Infinitesimal extensions of rank two vector bundles on small codimensional submanifolds in $\mathbb{P}^N$

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Notation and Setup

Let X be a submanifold of dimension n of a complex projective manifold P of dimension N, with $n < N$. For every $i \geq 0$ denote by $X(i)$ the i-th infinitesimal neighborhood of X in P, i.e. the subscheme of P defined by the sheaf of ideals $\mathcal{I}_X^{i+1}$, where $\mathcal{I}_X$ is the sheaf of ideals of X in $\mathcal{O}_P$. Note that $X(0) = X$. Fix an $i \geq 0$; if E be a vector bundle of rank r on $X(i)$, a natural problem is to give criteria for the extendability of E to the next infinitesimal neighborhood $X(i+1)$. Under the above hypotheses and notation we have the following general result of Grothendieck [3, SGA1, 1960–1961]:

Theorem (Grothendieck). Under the above setup, assume that

$$H^2(X, E \otimes E^\vee \otimes \mathbf{S}^{i+1}(N^\vee_{X|P})) = 0, \qquad (1)$$

where $\forall\ j \geq 1$, $\mathbf{S}^j(N^\vee_{X|P})) = \mathcal{I}^j_X/\mathcal{I}^{j+1}_X$ is the j-th symmetric power of the conormal bundle $N^\vee_{X|P} = \mathcal{I}_X/\mathcal{I}^2_X$ of X in P. Then E can be extended to a vector bundle $\mathcal{E}$ on $X(i+1)$. If moreover $H^1(X, E \otimes E^\vee \otimes \mathbf{S}^{i+1}(N^\vee_{X|P})) = 0$ then such an extension is also unique up to isomorphism.

If in Grothendieck's theorem above X is a curve and E a vector bundle on X then the vanishing (1) is automatically fulfilled, so that E can be extended to a vector bundle $\mathcal{E}_i$ on $X(i)$ for every $i \geq 1$. Note also that the vanishing (1) is only a sufficient condition for the extendability of the vector bundle E in Grothendieck's theorem above.

We proceed further with some historical motivation by mentioning the following very nice result:

Theorem (Griffiths-Harris 1983, Harris-Hulek 1983, Ellingsrud-Gruson-Peskine-Strømme 1985). Let X be a smooth projective complex surface embedded in $\mathbb{P}^n$ $(n \geq 3)$ as a complete intersection. Let Y be a smooth connected curve in X such that the exact sequence of normal bundles

$$0 \longrightarrow N_{Y|X} \longrightarrow N_{Y|\mathbb{P}^n} \longrightarrow N_{X|\mathbb{P}^n}|Y \longrightarrow 0$$

splits. Then $\exists$ a hypersurface H of $\mathbb{P}^n$ such that $Y = X \cap H$ (scheme-theoretically).

The proofs of Griffiths-Harris and Harris-Hulek make use of the Variation of the Hodge structures. Instead, the (very elegant) proof of Ellingsrud-Gruson-Peskine-Strømme (Invent. Math. 80 (1985), 181–184) is of a completely different nature. The crucial point in their proof is to extend Y to an effective Cartier divisor Y' on the first infinitesimal neighborhood of $X(1)$ of X in $\mathbb{P}^n$. As soon as they did that, some standard argument shows that Y' can be extended to a hypersurface

H of $\mathbb{P}^n$ such that $X \cap H = Y$ (scheme-theoretically).

Now let (more generally) X be a submanifold of $\mathbb{P}^N$ of dimension n and E a vector bundle rank r on X, with $1 \leq r \leq n-1$. Then consider the following condition regarding the triple $(\mathbb{P}^N, X, E)$:

$(*)$ There exists an integer $l_0 > 0$ such that for every $l \geq l_0$ there exists a section $s = s_l \in H^0(E(l))$ whose zero locus $Y := Z(s)$ is an r-codimensional submanifold of X such that the canonical exact sequence of normal bundles

$$0 \to N_{Y|X} \to N_{Y|\mathbb{P}^N} \to N_{X|\mathbb{P}^N}|Y \to 0 \tag{2}$$

splits. Note that $N_{Y|X} \cong E(l)|Y$.

First, we have the following general result:

Proposition 1. Under the above notation, let E be a vector bundle rank r, with $1 \leq r \leq n-1$, on an n-dimensional submanifold $X \subset \mathbb{P}^N$. If there exists a vector bundle $\mathcal{E}$ on $X(1)$ which extends E, there exists an integer $l_0 > 0$ such that for every $l \geq l_0$ and for every section $s \in H^0(E(l))$ whose zero locus Y is smooth and r-codimensional in X, the exact sequence (2) splits. In particular, the condition $(*)$ above holds true.

Proof: Consider the exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|X = E \longrightarrow 0,$$

where $F := \mathrm{Ker}(\mathcal{E} \longrightarrow E)$. By a well-known theorem of Serre, $H^1(X(1), F(l)) = 0$ for $l \gg 0$, the map $H^0(X(1), \mathcal{E}(l)) \longrightarrow H^0(X, E(l))$ is surjective for $l \gg 0$. Moreover, enlarging l enough, we can also assume that the vector bundle $E(l)$ is ample and generated by its global sections. Let $s \in H^0(X, E(l))$ be a general global section;

then by result of Bertini-Serre-Sommese, its zero locus $Y := Z(s)$ is smooth, connected and $(n-r)$-dimensional. As s lifts to a section $s' \in H^0(X(1), \mathcal{E}(l))$, let $Y' := Z(s')$ its zero locus of s' it follows that $Y' \cap X = Y$ (scheme-theoretic intersection in $X(1)$). Moreover, Y' is a local complete intersection of codimension r in $X(1)$ (because it is easy to check that $X(1)$ is locally Cohen-Macaulay). Then by the Key Lemma below the exact sequence (2) splits. $\square$

One of the main idea of the proof of Theorem 1 below is the following infinitesimal interpretation of the splitting of the exact sequence (2):

Key Lemma (Ellingsrud-Gruson-Peskine-Strømme, if $\operatorname{codim}_X Y = 1$, and by my former Ph.D. student F. Repetto, if $\operatorname{codim}_X Y > 1$). Let P, X and Y be three smooth projective varieties such that $Y \subsetneq X \subsetneq P$, with $\dim Y \geq 1$. Then the canonical exact sequence of normal bundles

$$0 \longrightarrow N_{Y|X} \longrightarrow N_{Y|P} \longrightarrow N_{X|P}|Y \longrightarrow 0$$

splits iff $\exists$ a closed subscheme Y' of the first infinitesimal neighborhood $X(1)$ of X

in P which is a local complete intersection in $X(1)$, with $\mathrm{codim}_{X(1)} Y' = \mathrm{codim}_X Y$ and $Y' \cap X = Y$ (scheme-theoretically in $X(1)$).

Then our first main result is the following partial converse of Proposition 1:

Theorem 1. Let $X \subset \mathbb{P}^N$ be a smooth n-dimensional subvariety, with $n \geq \frac{N+3}{2}$ and $n \geq 4$. Let E be a vector bundle of rank 2 on X which satisfies the condition $(*)$ above. Then E can be extended uniquely (up to isomorphism) to a vector bundle $\mathcal{E}$ of rank 2 on the first infinitesimal neighborhood $X(1)$ of X in $\mathbb{P}^N$.

The main ingredients in the proof of Theorem 1 are: the Barth-Lefschetz theorems for small codimensional submanifolds of $\mathbb{P}^N$ and for the zero loci of global sections of an ample vector bundle, the Kodaira-Le Potier vanishing theorem, the above Key Lemma, and the following generalized form of a result due to Serre and Hartshorne (which allows one to construct the desired extension $\mathcal{E}$ of E, and which may also have an interest in itself):

Theorem 2. (A general Serre–Hartshorne correspondence) Let $\mathcal{X}$ be an irreducible (not necessarily reduced) algebraic scheme over an field k, and $Y' \subset \mathcal{X}$ a local complete intersection subscheme of $\mathcal{X}$ of codimension 2 such that $\det(N_{Y'|\mathcal{X}})$ extends to a line bundle L on $\mathcal{X}$ such that $H^2(\mathcal{X}, L^\vee) = 0$. Then there exists a vector bundle $\mathcal{E}$ of rank two on $\mathcal{X}$ and a section $t \in H^0(\mathcal{X}, \mathcal{E})$ such that $\det(\mathcal{E}) = L$, and $Z(t) = Y'$, i.e. the zero locus of t is Y'. If moreover $H^1(\mathcal{X}, L^\vee) = 0$, then the pair $(\mathcal{E}, t)$ is unique up to isomorphism.

Note. The proof of Theorem 2 was worked out together with E. Arrondo (Madrid).

Examples of submanifolds X of P^N with $\dim X = \frac{N+3}{2}$. For every $m \geq 3$ consider the Plücker embedding $i_m \colon \mathbb{G}(1,m) \hookrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ of the Grassmann variety of lines in $\mathbb{P}^m$, and set $X'_m := i_m(\mathbb{G}(1,m))$. As is well-known, X'_m is a 4-defective subvariety of $\mathbb{P}^{\binom{m+1}{2}-1}$, meaning that there is a linear projection $\pi_{L_m} \colon \mathbb{P}^{\binom{m+1}{2}-1} \dashrightarrow \mathbb{P}^{4m-7}$ of center a linear subspace L_m of dimension $\binom{m+1}{2} - (4m-7) - 2$ which does not intersect X'_m such that the restriction $\pi_{L_m}|X'_m \colon X'_m \longrightarrow \pi_{L_m}(X'_m)$

is a biregular isomorphism (see J. Harris, Algebraic Geometry: A first Course, Graduate Texts in Math. 133, Springer-Verlag, 1992, Exercise 11.27, page 145). Then we get the closed embedding

$$\pi_{L_m} \circ i_m \colon \mathbb{G}(1,m) \hookrightarrow \mathbb{P}^{4m-7}. \tag{3}$$

Set $X_m = \pi_L(X'_m)$. Note that for $m = 3$ and $m = 4$ the embeddins (3) are just the Plücker embeddings $\mathbb{G}(1,3) \hookrightarrow \mathbb{P}^5$, respectively $\mathbb{G}(1,4) \hookrightarrow \mathbb{P}^9$.

It follows that if we set $N := 4m - 7$, X_m is an n-dimensional closed submanifold of $\mathbb{P}^N$ such that $\dim X_m = \frac{N+3}{2}$.

Note also that $\dim X_m \geq 4$ iff $m \geq 3$. In particular, Theorem 1 applies to every rank two vector bundle on X_m, with $m \geq 3$, which satisfies condition $(*)$ above.

Now consider the Grassmann variety $\mathbb{G}(k,m)$ of k-dimensional linear subspaces of $\mathbb{P}^m$, with $m \geq 3$ and $1 \leq k \leq m-2$ (hence $\mathbb{G}(k,m)$ is not a projective space). Then $\dim \mathbb{G}(k,m) = (k+1)(m-k)$. Let E denote the universal quotient bundle of $\mathcal{O}_{\mathbb{G}(k,m)}^{\oplus m+1}$ (of rank $m-k$). Fix an arbitrary projective embedding $\mathbb{G}(k,m) \hookrightarrow \mathbb{P}^N$ (for example, the Plücker embedding $i\colon X \hookrightarrow \mathbb{P}^{\binom{m+1}{k+1}-1}$), and denote by X the image of $\mathbb{G}(k,m)$ in $\mathbb{P}^N$.

Now we come up to our second main result:

Theorem 3. Under the above notation and hypotheses the universal quotient vector bundle E of $X \cong \mathbb{G}(k,m)$ (with $1 \leq k \leq m-2$) cannot be extended to a vector bundle on the first infinitesimal neighborhood $X(1)$ of X in $\mathbb{P}^N$.

Sketch of the proof. Assume by way of contradiction that there would exist a vector bundle $\mathcal{E}$ on $X(1)$ such that $\mathcal{E}|_X \cong E$.

Tensoring by $\mathcal{E}$ the exact sequence

$$0 \longrightarrow N^{\vee}_{X|\mathbb{P}^N} \longrightarrow \mathcal{O}_{X(1)} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and taking into account that $\mathcal{E} \otimes N^{\vee}_{X|\mathbb{P}^N} \cong E \otimes N^{\vee}_{X|\mathbb{P}^N}$ we get the exact sequence

$$0 \longrightarrow E \otimes N^{\vee}_{X|\mathbb{P}^N} \longrightarrow \mathcal{E} \longrightarrow E \longrightarrow 0. \quad (4)$$

Now assume for the moment that the following condition holds true

$$H^1(X, E \otimes N^{\vee}_{X|\mathbb{P}^N}) = 0. \quad (5)$$

Then (4) and (5) imply that the restriction map $H^0(X(1), \mathcal{E}) \longrightarrow H^0(X, E)$ is surjective. Considering the canonical surjection $\varphi\colon \mathcal{O}_X^{\oplus(m+1)} \twoheadrightarrow E$ given by $(s_0, s_1, \dots, s_m) \in H^0(X, E)^{\oplus(m+1)}$, it follows that there exists an $(m+1)$-uple $(s'_0, s'_1, \dots, s'_m) \in H^0(X(1), \mathcal{E})^{\oplus(m+1)}$ such that $s'_i|X = s_i$, $i = 0, 1, \dots, m$. Since φ is surjective, the sections $s_0, s_1, \dots, s_m$ generate E, hence by Nakayama's Lemma the sections $s'_0, s'_1, \dots, s'_m$ generate $\mathcal{E}$. In other words, the surjection φ lifts to a surjection $\varphi' \colon \mathcal{O}_{X(1)}^{\oplus(m+1)} \twoheadrightarrow \mathcal{E}$. Then by the universal property of the Grassmann variety $X = \mathbb{G}(k,m)$ there exists a morphism of schemes $\pi \colon X(1) \longrightarrow X$ such that $\pi^*(E) = \mathcal{E}$. Since $\mathcal{E}|_X = E$ it follows that π is a retraction of the canonical embedding $X \hookrightarrow X(1)$. By a well known result of Mustata–Popa, this latter fact is equivalent with the splitting of the canonical tangent exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^N}|_X \longrightarrow N_{X|\mathbb{P}^N} \longrightarrow 0.$$

Then by a result of Van de Ven [9] the splitting of the above sequence implies implies that X is a linear subspace of $\mathbb{P}^N$, which is impossible because X is not a projective space.

To prove (5) we first claim that (5) is equivalent to the following vanishing:

$$H^0(E \otimes F) = 0, \tag{6}$$

where F is is defined in the following commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N^{\vee}_{X|\mathbb{P}^N} \arrow[r] \arrow[d, "\mathrm{id}"'] & \Omega^1_{\mathbb{P}^N}|X \arrow[r] \arrow[d] & \Omega^1_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N^{\vee}_{X|\mathbb{P}^N} \arrow[r, "\varphi"] & \mathcal{O}_X(-1)^{\oplus(N+1)} \arrow[r] \arrow[d] & F := \mathrm{Coker}(\varphi) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_X \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{O}_X \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} .
$$

The first row in this diagram is the cotangent sequence of X in $\mathbb{P}^N$ and the second column is the Euler sequence of $\mathbb{P}^N$ restricted to X. Note that the sheaf F coincides with $\mathcal{P}^1(\mathcal{O}_X(1))(-1)$, where $\mathcal{P}^1(\mathcal{O}_X(1))$ is the sheaf of first-order principal parts of $\mathcal{O}_X(1)$

Tensoring this diagram by E we get the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E \otimes N^{\vee}_{X|\mathbb{P}^N} & \longrightarrow & E \otimes \Omega^1_{\mathbb{P}^N}|X & \longrightarrow & E \otimes \Omega^1_X & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E \otimes N^{\vee}_{X|\mathbb{P}^N} & \longrightarrow & E(-1)^{\oplus(N+1)} & \longrightarrow & E \otimes F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E & \xrightarrow{\mathrm{id}} & E & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{7}
$$

The second row of this diagram yields the cohomology sequence

$$H^0(X, E(-1)^{\oplus(N+1)}) \longrightarrow H^0(X, E \otimes F) \longrightarrow$$

$$\longrightarrow H^1(E \otimes N^{\vee}_{X|\mathbb{P}^N}) \longrightarrow H^1(E(-1)^{\oplus(N+1)}).$$

By [2, Corollary (4.11) and Theorem (4.17)] (whose proofs are based on some vanishing results for flag manifolds of Kempf) we have $H^i(E(-1)^{\oplus(N+1)}) = 0$ for $i = 0, 1$ (recall that $\mathrm{Pic}(X) = \mathbb{Z}$). Thus the canonical map $\delta\colon H^0(E \otimes F) \longrightarrow H^1(X, E \otimes N^{\vee}_{X|\mathbb{P}^N})$ is an isomorphism, which proves the claim.

Therefore it will be sufficient to prove (6). But, as Giorgio Ottaviani kindly explained to me, (6) is a special case of a general result of Ottaviani-Rubei (Duke Math. J. 132, (3) (2006), Theorem 6.11). Indeed, considering the coboundary map

$$\delta' \colon H^0(E) \longrightarrow H^1(E \otimes \Omega^1_X)$$

associated to the last column of diagram (7) as a quiver, it follows $\delta' \neq 0$. Since $H^0(E)$ is the standard la representation (and hence irreducible), it follows that $H^0(E \otimes F) = 0$. Note that the fact that $H^0(E)$ is irreducible was proved directly in J. Wehler, Math. Ann. 268 (1984), 519–532. $\square$

HAPPY BIRTHDAY FYODOR!

and THANK YOU!

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