))))
2 1
4 1
6 1
8 [691, 1]
10 [691, 1; 3617, 1; 43867, 1]
12 [131, 1; 283, 1; 593, 1; 617, 1; 691, 2; 3617, 1; 43867, 1]
14 [103, 1; 131, 1; 283, 1; 593, 1; 617, 1; 691, 1; 3617, 1; 43867, 1;
6579 31, 1; 2294797, 1]
16 [103, 1; 131, 1; 283, 1; 593, 1; 617, 1; 691, 1; 1721, 1; 3617, 2;
```

```
9349, 1; 43867, 1; 362903, 1; 657931, 1; 2294797, 1; 1001259881, 1]
18 [37, 1; 103, 1; 131, 1; 283, 1; 593, 1; 617, 1; 683, 1; 691, 1; 1721,
1; 3617, 1; 9349, 1; 43867, 2; 362903, 1; 657931, 1; 2294797, 1; 305065927,
1; 1001259881, 1; 151628697551, 1]
20 [103, 1; 131, 1; 283, 2; 593, 1; 617, 2; 683, 1; 691, 1; 1721, 1; 3617,
1; 9349, 1; 43867, 1; 362903, 1; 657931, 1; 2294797, 1; 305065927, 1;
1001259881, 1; 151628697551, 1; 154210205991661, 1; 26315271553053477373,
1]
```

5 Link to Shahidi's method for SL(2) and regular prime p

Methods of constructing p-adic L-functions

Our long term purposes are to define and to use the p-adic L-functions in a way similar to complex L-functions via the following methods:

(1) Tate, Godement-Jacquet;

(2) the method of Rankin-Selberg;

(3) the method of Euler subgroups of Piatetski-Shapiro and the doubling method of Rallis-Böcherer (integral representations on a subgroup of $G \times G$);

(4) Shimura's method (the convolution integral with theta series), and

(5) Shahidi's method.

There exist already advances for (1) to (4), and we are also trying to develop (5).

We use the Eisenstein series on classical groups and p-adic integral of Shahidi's type for the reciprocal of a product of certain L-functions.

Link to Shahidi's method in the case of SL(2) and regular prime p

The starting point here is the Eisenstein series

$$E(s, P, f, g) = \sum_{\gamma \in P \diagdown G} f_s(\gamma g),$$

on a reductive group G and a *maximal parabolic subgroup* $P = MU^P$ (decomposition of Levi).

This series generalizes

$$E(z, s) = \frac{1}{2} \sum \frac{y^s}{|cz+d|^{2s}}, \quad (c, d) = 1.$$

Here f_s is an appropriate function in the induced representation space $I(s, \pi) = Ind_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\pi \otimes |det_M(\cdot)|_{\mathbb{A}}^s))$, see (I.2.5.1) at p. 34 of [GeSha].

Computing a non-constant term (a Fourier coefficient) of this Eisenstein series provides an analytic continuation and the functional equation for many Langlands L functions $L(s, \pi, r_j)$.

In this way the ψ-th Fourier coefficient (with ψ of type $\psi(x) = \exp(2\pi i n x), n \in \mathbb{N}, n \neq 0$) of the series $E(s, P, f, e)$ is determined by the Whittaker functions W_v in the form (see [GeSha], (II.2.3.1), p.78):

$$E^{\psi}(e, f, s) = \prod_{v \in S} W_v(e_v) \prod_{j=1}^{m} \frac{1}{L^S(1 + js, \pi, r_j)},$$

where r_j are certains fundamental representations of the dual group ${}^L M$.

Theorem 5.1 (a complex version) *With the data $G = SL(2)$, $M = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\} \cong \mathrm{GL}_1$, $\pi = I$, and ψ a non-trivial character of the group $U(\mathbb{A})/U(\mathbb{Q})$, $U = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \cong \mathbb{G}_a$, let $E^{\psi}(s, f, e) = \int E(s, f, n)\psi(n)dn$, the integration on the quotient space of $U(\mathbb{A})$ by $U(\mathbb{Q})$. Then the first Fourier coefficient has the form*

$$E^{\psi}(s, f, e) = W_{\infty}(s) \frac{1}{\zeta(1 + s)},$$

for a certain Whittaker function $W_{\infty}(s)$ (see [Kub], p.46).

Theorem 5.2 (a p-adic version, a work in progress) *(with S.Gelbart, S.Miller, F.Shahidi)*

Let p be a regular prime. Then there exists an explicitly given distribution μ^* on $\mathbb{Z}_p^*$ such that for all $k \geq 3$ and for all primitive Dirichlet characters $\chi \bmod p^v$ with $\chi(-1) = (-1)^k$ one has

$$\int_{\mathbb{Z}_p^*} \chi y_p^k \mu^* = \frac{1}{(1 - \chi(p)p^{k-1})L(1 - k, \chi)},$$

where $L(s, \chi)$ is the Dirichlet L-function. More precisely, the distribution μ^* can be expressed through the non-constant Fourier coefficients of a certain Eisenstein series Φ^*.

Remark. Using Siegel's method for the symplectic groups GSp_m, and for all primes p, this result also follows from Main Theorem 2.1 by specializing it to the case of regular p and $m = 1$.

6 Doubling method and Ikeda's constructions

Further applications: we only mention the proof of the p-adic Miyawaki Modularity Lifting Conjecture

by pullback of families Siegel modular forms (jointly with Hisa-Aki Kawamura), see [Kawa], [PaIsr11].

Ikeda's constructions ([Ike01], [Ike06]) extend the doubling method to pullbacks of cusp forms instead of pullbacks of Eisenstein series.

In the Fall 1999 in IAS, Ilya was much inspired by the preprint of the first Ikeda's lifting, and tried to interpret it representation-theoretically.

Indeed, it extends his own work [PS1] on Saito-Kurokawa lifting from genus 2 to arbitrary genus $2m$.

In fact, there is a relation of Ikeda's work to Arthur's conjecture [Ar89].

In the same period, Ilya studied the preprint of *[KMS2000] on p-adic Rankin-Selberg L-functions in an informal seminar in his office in IAS together with me and other participants: Jim Cogdell, Siegfried Böcherer, Reiner Schulze-Pillot, ...*

The use of the The Eisenstein family $\mathcal{E}_k^{(n)}$

as above plays a crucial role in Ikeda's work: the idea was to substitute the Satake parameter $\alpha_p(k)$ of a cusp form in place of the parameter k in the Siegel-Eisenstein family.

Both p-adic and complex analytic L-functions are produced in this way.

Thus obtained cuspidal p-adic measures generalize the Eisenstein measure, and produce families of cusp forms.

A version of this construction produces Klingen-Eisenstein series and Langlands Eisenstein series, see [PaSE] (p-adic Peterson product of a cusp form with a pullback of the constructed family), more recently used by Skinner-Urban [MC].

For genus two, my student P.Guerzhoy found in 1998 a p-adic version of the *holomorphic Maass-Saito-Kurokawa lifting* [Gue], answering a question of E.Freitag. P.Guerzhoy visited Ilya here in Yale in 1999.

A Appendix. On p-adic L-functions for $GSp(4)$

talk by Alexei Panchishkin on December 2, 1999, at Automorphic Forms and L-functions Seminar in IAS.

A.0 Introduction.

The purpose of this talk is to describe a joint work in progress with I.I.Piatetski-Shapiro started in February 1998 in Jerusalem during the conference "p-Adic Aspects of the Theory of Automorphic Representations".

Let G be a semi-simple algebraic group over a number field F, and $p \geq 5$ be a fixed prime number. Recall that the *Iwasawa albebra* Λ is defined as $\mathbb{Z}_p[[T]]$ and let $\mathcal{L} = \mathrm{Quot}\Lambda$ denote its quotient field. Elements $a(T) \in \mathcal{L}$ represent some $\mathbb{C}_p$-meromorphic functions with finite number of poles on the unit disc $U_p = \{t \in \mathbb{C}_p \mid |t|_p < 1\} \subset \mathbb{C}_p$ where $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ the *Tate field*. We consider the following problem: how to attach to a (complex valued) Langlands L-functions $L(s, \pi, r)$ a certain p-adic valued meromorphic L function $L_{\pi,r,p}$ with a finite number of poles where π is an automorphic representation of the adelic group $G(\mathbb{A}_F)$ and r is a finite dimensional complex representation $r : {}^LG(\mathbb{C}) \to GL_m(\mathbb{C})$ of the Langlands group ${}^LG(\mathbb{C})$. The p-adic L-function $L_{\pi,r,p}$ should belong to $\mathcal{L}$ or to its finite extension. The first example of a function of this type comes from the work of Kubota and Leopoldt [Ku-Le] and Iwasawa [Iw]: there exists a unique element $g(T) \in \mathcal{L}$ such that for all positive integers $k \equiv 0(\bmod\,(p-1))$, $g((1+p)^k - 1) = \zeta^*(1-k)$, where $\zeta^*(s) = (1-p^{-s})\zeta(s)$ is the Riemann zeta function with the p-factor removed from its Euler product. The function $\zeta_p(s) = g((1+p)^{1-s} - 1)$ is analytic for all $s \in \mathbb{Z}_p \backslash 1$ with values in $\mathbb{Q}_p$ and it is called the *Kubota-Leopoldt p-adic zeta function*. It has the following properties: $\zeta_p(1-k) = \zeta^*(1-k)$ for all positive integers $k \equiv 0(\bmod\,(p-1))$, and $\mathrm{Res}_{s=1}\zeta_p(s) = 1 - \frac{1}{p}$. In this case we have actually $Tg(T) \in \Lambda^\times$ so that $\zeta_p(s)$ has no zeroes, unnike the complex zeta-function. However, one could start from another progrssion $k \equiv i(\mathrm{mod} p-1), k > 0$, i mod $(p-1)$ and obtain in the same way other branches $\zeta_{p,i}(s)$ of p-adic zeta function which have interesting zeroes important in the Iwasawa theory [Iw, Wi90].

Constructions of $L_{\pi,r} \in \mathcal{L}$ are known in a number of cases but there exists no general definition. For example, the standard L functions $L(s, \pi, St_{2n+1})$ of degree $2n+1$ for the group $GSp_{2n} \subset GL_{2n}$ over $F = \mathbb{Q}$ attached to the standard orthogonal representation of ${}^L\mathrm{GSp}_{2n}(\mathbb{C})$ and to a cuspidal irreducible representation $\pi = \pi_f$ coming from a holomorphic Siegel cusp eigenform f admits a p-adic analogue which was constructed using the *Rankin-Selberg method* in the p-ordinary case [PaLNM] for *even* n. This construction was extended by S.Böcherer and C.-G. Schmidt [Bo-Sch] to the general case of p-ordinary forms of arbitrary genus n and weight $k > n$, by using the *method of doubling of variables*. The critical values in the sense of Deligne [De79] of the L-function $L(s, \pi_f \otimes \chi, St_{2n+1})$ are $s \in \mathbb{Z}$ such that $1-k+n \leq s \leq k-n$ satisfying the following parity condition: $(-1)^s = \begin{cases} \chi(-1), & \text{if } s \geq 0 \\ -\chi(-1), & \text{if } s < 0 \end{cases}$

for a Dirichlet character $\chi \bmod p^N$. This description follow from the form of Γ-factor

$$L_\infty(s, \pi_f \otimes \chi, St_{2n+1}) = \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s + k - j)\Gamma(s + \delta), \tag{A.0.1}$$

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s), \Gamma_{\mathbb{R}}(s) = (\pi)^{-s/2}\Gamma(s/2), \delta = (1 - \chi(-1))/2.$$

In this case the algebraic numbers $L_{s,\chi} = \dfrac{L^*((s, \pi_f \otimes \chi, St_{2n+1})}{\langle f, f\rangle}$ can be interpolated to values of some Iwasawa-type series $g_{f,i}(\chi(1+p)(1+p)^k - 1)$ where $\langle f, f\rangle$ is the Petersson scalar product, i runs over residues $\bmod(p-1)$. In this case ${}^L GSp_{2n}(\mathbb{C}) = GSpin_{2n+1}$, the universal cover of the orthogonal group $GO_{2n+1}(\mathbb{C})$, $St_{2n+1} : GO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$.

In order to construct in general p-adic automorphic L-functions out of their complex critical special values one can successfully use p-adic integration along a (many variable) Eisenstein measure which was introduced by N.Katz [Ka78] and used by H.Hida [Hi91] in the case of $G = \mathrm{GL}_2$ over a totally real field F (i.e. for the elliptic modular forms and Hilbert modular forms). The application of such a measure to a given p-adic family of modular forms provides a general construction of p-adic L-functions of several variables. On the other hand, the evaluation of this measure at certain points gives another important source of p-adic L-functions [Ka78]. In the Siegel modular case the Eisenstein measure was constructed in [PaSE].

The goal of our work is to construct a p-adic version of the L-function $L(s, \pi_f, r_4)$ of degree 4 attached to a Siegel-Hilbert cusp egenform of degree 4 over a totally real field F, i.e. for the symplectic group

$$\mathrm{GSp}_4 = \left\{g \in \mathrm{GL}_4 \mid {}^t g J_4 g = \nu(g) J_4, \nu(g) \in \mathrm{GL}_1\right\},$$

over F where

$$J_4 = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$$

We use the Eisenstein measure and a p-adic analogue of the Petersson product for Λ-adic automorphic forms on GL_2 over a totally real field, see [Hi90, Hi94]. Instead of p-adic interpolation of critical values we try to imitate in the p-adic case a known complex analytic integral representation for $L(s, \pi_f, r_4)$. Main Theorem is given in Section 4.

A.1 Complex analytic L-functions for $GSp(4)$.

Let F be a global field of characteristic $\neq 2$, and V a four dimensional vector space over F endowed with a non-degenerate skew-symmetric form $\rho : V \times V \to F$,

$$G_\rho = \mathrm{GSp}_4 = \left\{g \in \mathrm{GL}(V) \mid \rho(gu, gv) = \nu_g \rho(u, v),\ \nu_g \in F^\times\right\},$$

the algebraic group of symplectic similitudes of ρ over F. Let $\pi = \otimes_v \pi_v$ be an irrreducible cuspidal automorphic representation of $G_\rho(\mathbb{A}_F)$ where v runs over all places of F, then according to Langlands' classification of irreducible supercuspidal representations π_v of $G_\rho(F_v)$ for almost all v π_v correspond to a semi-simple conjugacy class of a diagonal matrix

$$h_v = \mathrm{diag}\{\alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0\alpha_1\alpha_2\} \in {}^L G_\rho(\mathbb{C}) \overset{\sim}{\to} GSP_4(\mathbb{C}) \overset{r_4}{\to} GL_4(\mathbb{C})$$
$$(\alpha_j = \alpha_j(v), v \notin S, |S| < \infty).$$

The Andrianov L-function (or the *spinor L-function*) of π is then the following Euler product

$$L(s, \pi, r_4) = \prod_{v \notin S} \det\left(1_4 - r_4(h_v) \cdot Nv^{-s}\right)^{-1} \times \begin{pmatrix} \text{a finite Euler product} \\ \text{over } v \in S \end{pmatrix} \quad \text{(A.1.1)}$$

This L function plays an impotant role in arithmetic, in particular it is related to l-adic Galois representation on H^3 of the corresponding Siegel threefold [Tay], [Lau].

This L function was introduced by Andrianov [AndBud], [And74] in the classical fashion, for $F = \mathbb{Q}$, and for $\pi = \pi_f$ coming from a holomorphic Siegel cusp eigenform $f = \sum_\xi A_\xi q^\xi$ for the Siegel modular group $\Gamma_2 = Sp_4(\mathbb{Z})$ over the Siegel upper half plane of genus two

$$H_2 = \{z = {}^t z \in M_2(\mathbb{C}) \mid \mathrm{Im}\,(z) > 0\},$$

where ξ runs over the semi-group B_2 of semi-definite half integral symmetric 2×2-matrices ξ, $A_\xi \in \mathbb{C}$, so that $q^\xi = \exp(2\pi i \mathrm{Tr}(\xi z))$ form a multiplicative semi-group q^{B_2}. Consider the Hecke algebra $\mathcal{H} = \langle(\Gamma_2 g \Gamma_2)\rangle = \otimes_p \mathcal{H}_p$ generated by all double coset classes $(\Gamma_2 g \Gamma_2)$ with $g \in GSp_4(\mathbb{Q})$. Then we have that $\mathcal{H}_p = \mathbb{Q}[x_0^\pm, x_1^\pm, x_2^\pm]^{W_2}$ (W_2 the Weyl group) and one has a $\mathbb{Q}$-algebras homomorphism $\lambda_f : \mathcal{H} \to \mathbb{C}$ given by $f|X = \lambda_f(X)f$, $X \in \mathcal{H}$, and α_j are defined as $\lambda_f(x_j)$, $j = 0, 1, 2$. In the notation of Andrianov,

$$Z_f(s) = L(s - k + (3/2), \pi_f, r_4) = \prod_p \det\left(1_4 - h_p p^{k-(3/2)}\right)^{-1} p^{-s} \quad \text{(A.1.2)}$$

is called the *spinor L function* of f, and he proved that it coincides with a linear combination of the Dirichlet series $L(s, f, \xi_0) = \displaystyle\sum_{m=1}^{\infty} \frac{A_{m\xi_0}}{m^s}$ where $\xi_0 > 0$ is a positive definite matrix of a fixed discriminant $-\det \xi_0$. Starting from this identity, he obtained an integral representation for $Z_f(s)$ using the group $GL_{2,K}$ where $K = \mathbb{Q}(\sqrt{-\det \xi_0})$ an *imaginary quadratic field*. This integral representation implied an analytic continuation of $Z_f(s)$ to the whole complex plane and the functional equation of the type

$$\Psi_f(s) = \Gamma_\mathbb{C}(s)\Gamma_\mathbb{C}(s - k + 2)Z_f(s) = (-1)^k \Psi_f(2k - 2 - s). \quad \text{(A.1.3)}$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ is the standard Γ-factor. Its analytic properties were studied by A. N. Andrianov [And74] but still little is known about algebraic and arithmetic properties of critical values of this function; however, the general Deligne conjecture on critical values of L-functions predicts that algebraicity properties could exist only for $s = k - 1$ (see [Bo86, Fu-Sh, Ko-Ku] for evidences and discussions).

The work of A.N.Andrianov was extended by I.I.Piatetski-Shapiro [PShBud], [PshPac] to arbitrary F using a quadratic extension K/F and the following construction. Put

$$V = K^2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_j \in K, j = 1, 2 \right\}$$

then V may be viewed as a four dimensional F vector space, $\dim_F V = 4$, and define $\rho(x, y) = \mathrm{Tr}_{K/F}(x_1 y_2 - x_2 y_1)$. Let us consider the following F-algebraic group

$$G = \{g \in GL_{2,K} \mid \det g \in GL_{1,F}\}, \quad SL_2(K) \subset G(F) \subset GL_2(K) \tag{A.1.4}$$

then there is an imbedding of F-algebraic groups $i : G \hookrightarrow G_\rho$ because $x_1y_2 - x_2y_1 = \det(x, y)$ and $\det(gx, gy) = \det g \cdot \det(x, y)$, so that $\rho(gx, gy) = \det g \cdot \rho(x, y)$. Note that $SL_2(\mathbb{A}_K) \subset G(\mathbb{A}_F) \subset GL_2(\mathbb{A}_K)$ and $G(\mathbb{A}_F) \hookrightarrow G_\rho(\mathbb{A}_F) = GSp_4(\mathbb{A}_F)$. It turns out that there is an integral representation for $L(s, \pi, r)$ of the following type:

$$L(s, \pi, r) = \int_{G(F)C(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(i(g))E(g, s)dg := I_\pi(s) \tag{A.1.5}$$

where φ is an automorphic form on $G_\rho(\mathbb{A}_F) = GSp_4(\mathbb{A}_F)$ from the representation space of π, $C(\mathbb{A}_F)$ the center of $G(\mathbb{A}_F) \subset GL_2(\mathbb{A}_K)$, $E^\Phi(g, s)$ is a certain Eisenstein series on $G(\mathbb{A}_F) \subset GL_2(\mathbb{A}_K)$ attached to a Schwartz function $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ ([PshPac], §5).

A.2 Initial idea of a p-adic construction.

Let $p \geq 5$ be a prime number. We consider the case of two totally real fields $K \supset F$ and a representation π_f attached to a holomorphic Siegel-Hilbert cusp form $f(z) = \tilde{\varphi}$ of scalar weight $k = (k, \ldots, k)$ on the Siegel-Hilbert half plane

$$H_{2,F} = H_2 \times \cdots \times H_2 \;\; (n \text{ copies}); \tag{A.2.1}$$

in this case there is also a critical value $s = k - 1$ for L-functions of the type $L(s, \pi_f, \otimes\chi, r)$ where χ is a character of finite order of $\mathbb{A}_F^\times/F^\times$. According to general conjectures on motivic L-functions there should exist p-adic L-functions which interpolate p-adically their critical values, see [Co], [Co-PeRi], [PaIF]. However in our present construction instead of p-adic interpolation of their special values of the type $L(k-1, \pi_f \otimes \chi, r)$ we use directly a p-adic version of (A.1.5) using techniques of Λ-adic modular forms (see Section 3). We hope that the resulting p-adic L-function provide also the above p-adic interpolation.

A.3 Λ-adic modular forms.

Let us consider the Iwasawa algebra [Iw] $\Lambda = \mathbb{Z}_p[[T]] \cong \mathbb{Z}_p[[\Gamma]]$ as the completed group ring of the profinite group $\Gamma = 1 + p\mathbb{Z}_p = \langle 1+p \rangle \subset \mathbb{Z}_p^\times$. We shall view elements of its quotient field $\mathcal{L} = \mathrm{Quot}\Lambda$ as $\mathbb{C}_p$-meromorphic functions with a finite number of poles on the unit disc $U = \{t \in \mathbb{C}_p | \ |t|_p < 1\} \subset \mathbb{C}_p$. According to the theorem of Kubota-Leopoldt [Ku-Le], there exists a unique element $g(T) \in \mathcal{L}$ such that for all $k \geq 1$, $k \equiv 0 \bmod (p-1)$

$$g((1+p)^k - 1) = \zeta^*(1-k)$$

where $\zeta^*(1-k)$ denotes the special value at $s = 1-k$ of the Riemann zeta-function with a modified Euler p-factor: $\zeta^*(s) = (1-p^{-s})\zeta(s)$. One could also start from positive values $s = k, k \equiv 0 \bmod (p-1)$, and construct a p-adic zeta function $\zeta_{+,p}$ which interpolate $k \mapsto \zeta_+^*(k) = \dfrac{\Gamma(k)}{(2\pi i)^k}\zeta(k)(1-p^{k-1})$ (see [Colm98]) and satisfies the following "functional equation" $\zeta_{+,p}(s) = 2\zeta_p(1-s)$.

Definition A.1 (The Serre ring) *$\Lambda[[q]]$ is the ring of all formal q-expansions with coefficients in Λ:*

$$\Lambda[[q]] = \{f = \sum_{n=0}^{\infty} a_n(T)q^n \mid a_n(T) \in \Lambda\};$$

Definition A.2 *The Λ-module $M(\Lambda) \subset \Lambda[[q]]$ of* Λ-adic modular forms *(of some fixed level N, $(N,p) = 1$ is generated by all $f = \sum_{n=0}^{\infty} a_n(T)q^n \in \Lambda[[q]]$ such that for each $k \geq 5$, $k \gg 0$ the specialisation*

$$f_k = f|_{T=(1+p)^k-1} \in \mathbb{Z}_p[[q]]$$

is a classical modular form of weight k and level Np. In more precise terms f is given by a p-adic measure μ_f on $\mathbb{Z}_p^\times$ with values in $\mathbb{Z}_p[[q]]$ such that the integrals

$$\int_{\mathbb{Z}_p^\times} x_p^k \mu_f = f_k \tag{A.3.1}$$

are classical modular forms.

Example A.3 (The Λ-adic Eisenstein series) *$f \in M(\Lambda)$ (of level $N = 1$) is defined by*

$$f_k = \frac{\zeta^*(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}^*(n)q^n, \quad \sigma_{k-1}^*(n) = \sum_{d|n, p\nmid d} d^{k-1}. \tag{A.3.2}$$

Example A.4 (Hida's families) *f are elements of*

$$S^{\mathrm{ord}}(\Lambda) = eS(\Lambda), \quad e = \lim_{n\to\infty} U_p^{n!}$$

($U_p(\sum_{n\geq 0} a_n q^n) = \sum_{n \geq 0} a_{pn}q^n$ is the Atkin U-operator), $S(\Lambda)$ is the Λ-submodule of Λ-adic cusp forms.

The Hilbert modular case.

According to the classical theorem of Klingen [Kli], for a totally real field K and for $k \geq 1$ the special values $\zeta_K(1-k)$ are rational numbers where $\zeta_K(s)$ is the Dedekind zeta function of K.
The Deligne-Ribet p-adic zeta function [De-Ri] interpolates p-adically these special values as an element $g_K \in \mathcal{L}$: for all positive integers $k \equiv 0(\mathrm{mod}(p-1))$, $g_K((1+p)^k-1) = \zeta_K^*(1-k)$, where $\zeta_K^*(s) = \prod_{\mathfrak{p}\ |p}(1-\mathcal{N}\mathfrak{p}^{-s})\zeta_K(s)$ is the Dedekind zeta function of K with all the $\mathfrak{p}$-factors over p removed from its Euler product. The function $\zeta_{K,p}(s) = g_K((1+p)^{1-s}-1)$ is analytic for all $s \in \mathbb{Z}_p\backslash 1$ with values in $\mathbb{Q}_p$ and it is called the *Deligne-Ribet p-adic zeta function*. It has the following properties: $\zeta_{K,p}(1-k) = \zeta_K^*(1-k)$ for all positive integers $k \equiv 0(\mathrm{mod}(p-1))$, and its residue $\mathrm{Res}_{s=1}\zeta_{K,p}(s)$ was computed by Colmez [Colm88]: $\mathrm{Res}_{s=1}\zeta_{K,p} = \dfrac{2^d h_K R_p E_p(1)}{w_K\sqrt{D_K}}$ where $d = [K:\mathbb{Q}]$, $E_p(s) = \prod_{\mathfrak{p}\ |p}(1-\mathcal{N}\mathfrak{p}^{-s})$, R_p the p-adic regulator of K (which does not vanish according to the *Leopoldt conjecture*).
A Λ-adic Hilbert modular form could be defined as a formal Fourier expansion

$$f = \sum_{0 \leqq \eta \in L_K} a_\eta q^\eta \in \Lambda[[q^{L_F}]] \quad (L_K \subset K \text{ a lattice})$$

(η runs over totally positive elements or 0) whose appropriate specialisations are classical Hilbert modular form. When $h_K > 1$ one needs to consider collections of such series $\{f_\lambda\}$ $(\lambda = 1, 2, \ldots, h_K)$ in order to be able to use the action of the Hecke algebra. Λ-adic Hilbert modular forms were used by Wiles in his proof of the Iwasawa conjecture over totally real fields (see [Wi90] where a precise definition of a Λ-adic Hilbert modular form is contained in Section 3). It is required that for all appropriate sufficiently large k the specialization $f_k = f|_{T=(1+p)^k-1}$ is the Fourier expansion of a classical Hilbert modular form. As over $\mathbb{Q}$, the first natural example of a Λ-adic Hilbert modular form is given by a Λ-adic Eisenstein series (more precisely, this series is given by the Katz-Hilbert-Eisenstein measure, see [Ka78]). Also, Hida's theory could be extended to the Hilbert modular case and even to the general case of cohomological modular forms on $GL_{2,K}$ over an arbitrary number field K (see [Hi94]).

The Siegel-Hilbert modular case.

A Λ-adic Siegel-Hilbert modular form could be defined as a formal Fourier expansion

$$f = \sum_{\xi \in L_{2,F}} A_\xi q^\xi \in \Lambda[[q^{L_{2,F}}]] \quad (L_{2,F} \subset M_{2,F})$$

($L_{2,F}$ is the semi-group of all symmetric totally non-negative matrices ξ in a sublattice of $M_{2,F}$) whose appropriate specialisations $f_k = f|_{T=(1+p)^k-1}$ are classical Siegel-Hilbert modular form. The first example of a Λ-adic Siegel-Hilbert modular form is given by an Eisenstein series (for $F = \mathbb{Q}$ these series are described in

[PaSE]). It seems that Hida's theory also could be extended to the Siegel-Hilbert modular case [Hi98], [Til-U],[Til].

A.4 p-adic L-functions.

Recall that we consider the case of two totally real fields $K \supset F$ and an irreducible representation $\pi = \pi_f$ attached to a holomorphic Siegel-Hilbert cusp form $f(z) = \tilde{\varphi}$ of scalar weight $k = (k, \dots, k)$ on the Siegel-Hilbert half plane

$$H_{2,F} = H_2 \times \cdots \times H_2 \quad (n \text{ copies});$$

Then we rewrite the integral representation (1.5) in the form of the Petersson scalar product over $K = F(\sqrt{D})$:

$$I_\pi(1/2) = \langle \widetilde{i^* \tilde{\varphi}}, \tilde{E}(s, \mu) \rangle_K \tag{4.1}$$

where i denotes both the imbedding $i : G \hookrightarrow G_\rho$ and the corresponding modular imbedding

$$i : H_F \times H_F \to H_{2,F}, \quad H_F = H \times \cdots \times H \;; H_{2,F} = H_2 \times \cdots \times H_2 \quad (n \text{ copies}); \tag{4.2}$$

If we write this imbedding in coordinates it takes the form

$$(z_1, z_2) \mapsto Z(z_1, z_2) = C \mathrm{diag}\{z_1, z_2\}^t C \ (z_1, z_2 \in H_F)$$

(see [Shi78], [Wi90], p. 521), where we could take $C = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1/\sqrt{D} & -1/\sqrt{D} \end{pmatrix} \in M_2(F)$ so that $i^* \tilde{\varphi} = \tilde{\varphi} \circ i$ is a holomorphic Hilbert modular form with an explicitely given Fourier expansion. If $\tilde{\varphi} = \sum_{\xi \in L_{2,F}} A_\xi q^\xi$ then $i^* \tilde{\varphi} = \sum_{\eta \in L_F} a_\eta q^\eta$ where each Fourier coefficient a_η is a finite sum of certain A_ξ:

$$a_\eta = \sum_{\xi : \eta = \frac{1}{4}(\xi_{11} + \xi_{22} + 2\xi_{12}/\sqrt{D})} A_\xi,$$

so that the map $\tilde{\varphi} \mapsto i^* \tilde{\varphi}$ could be defined in terms of their formal q-expansion. For the Λ-adic construction let us take a Λ-adic Siegel-Hilbert cusp form $\tilde{\varphi}$ on $GSp_{4,F}$ then $i^* \tilde{\varphi}$ is a Λ-adic Hilbert modular form over K which is explicitely described as a formal Fourier expansion. Now let us take G to be the Λ-adic Hilbert-Eisenstein series for $GL_{2,K}$. In order to define the Petersson product

$$\langle \widetilde{i^* \varphi}, G \rangle_K \tag{4.3}$$

we use the Eisenstein projection $1_{\mathrm{Eis}}(i^* \tilde{\varphi})$ (the projection in the $\mathcal{L}$-vector space $M(\mathcal{L})$ to the (finite-dimensional) $\mathcal{L}$-subspace $Eis_K(\mathcal{L})$ of Hilbert-Eisenstein series with an explicitely given base coming from the Katz-Hilbert-Eisenstein p-adic measure). The projection $1_{\mathrm{Eis}}(i^* \tilde{\varphi})$ could be explicitely computed using the Fourier expansions of $i^* \tilde{\varphi}$ and of the Fourier expansions of a $\mathcal{L}$-basis of $Eis_K(\mathcal{L})$.

$$\langle \widetilde{i^* \varphi}, G \rangle_K = \langle 1_{\mathrm{Eis}}(i^* \tilde{\varphi}), G \rangle_K.$$

Then we are reduced to the case of $\langle G_1, G_2\rangle_K$, where G_1 and G_2 are two normalized Hilbert-Eisenstein series, and in order to define their Petersson product we use the method of Rankin-Selberg.

Let us recall a classical formula

$$(f,g) = \frac{\pi}{3}\frac{\Gamma(k)}{(4\pi)^k}\mathrm{Res}_{s=k}L_{f,g}(s)$$

for the Petersson product $(f,g) = \displaystyle\int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k-2}\,dx\,dy$ (see [Ra39, Za81]) where $L_{f,g}(s) = \sum_{n=1}^{\infty} a_n\overline{b}_n n^{-s}$ denotes the Rankin L-function of two holomorphic modular forms of weight k on $SL_2(\mathbb{Z})$, with at least one of them a cusp form (i.e. $a_0b_0 = 0$): $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$ and $g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi inz}$. This equality makes it possible to define the Petersson scalar product (a *renormalized value*) (G_k, G_k) where $G_k = -\dfrac{B_k}{2k} + \displaystyle\sum_{n=1}^{\infty}\sigma_{k-1}(n)$ $(k \geq 4, k \text{ even})$. We have [Za81, p.435]:

$$L_{G_k,G_k}(s) = \sum_{n=1}^{\infty}\sigma_{k-1}(n)\sigma_{k-1}(n)n^{-s} = \frac{\zeta(s)\zeta(s-k+1)^2\zeta(s-2k+2)}{\zeta(2s-2k+2)}$$

which implies

$$\begin{aligned}(G_k, G_k) &= (-1)^{k/2-1}\frac{\Gamma(k)\Gamma(k-1)}{2^{3k-3}\pi^{2k-1}}\zeta(k)\zeta(k-1)\\ &= i^{3k-3}2^{2-k}\frac{\Gamma(k)}{(2\pi i)^k}\zeta(k)\frac{\Gamma(k-1)}{(2\pi i)^{k-1}}\zeta(k-1).\end{aligned}$$

We see that if G_1, G_2 were two cusp forms of weight k their Petersson product would essentially coincide with a normalized residue of the Rankin zeta function $L_{G_1,G_2}(s)$ at $s = k$. In the case of normalised Eisenstein series the Rankin zeta function $L_{G_1,G_2}(s)$ is explicitely evaluated via Rankin's lemma as a product of abelian Dirichlet L-functions. Let now $G_1 = \{G_{1,k}\}$, $G_{2,k} = \{G_{2,k}\}$ denote two p-adic families of Hilbert Eisenstein series. We may define the $I_{G_1,G_2} = \langle G_1, G_2\rangle_K$ as an element of $\mathcal{L}$ such that for all $k \gg 0$

$$(G_1, G_2) = \mathrm{Res}_{s=k}L_{G_{1,k},G_{2,k}}(s) \in \mathbb{Q}_p \ (s \in \mathbb{Z}_p)$$

in a similar way as in [Za81] and [Ko-Za] as the normalised p-adic residue of the p-adic Rankin convolution $L_{G_1,G_2}(s)$ (which is defined in terms of the corresponding Deligne-Ribet p-adic zeta function).

Main Theorem A.5 *Let $\tilde{\varphi}$ be a Λ-adic Siegel-Hilbert modular eigenform then there exists a canonically defined element*

$$I_{\tilde{\varphi},p} = \langle\tilde{i}^*\varphi, G\rangle_K \in \mathcal{L}$$

$i^*\tilde{\varphi}$ the Λ-adic pullback of φ, $i^*\tilde{\varphi}$ is a Λ-adic Hilbert modular form over K explicitely described by its Fourier expansion, G is a certain Λ-adic Hilbert-Eisenstein series, such that the function $I_{\tilde{\varphi},p}$ gives a p-adic interpolation of the residue of the normalized p-adic Rankin L function $L^*_{i^*\tilde{\varphi}_k,G_k}(s)$ (at $s = k$), the scalar weight of a specialisation $\tilde{\varphi}_k$):

$$I_{\tilde{\varphi},p}|_{T=(p+1)^k-1} = \mathrm{Res}_{s=k} L^*_{i^*\tilde{\varphi}_k,G_k}(s) \ (s \in \mathbb{Z}_p)$$

A.5 p-adic families of automorphic representations.

We use the occasion to discuss here the following general definition of a *p-adic family of automorphic representations* (or of a *Λ-adic automorphic form*). We shall view the Iwasawa algebra Λ as the algebra $\mathrm{Meas}(\mathbb{Z}_p, \mathbb{Z}_p)$ of all $\mathbb{Z}_p$-measures on $\mathbb{Z}_p$ (with the additive convolution as a multiplication). Let $V_{\mathbb{Q}} \subset C(G(\mathbb{A}_F))$ be a certain $\mathbb{Q}$-vector space of (complex-valued) continuous functions on the adelic group $G(\mathbb{A}_F)$ over a number field F. We suppose that $V_{\mathbb{Q}}$ has an integral structure $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$ so that $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$. Put $V_p = V_{\mathbb{Z}} \hat{\otimes} \mathbb{Z}_p$ (the completed tensor product). Define $D_p(V_p) = \mathrm{Meas}(\mathbb{Z}_p, V_p)$ (as a module over $\Lambda = D_p(\mathbb{Z}_p)$).

Definition A.6 *A p-adic family of automorphic representations on G is a p-adic measure $\varphi \in D_p(V_p)$ such that for almost all positive integers k we have that the integral $\int_{\mathbb{Z}_p} x^k \varphi = \varphi_k \in V_p$ belongs to $V_{\mathbb{Z}}$ and the function φ_k generates an automorphic representation π_k of $G(\mathbb{A}_F)$. We call such φ a Λ-adic automorphic form on $G(\mathbb{A}_F)$.*

Let $AF_G(\Lambda)$ denote the Λ-module generated by such elements φ. An element φ is called an eigenform if the representations π_k are all irreducible.

A natural example of such a vector space V for the group GL_2 over $\mathbb{Q}$ comes from holomorphic functions $f = \sum_{n=0}^{\infty} a_n \exp(2\pi i n z)$ having rational Fourier coefficients $a_n \in \mathbb{Q}$ with bounded denominators, i.e. for which there exists a positive integer $N = N(f)$ such that $N a_n \in \mathbb{Z}$. However there are other ways to attach such a vector space V to G by considering cohomology groups of the corresponding locally-symmetric spaces and automorphic forms φ on $G(\mathbb{A}_F)$ represented by rational cohomology classes ([Ko-Za]). Put $AF_G(\mathcal{L}) = AF_G(\Lambda) \otimes \mathcal{L}$. We hope that one could find in this way a general construction of p-adic automorphic L functions $L_{\pi,r,p}$ as certain $\mathcal{L}$-linear forms $l = l_{G,r}$ on the $\mathcal{L}$-vector space $AF_G(\mathcal{L})$. Such a linear form should play a role of an integral representation for p-adic L-functions: $L_{\pi_k,r,p} = l_{G,r}(\varphi)|_{T=(1+p)^k-1}$. A natural example of such a linear form comes from the Λ-adic Petersson product of Hida which provides a construction of p-adic L-functions for $GL_2 \times GL_2$ [Hi91].

On the other hand, there exist nice constructions of p-adic families of Galois representations attached to automorphic forms (Λ-adic Galois representations, see [Hi86], [Til-U]) which played an important role in the work of Wiles [Wi95]. It would be interesting to formulate a general Λ-adic Langlands conjecture relating Λ-adic automorphic forms and Λ-adic Galois representations.

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