On some special actions of the symmetric group S4 on K3 surfaces

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• $K3$ **surface** X:

simply connected smooth compact complex surface with nowere vanishing holomorphic 2-form ω:

$$H^0(X, \Omega_X^2) = \mathbb{C}\omega, \qquad \omega(x) \neq 0 \quad \forall x \in X.$$

Assumption: symmetric group $\mathbb{S}_4 \subset Aut(X)$ s.t.

(1) the alternating group $\mathbb{A}_4 \subset \mathbb{S}_4$ acts symplectically on X, i.e.,

$$g^*(\omega) = \omega \text{ for } \forall g \in \mathbb{A}_4;$$

(2) $\exists$ equivariant bi-rational contraction $\bar{c} : X \to \overline{X}$ to a $K3$ surface $\overline{X}$ with ADE-singularities s.t. $\overline{X}/\mathbb{S}_4 \simeq \mathbb{P}^2$.

Denote by $\mathcal{Q} \subset \mathbb{P}^{14}$ the variety of rational quartics $C \subset \mathbb{P}^2$, $\deg C = 4$.

C.T.C. Wall ('95): The singularities of rational quartics:

$$3A_1,\ \ D_4,\ \ A_1+A_3,\ \ A_5,\ \ 2A_1+A_2,\ \ D_5,\ \ A_2+A_3,\ \ A_1+2A_2,\ \ 3A_2;$$

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$$A_1+A_4,\ \ A_2+A_4,\ \ A_6,\ \ E_6.$$

Agreement. *Simple inflection (resp., 2-tuple inflection) points of quartics are considered as singular points; their singularity type is denoted by* F_1 *(resp.,* F_2*).*

$\mathcal{Q}$ has a natural equisingular stratification with the strata determined by the collections S of singularity types, $\mathcal{Q} = \bigsqcup_S \mathcal{C}(S)$.

Quasi-equisingular stratification:

$$\mathcal{Q} = \bigsqcup_{\overline{S}} \mathcal{C}(\overline{S}),$$

where

$$\overline{3A_1 + aF_1 + bF_2} = \overline{D_4 + aF_1 + bF_2},$$

$$\overline{A_1 + A_3 + aF_1 + bF_2} = \overline{A_5 + aF_1 + bF_2},$$

and $\overline{S} = S$ in all other cases and

$$\mathcal{C}(\overline{3A_1 + aF_1 + bF_2}) = \mathcal{C}(3A_1 + aF_1 + bF_2) \cup \mathcal{C}(D_4 + aF_1 + bF_2);$$

$$\mathcal{C}(\overline{A_1 + A_3 + aF_1 + bF_2}) = \mathcal{C}(A_1 + A_3 + aF_1 + bF_2) \cup \mathcal{C}(A_5 + aF_1 + bF_2).$$

The main result is

Theorem 1. *Up to equivariant deformations of* $\mathbb{S}_4$*-manifolds, there are exactly* 15 *different actions of* $\mathbb{S}_4$ *on* $K3$ *surfaces satisfying conditions* (1) *and* (2) *and these actions are in one-to-one correspondence with the quasi-equisingular strata of plane rational quartics having no singularities of types* A_4, A_6, *and* E_6.

Let $G_x = \{g \in \mathbb{S}_4 \mid g(x) = x\}$ be the stabilizer of $x \in X$. A priori, G_x is either a cyclic group of order $q \leq 4$, or $\mathbb{S}_2 \times \mathbb{S}_2 \subset \mathbb{S}_4$, or Klien four group K_4, or $\mathbb{S}_3 \subset \mathbb{S}_4$, or $K_4 \ltimes \mathbb{S}_2$, or $\mathbb{A}_4$, or $\mathbb{S}_4$.

By Cartan Lemma, the action of G_x can be linearized, i.e., $G_x \subset GL(2, \mathbb{C})$. Let $I_g := (\theta_1(g), \theta_2(g))$ be the eigen values of $g \in G_x$.

Denote by ε_q a primitive q-root of the unity.

Lemma 1. *Let* $\mathbb{A}_4 \subset \mathbb{S}_4 \subset Aut(X)$ *be as above. Then:*

(i) *if the order of* $g \in G_x \cap \mathbb{A}_4$ *is two, then* $I_g = (-1, -1)$*;*

(ii) *if the order of* $g \in G_x$ *is three, then* $I_g = (\varepsilon_3, \varepsilon_3^{-1})$*;*

(iii) *if the order of* $g \in G_x$ *is four, then* $I_g = (\varepsilon_4, \varepsilon_4)$*;*

(iv) K_4, $K_4 \ltimes \mathbb{S}_2$, $\mathbb{A}_4$, *and* $\mathbb{S}_4$ *can not be the stabilizer of* $x \in X$.

Consider $f : X \to X/\mathbb{S}_4 = Z$, $f_1 : X \to X/\mathbb{A}_4 = Y$, and $f_2 : Y \to Z$, $f = f_2 \circ f_1$. Since $\mathbb{A}_4$ acts symplectically on X, then Y is a $K3$ surface with ADE-singularities. By Lemma 1, the singular points of Y are of types A_1 or A_2.

We say that a point $x \in X$ is **special** if G_x is non-trivial. Let S be the set of special points. Then $S = I \sqcup R$, where

. $$I = \{x \in X \mid x \text{ is an isolated fixed point of } G_x\}$$

and R is the union of curves.

Let $|G_x|$ be the order of G_x and

. $$I_n = \{x \in I \mid G_x \text{ is a cyclic group, } |G_x| = n\}.$$

By Lemma 1, if $x \in I_2$, then G_x is generated by the product of two commuting transpositions and $f_1(x)$ and $f(x)$ are singular points of type A_1 on Y and Z. Denote by $a_2 = |I_2|$ the cardinality of I_2.

If $x \in I_3$, then $f_1(x)$ and $f(x)$ are singular points of type A_2. Denote by $b_3 = |I_3|$ the cardinality of I_3.

If $x \in I_4$, then $f_1(x)$ is a singular point of type A_1 and $f(x)$ is a singular point of type $\frac{1}{4}(1,1)$. Denote by $a_4 = |I_4|$ the cardinality of I_4.

If $x \in R \setminus Sing\, R$, then $G_x = \mathbb{S}_2 \subset \mathbb{S}_4$ is generated by transposition eigen values of which (as an element of $GL(2,\mathbb{C})$) is $(-1,1)$ and we have

Claim 1. *The covering f is ramified along R with multiplicity two and for $\forall x \in R \setminus Sing\, R$ its stabilizer $G_x = \mathbb{S}_2 \subset \mathbb{S}_4$ is generated by a transposition.*

If $x \in Sing\, R$, then either $G_x = \mathbb{S}_2 \times \mathbb{S}_2 \subset \mathbb{S}_4$ and $f_1(x)$ is singular of type A_1, or $G_x = \mathbb{S}_3 \subset \mathbb{S}_4$ and $f_1(x)$ is singular of type A_2, but in the both cases Z is non-singular at $f(x)$. In the first case, $B = f(R)$ has a singularity of type A_1 at $f(x)$ and in the second case, B has a singularity of type A_2.

Denote by

$$J_{2,2} = \{x \in Sing\, R \mid G_x = \mathbb{S}_2 \times \mathbb{S}_2 \subset \mathbb{S}_4\}, \ \ J_6 = \{x \in Sing\, R \mid G_x = \mathbb{S}_3 \subset \mathbb{S}_4\}.$$

Let $a_{2,2} = |J_{2,2}|$ and $b_6 = |J_6|$ be the cardinalities of $J_{2,2}$ and J_6.

Lemma 2. $$a_2 + a_4 + a_{2,2} = b_3 + b_6 = 24$$

For a subgroup $G_{i,j} = \mathbb{S}_2 \subset \mathbb{S}_4$ generated by a transposition $g_{i,j} = (i,j)$, denote by $R_{g_{i,j}} = \overline{\{x \in R \mid G_x = G_{i,j}\}}$.

We have

. $$R = \bigcup_{1 \le i < j \le 4} R_{g_{i,j}} \quad \text{and} \quad g(R_{g_{i_1,j_1}}) = R_{g_{i_2,j_2}}$$

for $g_{i_1,j_1} = g^{-1} g_{i_2,j_2} g$.

Therefore $f_{(i,j)} = f_{|R_{g_{i,j}}} : R_{g_{i,j}} \to B$ is a covering of degree two.

Let $B = B_1 \cup \dots \cup B_k$ be the decomposition into the union of the irreducible components and let $R_{g_{i,j},l} = R_{g_{i,j}} \cap f^{-1}(B_l) \subset R$. Then $R_{g_{i,j},l}$ are non-singular curves, for each l the degree of the covering $f_{i,j} : R_{g_{i,j},l} \to B_l$ is two and it is ramified only at the points of $R_{g_{i,j},l} \cap J_{2,2}$. If $z \in Sing\, B_l$ and $x \in f^{-1}(z)$, then there are $\frac{|G_x|}{2}$ pairwise transversally intersecting curves $R_{g_{i,j},l}$ passing through x, where $g_{i,j}$ are the transpositions in G_x.

Denote by n_l (resp., c_l) the number of singular points of B_l of type A_1 (resp., A_2).

Lemma 3. $$(R^2_{g_{i,j},l})_X = (B_l^2)_Z - 4c_l - 2n_l.$$

The collection $Pas(X) = (a_2, a_4, a_{2,2}; b_3, b_6; \mathbf{r})$ is called the **pasport** of the action of $\mathbb{S}_4$ on the $K3$ surface X, where $\mathbf{r} = (r_1, \dots, r_k)$ is the non-ordered collection consisting of k intersection numberes $r_l = (R^2_{g_{i,j},l})_X$.

Proposition 1. *Let the action of* $\mathbb{S}_4$ *on* $K3$ *surfaces* X_1 *and* X_2 *satisfies conditions* (1) *and* (2). *If* X_1 *and* X_2 *are* $\mathbb{S}_4$*-deformation equivalent, then* $Pas(X_1) = Pas(X_2)$.

For a point $z \in Z$ we put

$$\alpha_2(z) := \begin{cases} 2 & \text{if } f^{-1}(z) \subset I_2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\alpha_4(z) := \begin{cases} 1 & \text{if } f^{-1}(z) \subset I_4, \\ 0 & \text{otherwise;} \end{cases}$$

$$\alpha_{2,2}(z) := \begin{cases} 1 & \text{if } f^{-1}(z) \subset J_{2,2}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\beta_3(z) := \begin{cases} 2 & \text{if } f^{-1}(z) \subset I_3, \\ 0 & \text{otherwise;} \end{cases}$$

$$\beta_6(z) := \begin{cases} 1 & \text{if } f^{-1}(z) \subset J_6, \\ 0 & \text{otherwise} \end{cases}$$

and let $\alpha(z) := \alpha_2(z) + \alpha_4(z) + \alpha_{2,2}(z), \quad \beta(z) := \beta_3(z) + \beta_6(z).$

By Lemma 2, we have

Proposition 2. $$\sum_{z \in Z} \alpha(z) = 4, \qquad \sum_{z \in Z} \beta(z) = 6.$$

We have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{c}} & \overline{X} \supset \overline{R} \\ \downarrow f & & \downarrow \overline{f} \\ Z & \xrightarrow{c} & \mathbb{P}^2 \supset \overline{B} \end{array}$$

It is easy to see that

. $$\alpha(z) \neq 0 \text{ or } \beta(z) \neq 0 \Rightarrow c(z) \in Sing\,\overline{B}.$$

For $\overline{z} \in Sing\,\overline{B}$, define

. $$\alpha(\overline{z}) := \sum_{z \in c^{-1}(\overline{z})} \alpha(z), \qquad \beta(\overline{z}) := \sum_{z \in c^{-1}(\overline{z})} \beta(z).$$

By Proposition 2,

. $$\alpha(\overline{B}) := \sum_{\overline{z} \in Sing\,\overline{B}} \alpha(\overline{z}) = 4, \qquad \beta(\overline{B}) := \sum_{\overline{z} \in Sing\,\overline{B}} \beta(\overline{z}) = 6.$$

Proposition 3. *Let* $\mathbb{A}_4 \subset \mathbb{S}_4 \subset Aut(X)$ *be as above. Then* $\deg \overline{B} = 6$ *and* $\overline{f}$ *is branched along* $\overline{B}$ *with multiplisity two.*

Let $\overline{z} \in Sing\,\overline{B}$ and $\overline{x} \in \overline{f}^{-1}(\overline{z})$ be a singular point (ADE-singularity).

Denote

. $$E_{\overline{x}} := \sharp\{\text{irreducible components of } \overline{c}^{-1}(\overline{x})\},$$

. $$I_{G_{\overline{x}}} := (\mathbb{S}_4 : G_{\overline{x}}),$$

. $$N_{\overline{z}} := E_{\overline{x}} \cdot I_{G_{\overline{x}}}.$$

Claim 1. $\sum_{\overline{z} \in Sing\,\overline{B}} N_{\overline{z}} \leq 19.$

For $\overline{x} \in \overline{X}$ and $\overline{z} = \overline{f}(\overline{x}) \in Sing\,\overline{B} \subset \mathbb{P}^2$, let

. $$U \subset \overline{X} \quad \text{and} \quad V \simeq D = \{|t| < 1\}^2 \subset \mathbb{P}^2$$

be connected small neibourhoods resp. of $\overline{x}$ and $\overline{z}$ s.t.

. $$\overline{f}_{|U} : U \twoheadrightarrow V \simeq U/G_{\overline{x}}.$$

Covering $\overline{f}_{|U}$ defines

. $$\overline{f}_{|U*} : \pi_1(V \setminus \overline{B}) \twoheadrightarrow G_{\overline{x}}.$$

The point $\overline{x} \in U$ is either smooth or ADE-singularity $\Rightarrow$ it is a factor-singularity:

. $$\exists\ H \subset GL(2, \mathbb{C}) \text{ and } h : D \to U = D/H,$$

where D is a bi-disc in $\mathbb{C}^2$.

$$D \xrightarrow{h} U = D/H \xrightarrow{\overline{f}_{|U}} V = U/G_{\overline{x}} \simeq D.$$

Lemma 3. *There is a finite group $\mathcal{R} \subset GL(2, \mathbb{C})$ generated by reflections of second order s.t.*

(i) $$1 \to H \xrightarrow{\varphi} \mathcal{R} \xrightarrow{\psi} G_{\overline{x}} \to 1;$$

(ii) $\exists \ \ \xi : \pi_1(V \setminus \overline{B}) \twoheadrightarrow \mathcal{R}$ *s.t.*

$(ii)_1$ $\overline{f}_{|U*} = \psi \circ \xi$,

$(ii)_2$ $\xi(\gamma)$ *are 2-reflections for the geometric generators* $\gamma \in \pi_1(V \setminus \overline{B})$.

G.C. Shephard, J.A. Todd ([Sh.-T.'54]) classified finite groups generated by reflections. In particular, the groups generated by 2-reflections are:

$\mathcal{R} \simeq \mathbb{Z}_2$;

$\mathcal{R} = \mathcal{D}_p = \langle y_1, y_2 \mid y_1^2 = y_2^2 = (y_1 y_2)^p = 1\rangle$, $p \geq 2$

$\mathcal{R} = \mathcal{D}_{2p,2} =$

$\langle y_1, y_2, y_3 \mid y_1^2 = y_2^2 = y_3^2 = (y_1 y_2)^{2p} = [y_1 y_2, y_3] = 1, y_1 y_3 y_1 = y_3 (y_1 y_2)^p\rangle$,

$p \geq 2$;

$\mathcal{R} = \mathcal{RO}_{(2)}$ is the group number 12 in [Sh.-T.'54];

$\mathcal{R} = \mathcal{RO}_{(2,2)}$ is the group number 13 in [Sh.-T.'54];

$\mathcal{R} = \mathcal{RI}_{(2)}$ is the group number 22 in [Sh.-T.'54].

Reminder: $\alpha(\overline{z}) \leq 4$, $\beta(\overline{z}) \leq 6$, $N_{\overline{z}} \leq 19$.

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Table 1

$\overline{z} \in Sing\,\overline{B}$	$G_{\overline{x}}$	$\alpha(\overline{z})$	$\beta(\overline{z})$	$\delta(\overline{z})$	$N_{\overline{z}}$
A_1	$\mathbb{Z}_2$	0	0	1	12
A_1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	0	1	0
A_2	$\mathbb{S}_3$	0	1	1	0
A_3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	0	2	6
A_5	$\mathbb{Z}_2 \times \mathbb{Z}_2$	3	0	3	12
A_5	$\mathbb{S}_3$	0	2	3	4
A_7	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	0	4	18
A_8	$\mathbb{S}_3$	0	3	4	8
A_{11}	$\mathbb{S}_3$	0	4	6	12
A_{14}	$\mathbb{S}_3$	0	5	7	16
D_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	0	3	18
E_6	$\mathbb{S}_3$	0	2	3	16
E_6	$\mathbb{S}_4$	1	2	3	1

Proposition 4. *Let* $\overline{f} : \overline{X} \to \mathbb{P}^2$ *be as above. Then the branch curve* $\overline{B} \subset \mathbb{P}^2$ *of* $\overline{f}$ *is the union*

$$\overline{B} = \widehat{C} \cup \overline{L},$$

where $\widehat{C}$ *is a curve dual to a plane rational quartic* C *having no singulsrities of types* A_4, A_6, *and* E_6; *and* $\overline{L} = \cup L_P$, *where* $L_P \subset \mathbb{P}^2$ *is the line dual to the point* $P \in \mathbb{P}^2$ *and the union is taken over all singular points* $P \in C$ *of types* A_2 *and* D_5.

Notation. $Sing\,\overline{L}$ is the set of singularities of $\overline{B}$ belonging to $\overline{L}$.

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Table 2

no.	$Sing\,C$	deg $\widehat{C}$	$Sing\,\overline{B}$	
			$Sing\,\overline{L}$	$Sing\,\widehat{C}$
1.1	$3A_1+6F_1$	6		$4A_1+6A_2$
1.2	D_4+6F_1	6		$4A_1+6A_2$
2.1	$A_1+A_3+6F_1$	6		$2A_1+A_3+6A_2$
2.2	A_5+6F_1	6		$A_1+A_5+6A_2$
3.1	$3A_1+4F_1+F_2$	6		$3A_1+4A_2+E_6$
3.2	$D_4+4F_1+F_2$	6		$3A_1+4A_2+E_6$
4.1	$A_1+A_3+4F_1+F_2$	6		$A_1+A_3+4A_2+E_6$
4.2	$A_5+4F_1+F_2$	6		$A_5+4A_2+E_6$
5.1	$3A_1+2F_1+2F_2$	6		$2A_1+2A_2+2E_6$
5.2	$D_4+2F_1+2F_2$	6		$2A_1+2A_2+2E_6$
6	$A_1+A_3+2F_1+2F_2$	6		$A_3+2A_2+2E_6$
7	$3A_1+3F_2$	6		A_1+3E_6

no.	$Sing\,C$	$\deg\widehat{C}$	$Sing\,\overline{B}$	
			$Sing\,\overline{L}$	$Sing\,\widehat{C}$
8	$2A_1+A_2+4F_1$	5	$2A_1+A_5$	$2A_1+4A_2$
9	D_5+4F_1	5	A_3+A_5	$2A_1+4A_2$
10	$A_2+A_3+4F_1$	5	$2A_1+A_5$	A_3+4A_2
11	$2A_1+A_2+2F_1+F_2$	5	$2A_1+A_5$	$A_1+2A_2+E_6$
12	$D_5+2F_1+F_2$	5	A_3+A_5	$A_1+2A_2+E_6$
13	$A_1+2A_2+2F_1$	4	$3A_1+2A_5$	A_1+2A_2
14	$A_1+2A_2+F_2$	4	$3A_1+2A_5$	E_6
15	$3A_2$	3	$3A_1+3A_5$	A_1

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Dualizing coverings

Let $C \subset \mathbb{P}^2$ be an irreducible reduced curve, $\deg C \geq 2$, $\nu : \overline{C} \to C$ the normalization of C. For $p \in \overline{C}$ and $P = \nu(p) \in \mathbb{P}^2$ let us choose homogeneous coordinates (x_1, x_2, x_3) in $\mathbb{P}^2$ and a local parameter t in a complex analytic neighborhood $U \subset \overline{C}$ of p s.t. ν is given by

$$x_1 = \sum_{i=s_p}^{\infty} a_i t^i,\ x_2 = t^{r_p},\ x_3 = 1, \qquad (1)$$

where $a_{s_p} \neq 0$ and $s_p > r_p \geq 1$.

The integer r_p is the *multiplicity* of the germ $\nu(U)$ of C at $P = \nu(p)$,

$l_p = \{x_1 = 0\}$ is a *tangent line* to C at P,

s_p is called the *tangent multiplicity* of the germ $\nu(U)$ at P.

Let $\widehat{C} \subset \widehat{\mathbb{P}}^2$ be the dual curve to C.

Denote $I = \{(P, l) \in \mathbb{P}^2 \times \widehat{\mathbb{P}}^2 \mid P \in l\}$ the incidence variety.

The graph of the correspondence between C and $\widehat{C}$ is a curve

$\check{C} = \{(\nu(p), l_p) \in I \mid p \in \overline{C}$ and l_p is the tangent line to C at $\nu(p) \in C\}$.

Denote $L_p \subset \widehat{\mathbb{P}}^2$ the line dual to a point $\nu(p) \in C \subset \mathbb{P}^2$.

Let $\mathrm{pr}_1 : \mathbb{P}^2 \times \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ and $\mathrm{pr}_2 : \mathbb{P}^2 \times \widehat{\mathbb{P}}^2 \to \widehat{\mathbb{P}}^2$ be the projections,

. $$X' = \mathrm{pr}_1^{-1}(C) \cap I, \qquad f' = \mathrm{pr}_{2|X'} : X' \to \widehat{\mathbb{P}}^2.$$

$f'^{-1}(l) = \{(P, l) \in \mathbb{P}^2 \times \widehat{\mathbb{P}}^2 \mid P \in C \cap l\} \Rightarrow \deg f' = \deg C = d$.

The projection $\mathrm{pr}_1 : X' \to C$ defines on X' a ruled structure.

Denote by $\nu' : \widetilde{X} \to X'$ the normalization of X'.

$\widetilde{f} = f' \circ \nu' : \widetilde{X} \to \widehat{\mathbb{P}}^2$ is called the **dualizing covering** for $C \subset \mathbb{P}^2$.

We have:

(1) $\deg \widetilde{f} = d$.

(2) $\widetilde{X} \simeq \overline{C} \times_C X'$ is the fibre product of the normalization $\nu : \overline{C} \to C$ and the projection $\mathrm{pr}_1 : X' \to C$;

(3) $\widetilde{X}$ is a non-singular ruled surface over $\overline{C}$, $\rho := \mathrm{pr}_1 : \widetilde{X} \to \overline{C}$, the fiber $\rho^{-1}(p) := F_p \simeq \mathbb{P}^1$ over $p \in \overline{C}$;

(4) $\widetilde{f}(F_p) = L_p \subset \widehat{\mathbb{P}}^2$, where L_p is dual to $\nu(p) \in C \subset \mathbb{P}^2$.

(5) $\tilde{C} = \nu'^{-1}(\check{C}) \subset \widetilde{X}$ is a section of this ruled structure;

Theorem 2. *Let $\widetilde{f} : \widetilde{X} \to \widehat{\mathbb{P}}^2$ be the dualizing covering for C. Then*

(1) the branch locus of $\widetilde{f}$ is $\overline{B} = \widehat{C} \cup \overline{L}$, where $\overline{L} = \bigcup_{r_p \geq 2} L_p$, L_p are the lines dual to the points $\nu(p) \in C$ and the union is taken over all $p \in \overline{C}$ for which the multiplicity $r_p \geq 2$;

(2) the ramification locus of $\widetilde{f}$ is $\widetilde{R} = \widetilde{C} \cup \widetilde{F}$, where $\widetilde{F} = \bigcup_{r_p \geq 2} F_p$ and the union is taken over all $p \in \overline{C}$ for which $r_p \geq 2$;

(3) the local degree $\deg_q \widetilde{f}$ of $\widetilde{f}$ at $q = F_p \cap \widetilde{C}$ is equal to the tangent multiplicity s_p, and $\deg_q \widetilde{f} = r_p$ at all points $q \in F_p \setminus \widetilde{C}$; for all $q \in \widetilde{C} \setminus \widetilde{F}$ the local degree $\deg_q \widetilde{f} = 2$.

The dualizing covering $\widetilde{f}:\widetilde{X}\to\widehat{\mathbb{P}}^2$ defines a homomorphism $\widetilde{f}_*:\pi_1(\widehat{\mathbb{P}}^2\setminus\overline{B},p_0)\to\mathbb{S}_d$ ($\mathbb{S}_d$ acts on the points of $f^{-1}(p_0)$). The group $Im\,\widetilde{f}_*:=G_{\widetilde{f}}\subset\mathbb{S}_d$ is called the *Galois group* of the covering $\widetilde{f}$.

Theorem 3. *The Galois group $G_{\widetilde{f}}$ of the dualizing covering $\widetilde{f}:\widetilde{X}\to\widehat{\mathbb{P}}^2$ associated with $C\subset\mathbb{P}^2$,* deg $C=d$, *is the symmetric group $\mathbb{S}_d$.*

Cayley's imbedding $i:G_{\widetilde{f}}\hookrightarrow\mathbb{S}_{|G_{\widetilde{f}}|}$ (G acts on itself by multiplication (from the right)) and $\widetilde{f}_*$ defines a homomorphism $\overline{f}_*=i\circ\widetilde{f}_*:\pi_1(\widehat{\mathbb{P}}^2\setminus\overline{B})\to\mathbb{S}_{|G_{\widetilde{f}}|}$ which defines a Galois covering $\overline{f}:\overline{X}\to\widehat{\mathbb{P}}^2$ wigh Galois group $G_{\widetilde{f}}$, where $\overline{X}$ is a normal variety. The group $G_{\widetilde{f}}$ acts on $\overline{X}$ s.t. $\overline{f}:\overline{X}\to\overline{X}/G_{\widetilde{f}}=\widehat{\mathbb{P}}^2$.

Let $\overline{V} \simeq D = \{|t| < 1\}^2 \subset \widehat{\mathbb{P}}^2$ be a small neibourhood of $\overline{z} \in \widehat{\mathbb{P}}^2$. The imbedding $i : \overline{V} \hookrightarrow \widehat{\mathbb{P}}^2$ defines homomorphisms

$i_* : \pi_1(\overline{V} \setminus \overline{B}, p) \to \pi_1(\widehat{\mathbb{P}}^2 \setminus \overline{B}, p)$ and $\varphi = h_* \circ i_* : \pi_1(\overline{V} \setminus \overline{B}, p) \to G_{\widetilde{f}}$

(φ is defined uniquelly up to automorphisms of $G_{\widetilde{f}}$).

The image $Im\,\varphi := G_{\mathrm{loc},\overline{z}} \subset G_{\widetilde{f}}$ is called the *local monodromy group* of the covering $\overline{f}$ at the point $\overline{z}$.

$\overline{f}^{-1}(\overline{V}) = \sqcup_{i=1}^{k} \overline{U}_i$, where $k = (G_{\widetilde{f}} : G_{\mathrm{loc},\overline{z}})$ is the index of $G_{\mathrm{loc},\overline{z}}$ in $G_{\widetilde{f}}$, and for each $\overline{x}_i \in \overline{U}_i \cap \overline{f}^{-1}(\overline{z})$ its stabilizer $G_{\overline{x}_i}$ is isomorphic to $G_{\mathrm{loc},\overline{z}}$.

Lemma *Let $C \subset \mathbb{P}^2$ be a rational quartic having no singularities of types A_4, A_6, and E_6 and let $\overline{f} : \overline{X} \to \widehat{\mathbb{P}}^2$ be the Galois normal closure of $\widetilde{f} : \widetilde{X} \to \widehat{\mathbb{P}}^2$. Let P be a singular point of type $S_C(P)$ of C, $l = \overline{z}$ is the tangent line to C at P, and $S_{\overline{B}}(\overline{z})$ the type of singularity at $\overline{z} \in \overline{B}$. Then*

$$G_{loc,\overline{z}} = \begin{cases} \mathbb{S}_2 & \text{if } S_{\overline{B}}(\overline{z}) = A_0; \\ \mathbb{S}_2 \times \mathbb{S}_2 & \text{if } S_{\overline{B}}(\overline{z}) = A_{2p-1},\ p = 1, 2; \\ \mathbb{S}_3 & \text{if } S_{\overline{B}}(\overline{z}) = A_2; \\ \mathbb{S}_4 & \text{if } S_{\overline{B}}(\overline{z}) = E_6 \end{cases}$$

and if $S_{\overline{B}}(\overline{z}) = A_5$, then

$$G_{loc,\overline{z}} = \begin{cases} \mathbb{S}_2 \times \mathbb{S}_2 & \text{if } S_C(P) = A_5; \\ \mathbb{S}_3 & \text{if } S_C(P) = A_2, \end{cases} .$$

In all cases $\overline{f}_*(\gamma)$ are transpositions for geometric generators $\gamma \in \pi_1(\overline{V} \setminus \overline{B})$.

Proposition 5. *Let C be a rational quartic having no singularities of types A_4, A_6, and E_6 and let $\overline{f} : \overline{X} \to \widehat{\mathbb{P}}^2$ be the Galois normal closure of the dualizing covering $\widetilde{f} : \widetilde{X} \to \widehat{\mathbb{P}}^2$. Then $\overline{X}$ is a $K3$ surface with ADE-singularities and if $\overline{c} : X \to \overline{X}$ is the minimal resolution of singularities, then the pasport $Pas(X) = (a_2, a_4, a_{2,2}; b_3, b_6; \mathbf{r})$ of the action of $\mathbb{S}_4$ on $K3$ surface X depends on the singularities type of C and it is given in Table 3.*

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Table 3

no.	$Sing\,C$	$Pas(X)$					
		a_2	a_4	$a_{2,2}$	b_3	b_6	$\mathbf{r}$
1.1	$3A_1+6F_1$	0	0	24	0	24	(4)
1.2	D_4+6F_1	0	0	24	0	24	(4)
2.1	$A_1+A_3+6F_1$	12	0	12	0	24	(0)
2.2	A_5+6F_1	12	0	12	0	24	(0)
3.1	$3A_1+4F_2+F_2$	0	6	18	8	16	(2)
3.2	$D_4+4F_1+F_2$	0	6	18	8	16	(2)
4.1	$A_1+A_3+4F_1+F_2$	12	6	6	8	16	(−2)
4.2	$A_5+4F_1+F_2$	12	6	6	8	16	(−2)
5.1	$3A_1+2F_1+2F_2$	0	12	12	16	8	(0)
5.2	$D_4+2F_1+2F_2$	0	12	12	16	8	(0)
6	$A_1+A_3+2F_1+2F_2$	12	0	12	16	8	(−4)
7	$3A_1+3F_2$	0	18	6	24	0	(−2)

no.	$Sing\,C$	$Pas(X)$					
		a_2	a_4	$a_{2,2}$	b_3	b_6	$\mathbf{r}$
8	$A_1 + A_2 + 4F_1$	0	24	0	8	16	$(2, -2)$
9	$D_5 + 4F_1$	12	12	0	8	16	$(0, -4)$
10	$A_2 + A_3 + 4F_1$	12	0	12	8	16	$(-2, -2)$
11	$2A_1 + A_2 + 4F_1$	0	6	18	16	8	$(0, -2)$
12	$D_5 + 4F_1$	12	6	6	16	8	$(-2, -4)$
13	$A_1 + 2A_2 + 2F_1$	0	0	24	16	8	$(0, -2, -2)$
14	$A_1 + 2A_2 + F_2$	0	6	18	24	0	$(-2, -2, -2)$
15	$3A_2$	0	0	24	24	0	$(-2, -2, -2, -2)$

Proposition 6. *Let* $C_{i,1}$ *and* $C_{i,2}$, $i = 1, \dots, 5$, *be rational quartics having collections of singularities of types* $S_{i,1}$ *and* $S_{i,2}$, *respectively, and let* $\overline{c} : X_{i,j} \to \overline{X}_{i,j}$, $j = 1, 2$, *be the minimal resolutions of singularities of the Galois normal closures* $\overline{f} : \overline{X}_{i,j} \to \widehat{\mathbb{P}}^2$ *of the dualizing coverings* $\widetilde{f} : \widetilde{X}_{i,j} \to \widehat{\mathbb{P}}^2$. *Then* $X_{i,1}$ *and* $X_{i,2}$ *are* $\mathbb{S}_4$*-deformation equivalent.*