Moduli spaces of framed perverse instantons on $\mathbb{P}^3$

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Outline

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Framed modules

$(E, \alpha : E \to F)$ a framed module
F a torsion free sheaf on a divisor $D \subset X$

Lemma

E slope semistable, $E|_D \simeq F$ a framing
Then the corresponding framed module (E, α), where $\alpha : E \to E|_D \simeq F$, is slope δ_{n-1}-stable for any small $\delta_{n-1} > 0$. In particular, (E, α) is Gieseker δ-stable for all polynomials δ of degree $n - 1$ with a small positive leading coefficient.

Moduli space of framed modules

Corollary

There exists $M(X; D, F, P)$ fine moduli scheme for $(E, E|_D \simeq F)$, where E slope semistable torsion free on X with fixed Hilbert polynomial P.
It is an open subscheme of the projective moduli scheme of Gieseker δ-stable framed modules $M^s_\delta(X; D, F, P) = M^{ss}_\delta(X; D, F, P)$ for any polynomial δ of degree $n-1$ with a small positive leading coefficient.

Framed instantons are not semistable as framed modules

Lemma

Let $\tilde{E}$ be an $(r-1, c)$-instanton on $\mathbb{P}^3$ and let $E|_{l_\infty} \simeq \mathcal{O}_{l_\infty}^r$ be a framing of $E = \tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3}$. If $c = c_2(E) > r(r-1)$ then E is an (r, c)-instanton but the corresponding framed module (E, α) is not Gieseker δ-semistable for any positive polynomial δ.

Nevertheless, framed instantons are semistable as framed modules

X the blow up of $\mathbb{P}^3$ along a line
D exceptional divisor

Theorem

There exists a quasi-projective scheme $\mathcal{M}^f(\mathbb{P}^3; r, c)$ which represents the moduli functor $\tilde{\mathcal{M}}^f(\mathbb{P}^3; r, c) : \mathrm{Sch}/k \to \mathrm{Sets}$ given by

$$S \to \left\{ \begin{array}{c} \text{Isomorphism classes of } S\text{-flat families} \\ \text{of framed } (r, c)\text{-instantons } E \text{ on } \mathbb{P}^3. \end{array} \right\}$$

It is isomorphic to $M(X; D, \mathcal{O}_D^r, P)$ for a suitably chosen Hilbert polynomial P and an arbitrary polarization.

Outline

ADHM data

V vector space of dim c
W vector space dim r

$$\mathbf{B} = \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$$

Moment map $\mu : \mathbf{B} \to \mathrm{Hom}(V, V)$:

$$\mu(B_1, B_2, i, j) = [B_1, B_2] + ij$$

Definition

$(B_1, B_2, i, j) \in \mathbf{B}$ satisfies the *ADHM equation* if $[B_1, B_2] + ij = 0$.
Then we say that (B_1, B_2, i, j) is an *ADHM datum*.

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Stability of ADHM data

Definition

We say that an ADHM datum is

1. *stable*, if for every subspace $S \subsetneq V$ (note that we allow $S = 0$) such that $B_k(S) \subset S$ for $k = 1, 2$ we have $\mathrm{im}\, i \not\subset S$.
2. *costable*, if for every no non-zero subspace $S \subset V$ such that $B_k(S) \subset S$ for $k = 1, 2$ we have $S \not\subset \ker j$,
3. *regular*, if it is stable and costable.

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Group action on ADHM data

$G = \mathrm{GL}(V)$ acts on **B**

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

μ is G-equivariant (for the adjoint action of G on $\mathrm{End}(V)$), so G acts on $\mu^{-1}(0)$

Consider stability with respect to the determinant $\chi : G \to \mathbb{G}_m$

Lemma

$\mu^{-1}(0)^{ss}_{\chi} = \mu^{-1}(0)^{s}_{\chi} =$ stable ADHM data

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The Donaldson-Nakajima theorem

Theorem

The moduli space $\mathcal{M}(\mathbb{P}^2; r, c)$ of rank r torsion free sheaves on $\mathbb{P}^2$ with $c_2 = c$, framed along a line l_∞ is isomorphic to the GIT quotient $\mu^{-1}(0)/\!\!/_\chi G$.

Analogue of the Hilbert–Chow morphism

$\mathcal{M}_0^{\mathrm{reg}}(\mathbb{P}^2; r, c)$ rank r locally free sheaves on $\mathbb{P}^2$ with $c_2 = c$, framed along l_∞

We have a partial Donaldson–Uhlenbeck compactification:

$$\mathcal{M}_0(\mathbb{P}^2; r, c) = \bigsqcup_{0 \le d \le c} \mathcal{M}_0^{\mathrm{reg}}(\mathbb{P}^2; r, c - d) \times S^d(\mathbb{A}^2),$$

where $\mathbb{A}^2 = \mathbb{P}^2 - l_\infty$.

Theorem

$$\mathcal{M}(\mathbb{P}^2; r, c) \simeq \mu^{-1}(0) /\!\!/_\chi G \to \mu^{-1}(0)/G \simeq \mathcal{M}_0(\mathbb{P}^2; r, c)$$

can be identified with

$$(E, \Phi) \to ((E^{**}, \Phi), \mathrm{Supp}(E^{**}/E))$$

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Outline

1. Relation to moduli spaces of framed modules
2. Framed sheaves on $\mathbb{P}^2$ and ADHM data
3. Generalized ADHM data after Frenkel–Jardim
4. Perverse instantons on $\mathbb{P}^3$
5. Some remarks on smoothness

Generalized ADHM datum $(\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j}) \in \tilde{\mathbf{B}} = \mathbf{B} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, where

$$\tilde{B}_1, \tilde{B}_2 : V \to V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\tilde{i} : W \to V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\tilde{j} : V \to W \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$

For $p \in \mathbb{P}^1$ and $x = (\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j})$ we set

$$x(p) := (\tilde{B}_1(p), \tilde{B}_2(p), \tilde{i}(p), \tilde{j}(p))$$

Analogue of the moment map:
$\tilde{\mu} = \tilde{\mu}_{W,V} : \tilde{\mathbf{B}} \to \mathrm{End}(V) \otimes H^0(\mathcal{O}_X(2))$

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Definition of FJ-stability etc.

Definition

$x \in \tilde{\mathbf{B}}$ is

1. *FJ-stable* (*FJ-costable*, *FJ-regular*), if $x(p)$ is stable (respectively: costable, regular) for all $p \in \mathbb{P}^1$,
2. *FJ-semistable*, if there exists a point $p \in \mathbb{P}^1$ such that $x(p)$ is stable,
3. *FJ-semiregular*, if it is FJ-stable and there exists a point $p \in \mathbb{P}^1$ such that $x(p)$ is regular.

Notation

$\tilde{W} = V \oplus V \oplus W$

$x \in \tilde{\mathbf{B}} = \mathbf{B} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ gives $\alpha : V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3}$ by

$$\alpha = \begin{pmatrix} \tilde{B}_1 + 1 \otimes x_2 \\ \tilde{B}_2 + 1 \otimes x_3 \\ \tilde{j} \end{pmatrix} \tag{1}$$

and $\beta : \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \to V \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ by

$$\beta = \begin{pmatrix} -\tilde{B}_2 - 1 \otimes x_3 & \tilde{B}_1 + 1 \otimes x_2 & \tilde{i} \end{pmatrix}. \tag{2}$$

Lemma

$$\mathcal{C}_x^\bullet = (V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \tag{3}$$

is a monad if and only if x is FJ-stable generalized ADHM datum.

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The Frenkel–Jardim theorem

Theorem

We have set-theoretical bijections between

- *FJ-stable ADHM data and framed torsion free instantons;*
- *FJ-semiregular ADHM data and framed reflexive instantons;*
- *FJ-regular ADHM data and framed locally free instantons.*

Outline

Definition

Rank r perverse instanton — object $\mathcal{C} \in D^b(\mathbb{P}^3)$ such that:

1. $H^p(\mathbb{P}^3, \mathcal{C} \otimes \mathcal{O}_{\mathbb{P}^3}(q)) = 0$ if either $p = 0, 1$ and $p + q < 0$ or $p = 2, 3$ and $p + q \geq 0$,
2. $\mathcal{H}^p(\mathcal{C}) = 0$ for $p \neq 0, 1$,
3. $\exists j : l \hookrightarrow \mathbb{P}^3$ such that $Lj^*\mathcal{C} \simeq \mathcal{O}_l^{\oplus r}$.

Framed perverse instanton — pair $\mathcal{C}$ and a framing $\Phi : Lj^*\mathcal{C} \simeq \mathcal{O}_l^{\oplus r}$

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Stacky approach

Analogue of Drinfeld's theorem on representability of the stack of framed perverse sheaves on $\mathbb{P}^2$:

Theorem

The moduli stack $\mathrm{Perv}_r^c(\mathbb{P}^3, l_\infty)$ *of framed perverse* (r, c)*-instantons is isomorphic to the quotient stack* $[\tilde{\mu}^{-1}(0)/\,\mathrm{GL}(V)]$.

Corollary

We have bijection between isomorphism classes of perverse instantons $(\mathcal{C}, \Phi)$ *with* $\mathrm{ch}(\mathcal{C}) = r - c[H]^2$ *framed along* l_∞ *and* $\mathrm{GL}(c)$*-orbits of generalized ADHM* (r, c)*-data.*

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2. *costable*, if for every no non-zero subspace $S \subset V$ such that $\tilde{B}_k(S) \subset S \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ for $k = 1, 2$ we have $S \not\subset \ker \tilde{j}$,
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A perverse instanton is *stable* if it comes from stable ADHM datum.

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A perverse instanton is *stable* if it comes from stable ADHM datum.

Theorem

The quotient $\mathcal{M}(\mathbb{P}^3; r, c) := \tilde{\mu}^{-1}(0)/\!\!/_\chi G$ is a fine moduli scheme for the moduli functor $\overline{\mathcal{M}}(\mathbb{P}^3; r, c)$ of stable framed perverse instantons.

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There is bijection between G-orbits of stable ADHM data and isomorphism classes of stable framed perverse instantons.

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Lemma

For every stable rank $r > 0$ perverse instanton $\mathcal{C}$ on $\mathbb{P}^3$ there exists a regular rank r perverse instanton $\mathcal{C}'$ and a rank 0 perverse instanton $\mathcal{C}''$ such that we have a distinguished triangle

$$\mathcal{C}'' \to \mathcal{C} \to \mathcal{C}' \to \mathcal{C}''[1].$$

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E rank $r > 0$ instanton on $\mathbb{P}^3$.
There exists a unique regular instanton E' such that

$$0 \to E \to E' \to E'' \to 0,$$

where E'' is a rank 0 instanton (needs definition!).

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Partial Donaldson–Uhlenbeck compactification

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$$\mathcal{M}_0(\mathbb{P}^3; r, c) = \bigsqcup_{0 \le d \le c} \mathcal{M}_0^{\mathrm{reg}}(\mathbb{P}^3; r, c-d) \times \mathcal{M}_0(\mathbb{P}^3; 0, d).$$

The natural morphism

$$\mathcal{M}(\mathbb{P}^3; r, c) \simeq \tilde{\mu}^{-1}(0)/\!\!/_{\chi} G \to \tilde{\mu}^{-1}(0)/G \simeq \mathcal{M}_0(\mathbb{P}^3; r, c)$$

coming from GIT can be described as

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Definition

A *rank* 0 *instanton* E on $\mathbb{P}^3$ is a pure sheaf of dimension 1 such that $H^0(\mathbb{P}^3, E(-2)) = 0$ and $H^1(\mathbb{P}^3, E(-2)) = 0$.

Lemma

$\mathcal{C} \in D^b(\mathbb{P}^3)$ is a rank 0 perverse instanton if and only if $\mathcal{C}[1]$ is a rank 0 instanton.

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Moduli interpretation for instantons of rank 0.

$\mathcal{M}_0(\mathbb{P}^3; 0, c) = \tilde{\mu}^{-1}(0)/\operatorname{GL}(V)$ is isomorphic to the scheme Q_R^c of equivalence classes of c-dimensional R-modules for a non-commutative k-algebra

$$R = k\langle y_1, y_2, z_1, z_2\rangle/(y_1y_2 - y_2y_1, z_1z_2 - z_2z_1, y_1z_2 - z_2y_1 + y_2z_1 - z_1y_2).$$

$$I = (y_1z_2 - z_2y_1, y_1z_1 - z_1y_1, y_2z_2 - z_2y_2).$$

We have surjection $R \to R' = k[y_1, y_2, z_1, z_2]$ inducing

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This is a closed embedding in char zero.

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Reducibility of the moduli space

We have a set-theoretical injection:

$$\varphi : S^c\mathbb{A}^4 \to Q_R^c \simeq \mathcal{M}_0(\mathbb{P}^3; 0, c).$$

$\mathbb{A}^4$ — lines $l_1, \ldots, l_c$ in $\mathbb{P}^3$ not intersecting l_∞
point in $S^c\mathbb{A}^4$ corresponds to $E = \mathcal{O}_{l_1}(1) \oplus \cdots \oplus \mathcal{O}_{l_c}(1)$

The image of φ is an irreducible component of $\mathcal{M}_0(\mathbb{P}^3; 0, c)$ of dimension $4c$.

$\mathcal{M}_0(\mathbb{P}^3; 0, 2)$ has two 8-dimensional irreducible components intersecting along a 7-dimensional variety.

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Reducibility of the moduli space

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$$\varphi : S^c\mathbb{A}^4 \to Q_R^c \simeq \mathcal{M}_0(\mathbb{P}^3; 0, c).$$

$\mathbb{A}^4$ — lines $l_1, \ldots, l_c$ in $\mathbb{P}^3$ not intersecting l_∞
point in $S^c\mathbb{A}^4$ corresponds to $E = \mathcal{O}_{l_1}(1) \oplus \cdots \oplus \mathcal{O}_{l_c}(1)$

The image of φ is an irreducible component of $\mathcal{M}_0(\mathbb{P}^3; 0, c)$ of dimension $4c$.

$\mathcal{M}_0(\mathbb{P}^3; 0, 2)$ has two 8-dimensional irreducible components intersecting along a 7-dimensional variety.

Relation between different notions of stability

Lemma

If a generalized ADHM datum is FJ-semistable then it is also stable.

Implication in the opposite direction is false starting from $r = 1$, $c = 2$

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Outline

$(\mathcal{C}, \Phi)$ stable framed perverse instanton corresponding to $x \in \tilde{\mathbf{B}}$
$\varphi : G \to \tilde{\mathbf{B}}$, $g \to gx$, the orbit map

$$0 \to K^0 = \mathfrak{g} \xrightarrow{d\varphi_e} K^1 = T_x\tilde{\mathbf{B}} \xrightarrow{d\tilde{\mu}_x} K^2 = T_0(\mathrm{End}(V) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2))) \to 0$$

Theorem

$H^i(K) = 0$ *for* $i \neq 1, 2$
$H^1(K) = \mathrm{Ext}^1(\mathcal{C}, J_{l_\infty} \otimes \mathcal{C})$
$H^2(K) = \mathrm{Ext}^2(\mathcal{C}, \mathcal{C})$

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Frenkel–Jardim's conjecture

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The moduli space of framed instantons is smooth and irreducible.

This is false: take E

$$0 \to E \longrightarrow \mathcal{O}_{\mathbb{P}^3}^4 \xrightarrow{\tilde{\varphi}} \mathcal{O}_m(1)^3 \to 0,$$

where $m := (x_0 = x_1 = 0)$ and $\tilde{\varphi}$ comes from

$$\varphi = \begin{pmatrix} x_2 & x_3 & 0 & 0 \\ 0 & x_2 & x_3 & 0 \\ 0 & 0 & x_2 & x_3 \end{pmatrix}.$$

Then $\dim \mathrm{Ext}^2(E, E) = 3$ and E comes from FJ-stable ADHM datum.

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Smoothness

Theorem

Let E_i be a locally free (r_i, c_i)-instanton on $\mathbb{P}^3$, where $i = 1, 2$. If $r_1, r_2 \leq 3$ and $c_1 c_2 \leq 6$ then $\mathrm{Ext}^2(E_1, E_2) = 0$.

Corollary

For $r \leq 3$ and $c \leq 2$ the moduli space of framed locally free (r, c)-instantons is smooth of dimension $4cr$.

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For $r \leq 3$ and $c \leq 2$ the moduli space of framed locally free (r, c)-instantons is smooth of dimension $4cr$.

$M_{\mathbb{P}^3}(r, c)$ the moduli space of Gieseker stable loc. free (r, c)-instantons on $\mathbb{P}^3$

$S \subset \mathbb{P}^3$ a smooth quartic with $\mathrm{Pic}\, S = \mathbb{Z}$.

$M_S(r, 4c)$ the moduli space of rk r slope stable v. b. on S with $c_1 = 0$ and $c_2 = c \cdot h^2|_S = 4c$

Using idea of A. Tyurin

Theorem

Let $r \leq 3$ and $c \leq 2$. Then $M_{\mathbb{P}^3}(r, c)$ is smooth and the restriction $r : M_{\mathbb{P}^3}(r, c) \to M_S(r, 4c)$ is a morphism which induces an isomorphism of $M_{\mathbb{P}^3}(r, c)$ onto a Lagrangian submanifold of $M_S(r, 4c)$.

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