Cohomological characterization of vector bundles

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- Horrocks (1980): Cohomological characterization of the p-differential bundle $\Omega^p_{\mathbb{P}^n}$.
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1. Horrocks' Theorem.

2. Beilinson's type spectral sequence.

3. Applications to cohomological characterization of vector bundles.
   - Vector bundles on multiprojective spaces
   - Steiner vector bundles on algebraic varieties

4. Questions/Open Problems.

- L. Costa and R.M. Miró-Roig, Cohomological characterization of vector bundles on multiprojective spaces and $\mathbf{P}^d$-bundles. J. Alg. 294 (2005), 73-96.
- L. Costa and R.M. Miró-Roig, m-blocks collections and Castelnuovo-Mumford regularity. Nagoya J. Math. 186 (2007), 119-155.
- L. Costa and R.M. Miró-Roig, Geometric collections and Castelnuovo-Mumford regularity. Proc. Cambridge Math. Soc. 143 (2007), 557-578.
- R.M. Miró-Roig and H. Soares, Cohomological characterization of Steiner bundles. Forum Math. 21 (2009), 871-891.

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Horrocks' Theorem.

Beilinson's Theorem: $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \cdots, \mathcal{O}_{\mathbb{P}^n}(n))$ is a full strongly exceptional collection or, equivalently, $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \cdots, \mathcal{O}_{\mathbb{P}^n}(n))$ is an orthogonal basis of $D^b(\mathcal{O}_{\mathbb{P}^n} - \mathrm{mod})$ and its left dual is $(\mathcal{O}_{\mathbb{P}^n}(n), T_{\mathbb{P}^n}(n-1), \wedge^2 T_{\mathbb{P}^n}(n-2), \cdots, \wedge^n T_{\mathbb{P}^n})$.

Beilinson's Theorem: Let F be a coherent sheaf on $\mathbb{P}^n$. $\exists$ a spectral sequence situated in $-n \leq p \leq 0$, $0 \leq q \leq n$ and with E_1-term

$$E_1^{pq} = H^q(\mathbb{P}^n, F(p)) \otimes \Omega^{-p}(-p)$$

which converges to

$$E_\infty^i = \begin{cases} F \text{ for } i = 0 \\ 0 \text{ for } i \neq 0. \end{cases}$$

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Horrocks' Theorem: Let E be a vector bundle on $\mathbb{P}^n$. The following conditions are equivalent:

(i) E splits into a sum of line bundles.

(ii) E has no intermediate cohomology; i.e. $H^i(\mathbb{P}^n, E(t)) = 0$ for $1 \leq i \leq n-1$ and for all $t \in \mathbb{Z}$.

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Beilinson's type spectral sequence.

Let X be a smooth projective variety of dimension n.

- A coherent sheaf F on X is exceptional if $Hom(F,F) = \mathbb{C}$ and $Ext^i_X(F,F) = 0$ for $i > 0$.
- An ordered collection $(F_0, F_1, \ldots, F_m)$ of coherent sheaves on X is an exceptional collection if each sheaf F_i is exceptional and $Ext^i_X(F_k, F_j) = 0$ for $j < k$, $i \geq 0$.
- An exceptional collection $(F_0, F_1, \ldots, F_m)$ is a strongly exceptional collection if in addition $Ext^i_X(F_j, F_k) = 0$ for $i \geq 1$, $j \leq k$.
- An ordered collection $(F_0, \ldots, F_m)$ is a full (strongly) exceptional collection if it is a (strongly) exceptional collection and $F_0, \ldots, F_m$ generate $D^b(\mathcal{O}_X - mod)$.

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EXAMPLES

- $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$ is a full strongly exceptional collection on a projective space $\mathbb{P}^n$.
- $(\mathcal{O}_{\mathbb{P}^n}, \Omega^1_{\mathbb{P}^n}(1), \dots, \Omega^n_{\mathbb{P}^n}(n))$ is a full strongly exceptional collection on a projective space $\mathbb{P}^n$.
- Let $\pi : \widetilde{\mathbb{P}^2}(1) \rightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at one point $p \in \mathbb{P}^2$. Let H be the pullback of the hyperplane divisor in $\mathbb{P}^2$ and let $E = \pi^{-1}(p)$ be the exceptional divisor. Then the collection of divisors $(0, E, H, 2H)$ is a full strongly exceptional collection on $\widetilde{\mathbb{P}^2}(1)$.

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- (Kapranov) Let $Q_n \subset \mathbb{P}^{n+1}$, $n > 2$, be a hyperquadric. If n is even and Σ_1, Σ_2 are the Spinor bundles on Q_n, then

$$(\Sigma_1(-n), \Sigma_2(-n), \mathcal{O}_{Q_n}(-n+1), \cdots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection on Q_n; and if n is odd and Σ is the Spinor bundle on Q_n, then

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- (Kapranov) Take $X = Gr(k,n)$. Denotes by $\mathcal{S}$ the tautological k-dimensional bundle and $\Sigma^{\alpha}\mathcal{S}$ the space of the irreducible representations of $GL(\mathcal{S})$ with highest weight $\alpha = (\alpha_1, \dots, \alpha_s)$. Let $A(k,n)$ be the set of locally free sheaves $\Sigma^{\alpha}\mathcal{S}$ on $Gr(k,n)$ where α runs over Young diagrams fitting inside a $k \times (n-k)$ rectangle. $A(k,n)$ can be totally ordered in such a way that we obtain a full strongly exceptional collection $(E_1, \dots, E_{\rho(k,n)})$ of sheaves on X.

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REMARKS

- In all these collections the order is very important.
- The length of any full strongly exceptional collection is $\geq$ dim$(X)+1$.
- All full strongly exceptional collections on X have the same length and it coincides with the rank of the Grothendieck group $K_0(X)$ as $\mathbb{Z}$-module.
- Not all full strongly exceptional collections are made up of line bundles.

Definition

A GEOMETRIC COLLECTION of coherent sheaves $(E_0, \cdots, E_n)$ on a smooth algebraic variety X is a full exceptional collection of minimal length, dim $(X)+1$.

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Problem 1. To characterize smooth projective varieties which have a geometric collection.

Problem 2. To characterize smooth projective varieties which have a full strongly exceptional collection.

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Theorem (Costa and MR)

Let X be a smooth projective variety of dim n with a geometric collection $(E_0, \cdots, E_n)$ and let F be a coherent sheaf on X. $\exists$ two spectral sequences with E_1-term

$$_{I}E_1^{pq} = Ext^q(R^{(-p)}E_{n+p}, F) \otimes E_{p+n}$$

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Let X be a smooth projective variety of dimension n.

- An exceptional collection $(F_0, F_1, \cdots, F_m)$ of coherent sheaves on X is a block if $Ext^i(F_j, F_k) = 0$ for any i and $j \neq k$.
- An m-block collection of type $(\alpha_0, \alpha_1, \cdots, \alpha_m)$ is an exceptional collection $(\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m) = (E_1^0, \cdots, E_{\alpha_0}^0, \cdots, E_1^m, \cdots, E_{\alpha_m}^m)$ such that all the subcollections $\mathcal{E}_i = (E_1^i, E_2^i, \cdots, E_{\alpha_i}^i)$ are blocks.

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EXAMPLE

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, $d = n_1 + \cdots + n_s$ and denote

$\mathcal{O}_X(a_1, a_2, \cdots, a_s) := p_1^* \mathcal{O}_{\mathbb{P}^{n_1}}(a_1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^{n_2}}(a_2) \otimes \cdots \otimes p_s^* \mathcal{O}_{\mathbb{P}^{n_s}}(a_s)$.

For any $0 \leq j \leq d$, denote by $\mathcal{E}_j$ the collection of all line bundles

$$\mathcal{O}_X(a_1^j, a_2^j, \cdots, a_s^j)$$

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Consider $X = \mathbb{P}^2 \times \mathbb{P}^3$. The collection of line bundles

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is a full strongly exceptional collection of length $10 > dim(X) + 1 = 6$ and we pack in 6 blocks:

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Definition

Let $\sigma = (\mathcal{E}_0, \cdots, \mathcal{E}_m)$ be an *m*-block collection of coherent sheaves which generates $D^b(X)$.. The *m*-block $\mathcal{H} = (\mathcal{H}_0, \cdots, \mathcal{H}_m)$ is called the left dual *m*-block collection of σ if

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$$\mathcal{B} = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \cdots, \mathcal{O}_{\mathbb{P}^n}(n)).$$

Using the exterior powers

$$0 \longrightarrow \wedge^{k-1} T_{\mathbb{P}^n} \longrightarrow \wedge^k V \otimes \mathcal{O}_{\mathbb{P}^n}(k) \longrightarrow \wedge^k T_{\mathbb{P}^n} \longrightarrow 0$$

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we compute the left dual n-block collection of $\mathcal{B}$ and we get

$$\begin{aligned} \mathcal{H} &= (\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_j, \cdots, \mathcal{H}_n) \\ &= (\mathcal{O}_{\mathbb{P}^n}(n), T_{\mathbb{P}^n}(n-1), \cdots, \wedge^j T_{\mathbb{P}^n}(n-j), \cdots, \wedge^n T_{\mathbb{P}^n}). \end{aligned}$$

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Proposition

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ *be a multiprojective space of dimension* $d = n_1 + \cdots + n_s$. *Consider* $\mathcal{B} = (\mathcal{E}_0, \cdots, \mathcal{E}_d)$ *the d-block collection where for any* $0 \leq j \leq d$, $\mathcal{E}_j$ *is the set of line bundles*

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is the left dual d*-block collection of* $\mathcal{B} = (\mathcal{E}_0, \cdots, \mathcal{E}_d)$.

Sketch of the proof.

For any $0 \leq i \leq d$, we take $\mathcal{O}_X(a_1^i, \cdots, a_s^i) \in \mathcal{E}_i$ and we apply the Künneth formula,

$$H^\alpha(X, \bigwedge^{-t_1} \Omega_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} \Omega_{\mathbb{P}^{n_s}}(-t_s) \otimes \mathcal{O}_X(a_1^i, \cdots, a_s^i))$$

$$= \bigoplus_{\alpha_1 + \cdots + \alpha_s = \alpha} H^{\alpha_1}(\mathbb{P}^{n_1}, \bigwedge^{-t_1} \Omega(a_1^i - t_1)) \otimes \cdots \otimes H^{\alpha_s}(\mathbb{P}^{n_s}, \bigwedge^{-t_s} \Omega(a_s^i - t_s)).$$

Using Bott's formula, it is zero unless $\alpha = k$, $i = d - k$ and $\mathcal{O}_X(a_1^i, \cdots, a_s^i) = \mathcal{O}_X(t_1, \cdots, t_s)$, which proves what we want.

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Theorem (Beilinson type spectral sequence).

Let X be a smooth projective variety of dim n with an n-block collection $(\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \dots, E_{\alpha_i}^i)$ of coherent sheaves which generates $D^b(X)$. Denote by $(\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_n)$, $\mathcal{H}_i = (H_1^i, \dots, H_{\alpha_i}^i)$ the left dual n-block collection . $\forall F$ coherent sheaf , $\exists$ spectral sequences

$$ {}_I E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+n}} Ext^q(H_i^p, F) \otimes E_i^{p+n} & \text{if} \quad -n \le p \le -1 \\ \bigoplus_{i=1}^{\alpha_n} Ext^q(E_i^n, F) \otimes E_i^n & \text{if} \quad p = 0 \end{cases} $$

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- Vector bundles on multiprojective spaces
- Steiner vector bundles on algebraic varieties

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Theorem

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$H^j(F \otimes E_i^{d-j}) = \mathbb{C}$ *for* $1 \leq i \leq \alpha_{d-j}$. *Then*

$$F \cong \bigoplus_{t_1+\cdots+t_s=j-d} \bigwedge^{-t_1} \Omega_{\mathbb{P}^{n_1}}(-t_1) \boxtimes \cdots \boxtimes \bigwedge^{-t_s} \Omega_{\mathbb{P}^{n_s}}(-t_s)$$

$$\cong \bigwedge^{d-j} (\Omega_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}}(1, \cdots, 1)) \text{ with } -n_i \leq t_i \leq 0$$

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Sketch of the proof. We apply to F the spectral sequence with E_1-term

$$_{II}E_1^{pq} = \begin{cases} \bigoplus_{i=1}^{\alpha_{p+d}} Ext^q((E_i^{p+d})^*, F) \otimes (H_i^p)^* & \text{if} \quad -d \leq p \leq -1 \\ \bigoplus_{i=1}^{\alpha_d} Ext^q(E_i^{d^*}, F) \otimes E_i^{d^*} & \text{if} \quad p = 0 \end{cases}$$

By assumption, there is an integer j, $0 < j < d$, such that $_{II}E_1^{p,-p-1} = 0$ for any $-n \leq p \leq -j-1$ and $_{II}E_1^{p,-p+1} = 0$ for any $-j+1 \leq p \leq 0$.
So, F contains $_{II}E_1^{jj}$, i.e. F contains

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STEINER BUNDLES were first defined by Dolgachev and Kapranov as vector bundles E on $\mathbb{P}^n$ defined by an exact sequence of the form (Schwarzenberger: $t = s + n$)

$$(*) \quad 0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^s \to \mathcal{O}_{\mathbb{P}^n}^t \to E \to 0.$$

- They used Steiner bundles to study logarithmic bundles $\Omega(\log \mathcal{H})$ of meromorphic forms on $\mathbb{P}^n$ having at most logarithmic poles on a finite union $\mathcal{H}$ of hyperplanes with normal crossing.
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Definition

A vector bundle E on a smooth irreducible algebraic variety X is called a Steiner bundle if it is defined by an exact sequence of the form

$$0 \longrightarrow F_0^s \xrightarrow{\varphi} F_1^t \longrightarrow E \longrightarrow 0,$$

where $s, t \geq 1$ and (F_0, F_1) is an ordered pair of vector bundles on X satisfying the following two conditions:

(i) (F_0, F_1) is strongly exceptional;

(ii) $F_0^\vee \otimes F_1$ is generated by global sections.

- When $X = \mathbb{P}^n$, $F_0 = \mathcal{O}_{\mathbb{P}^n}(-1)$ and $F_1 = \mathcal{O}_{\mathbb{P}^n}$ we obtain the classical Steiner bundles as defined by Dolgachev and Kapranov.

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EXAMPLES OF STEINER BUNDLES.

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$$0 \to \mathcal{O}_{\mathbb{P}^n}(a)^s \to \mathcal{O}_{\mathbb{P}^n}(b)^t \to E \to 0,$$

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EXAMPLES OF STEINER BUNDLES.

- Let $Q_n \subset \mathbb{P}^{+1}$, $n \geq 2$, be the smooth hyperquadric. Let Σ_* denote the Spinor bundle Σ on Q_n if n is odd, and one of the Spinor bundles Σ_1 or Σ_2 on Q_n if n is even.
  The short exact sequences

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where $0 \leq a \leq n-1$, and

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Theorem (MR - Soares)

Let X be a smooth projective variety of dim n with an n-block collection $\mathcal{B} = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \dots, E_{\alpha_i}^i)$, of vector bundles on X which generate $\mathcal{D}^b(X)$. Let $E_{i_0}^a \in \mathcal{E}_a$, $E_{j_0}^b \in \mathcal{E}_b$, where $0 \leq a < b \leq n$ and $1 \leq i_0 \leq \alpha_a$, $1 \leq j_0 \leq \alpha_b$, and let E be a vector bundle on X. Then E is a Steiner bundle of type $(E_{i_0}^a, E_{j_0}^b)$ defined by

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iff $(E_{i_0}^a)^\vee \otimes E_{j_0}^b$ is globally generated and all $h^k(E \otimes (R^{(m)} E_i^{n-m})^\vee)$ vanish, with the only exceptions of

$$h^{n-a-1}(E \otimes (R^{(n-a)} E_{i_0}^a)^\vee) = s \text{ and } h^{n-b}(E \otimes (R^{(n-b)} E_{j_0}^b)^\vee) = t.$$

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Corollary

Let X be a smooth projective variety X with an n-block collection $\mathcal{B} = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \ldots, E_{\alpha_i}^i)$, of locally free sheaves on X which generate $\mathcal{D}$. Let $E_{i_0}^a \in \mathcal{E}_a$, $E_{j_0}^b \in \mathcal{E}_b$, where $0 \leq a < b \leq n$ and $1 \leq i_0 \leq \alpha_a$, $1 \leq j_0 \leq \alpha_b$.

If E and F are Steiner bundles of type $(E_{i_0}^a, E_{j_0}^b)$ on X then any extension G of E by F,

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Let X be a smooth projective variety X with an n-block collection $\mathcal{B} = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n)$, $\mathcal{E}_i = (E^i_1, \ldots, E^i_{\alpha_i})$, of locally free sheaves on X which generate $\mathcal{D}$. Let $E^a_{i_0} \in \mathcal{E}_a$, $E^b_{j_0} \in \mathcal{E}_b$, where $0 \leq a < b \leq n$ and $1 \leq i_0 \leq \alpha_a$, $1 \leq j_0 \leq \alpha_b$.

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THANK YOU!