Linear monads and instanton bundles on hyperquadrics

Laura Costa

March 2011

Join work with Rosa Maria Miró-Roig

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- Monads appear in a wide variety of context.
- Linear monads as a tool for constructing indecomposable vector bundles on hyperquadrics.
- Instanton bundles

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Notation:

- We will work over $\mathbb{C}$.
- Let $Q_n \subset \mathbb{P}^{n+1} = \mathbb{P}(V^\vee)$, $n > 2$, be a smooth quadric hypersurface.
- It is well known that

$$\mathrm{Pic}(Q_n) \cong \mathbb{Z} \quad \text{and} \quad \omega_{Q_n} \cong \mathcal{O}_{Q_n}(-n).$$

- Set $\Omega^j := \Omega^j_{\mathbb{P}^{n+1}}$ and we define inductively ψ_j:

$$\psi_0 := \mathcal{O}_{Q_n}, \quad \psi_1 := \Omega^1(1)_{|Q_n}$$

and, for all $j \geq 2$, we define ψ_j as

$$0 \longrightarrow \Omega^j(j)_{|Q_n} \longrightarrow \psi_j \longrightarrow \psi_{j-2} \longrightarrow 0.$$

- In particular we have the exact sequence:

$$0 \to \psi_1 \to \mathcal{O}_{Q_n} \otimes V^\vee \to \mathcal{O}_{Q_n} \to 0.$$

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Definition: Let X be a smooth projective variety. A **monad** on X is a complex of vector bundles:

$$M_\bullet : \quad F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

which is exact at F and at H. The sheaf

$$E := Ker(\beta)/Im(\alpha)$$

is called the **cohomology sheaf** of the monad $M_\bullet$.

A monad $M_\bullet : F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ has a so-called display:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \xrightarrow{\alpha} & G & \longrightarrow & Q & \longrightarrow & 0 \\
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From the display of a monad $M_\bullet$ one easily computes the rank and the Chern character of its cohomology sheaf. We have

(i) $rk(E) = rk(G) - rk(F) - rk(H)$, and

(ii) $c_t(E) = c_t(G)c_t(F)^{-1}c_t(H)^{-1}$.

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Monads were first introduced by Horrocks who showed that all vector bundles E on $\mathbb{P}^3$ can be obtained as the cohomology of

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \longrightarrow \oplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \longrightarrow \oplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0.$$

Monads appeared in a wide variety of contexts within algebraic-geometry, like the construction of locally free sheaves.

GOAL:

Linear monads

$$A \otimes \mathcal{O}_{Q_n}(-1) \to B \otimes \mathcal{O}_{Q_n} \to C \otimes \mathcal{O}_{Q_n}(1)$$

on $Q_n \subset \mathbb{P}^{n+1}$ where A, B and C are vector spaces of dimension a, b and c respectively.

Proposition:

Let $n \geq 3$. There exist monads on Q_n whose entries are linear maps:

$$\mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{Q_n}^b \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c$$

if and only if at least one of the following conditions holds:

(1) $b \geq 2c + n - 1$ and $b \geq a + c$.

(2) $b \geq a + c + n$.

If so, there actually exists a monad with the map α such that α_x is injective for all $x \in X$.

Sketch of the Proof:

Existence part: We may assume that Q_n is the quadric hypersurface in $\mathbb{P}^{n+1}$ defined by $x_0^2 + x_1^2 + \cdots + x_{n+1}^2 = 0$.

By Floystad, if $b \geq 2c + n$ and $b \geq a + c$ or $b \geq a + c + n + 1$ then there exist

$$\mathcal{O}_{\mathbb{P}^{n+1}}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^{n+1}}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^{n+1}}(1)^c \qquad (1)$$

with the map α such that α_x is injective for all $x \in X$.
So, restricting a general monad (1) to Q_n we get a monad

$$\mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{Q_n}^b \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c$$

with the map α such that α_x is injective for all $x \in X$.
So, it is enough to consider the cases

(a) $b = 2c + n - 1$ and $b \geq a + c$.

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Case $b = 2c + n - 1$ and $b = a + c$.

Set $n_1 = \frac{n-1}{2}$ if n is odd and $n_1 = \frac{n-2}{2}$ if n is even. Consider the $(n_1 + c) \times c$, $(n - 1 - n_1 + c) \times c$, $(n - 1 + c) \times (n - 1 - n_1 + c)$ and $(n - 1 + c) \times (n_1 + c)$ matrices

$$A_1 = \begin{pmatrix} x_0 & x_1 & \dots & \dots & x_{n_1} & 0 & 0 & \dots & \dots & 0 \\ 0 & x_0 & x_1 & \dots & \dots & x_{n_1} & 0 & 0 & \dots & 0 \\ \dots. & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x_0 & x_1 & \dots & \dots & x_{n_1} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n & 0 & 0 & \dots & \dots & 0 \\ 0 & x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n & 0 & 0 & \dots & 0 \\ \dots. & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n \end{pmatrix}$$

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Define the complex

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{Q_n}^b \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c \longrightarrow 0 \qquad (2)$$

where β is the map given by the matrix $B = (A_1 \quad A_2)$ and α is the map given by

$$A = \begin{pmatrix} A_2 \\ -A_1 \end{pmatrix}.$$

It is not difficult to see that α is such that α_x is injective for all $x \in X$.

Necessary Conditions: Pursuing the ideas developed by Fløystead and changing the role of $\mathbb{P}^n$ by Q_n we get that the numerical conditions on a, b, c and n are indeed necessary.

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Using linear monads to construct indecomposable vector bundles on Q_n, the following come up

Questions:

- How are the vector bundles obtained as cohomology sheaves of linear monads? Are they simple ? stable ?
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Definition:

Let X be a smooth projective variety and $\sigma = (F_0, \cdots, F_r)$ a collection of vector bundles on X. A vector bundle F on X has natural cohomology with respect to σ if for all i, at most one

$$H^q(X, F \otimes F_i)$$

is different from 0.

Use Σ_* meaning that for even n both Spinor bundles Σ_1 and Σ_2 are considered, and for odd n, the Spinor bundle Σ.

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Use Σ_* meaning that for even n both Spinor bundles Σ_1 and Σ_2 are considered, and for odd n, the Spinor bundle Σ.

Proposition:

Let $Q_n \subset \mathbb{P}^{n+1}$ be a quadric hypersurface and let E be a rank $b-a-c$ torsion free sheaf on Q_n with Chern polynomial $c_t(E) = \frac{1}{(1-e_1 t)^a (1+e_1 t)^b}$. It holds:

(a) If $b - c(n+2) < 0$, E has natural cohomology with respect to

$$\sigma = (\Sigma_*(-n), \mathcal{O}_{Q_n}(-n+1), \cdots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

and $H^i(Q_n, E \otimes \Sigma_*(-n+1)) = 0$ for all $i \geq 0$, then E is the cohomology bundle of a linear monad of the following type

$$S_\bullet : \quad \mathcal{O}_{Q_n}(-1)^a \longrightarrow \mathcal{O}_{Q_n}^b \longrightarrow \mathcal{O}_{Q_n}(1)^c.$$

Proposition:

(b) If E is the cohomology bundle of a linear monad of the following type

$$S_\bullet : \quad \mathcal{O}_{Q_n}(-1)^a \longrightarrow \mathcal{O}_{Q_n}^b \longrightarrow \mathcal{O}_{Q_n}(1)^c$$

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and $H^i(Q_n, E \otimes \Sigma_*(-n+1)) = 0$ for all $i \geq 0$.

Definition

A **mathematical instanton bundle** on Q_{2l+1} with quantum number k is a rank $2l$ vector bundle E on Q_{2l+1} with trivial splitting type (i.e. for a general line $L \subset Q_{2l+1}$ we have $E_{|L} \cong {\mathcal{O}_L}^{2l}$) and defined as the cohomology bundle of a monad

$$S_\bullet : \mathcal{O}_{Q_{2l+1}}(-1)^k \xrightarrow{A} \mathcal{O}_{Q_{2l+1}}^{2k+2l} \xrightarrow{B^t} \mathcal{O}_{Q_{2l+1}}(1)^k$$

where A and B are $k \times (2l + 2k)$ matrices with linear entries

Remark:

- The fact that $S_\bullet$ is a monad is equivalent to the following conditions on A, B

  (i) A, B have rank k at every point of Q_{2l+1},
  (ii) $AB^t = 0$.

- Let $L \subset Q_{2l+1}$ be a line joining the points $p \neq q \in Q_{2l+1}$. Then

$$E_{|L} \cong \mathcal{O}_L^{2l} \Leftrightarrow B(q)A(p) \quad \text{is invertible}$$

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Example:

Let $f \in \mathbb{C}[x_0, \cdots, x_{2l+2}]$ defining Q_{2l+1} be $f = x_0^2 + x_1^2 + \cdots + x_{2l+2}^2$. Consider the $k \times (l + k)$ matrices

$$A_1 = \begin{pmatrix} x_0 & x_1 & \dots & \dots & x_l & 0 & 0 & \dots & \dots & 0 \\ 0 & x_0 & x_1 & \dots & \dots & x_l & 0 & 0 & \dots & 0 \\ 0 & 0 & x_0 & x_1 & \dots & \dots & x_l & 0 & \dots & 0 \\ \dots. & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x_0 & x_1 & \dots & \dots & x_l \end{pmatrix}$$

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Let $\alpha : \mathcal{O}_{Q_{2l+1}}(-1)^k \longrightarrow \mathcal{O}^{2l+2k}_{Q_{2l+1}}$ be associated to

$$A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$$

and let $\beta : \mathcal{O}^{2k+2l}_{Q_{2l+1}} \longrightarrow \mathcal{O}_{Q_{2l+1}}(1)^k$ be associated to $B = A^t$, transpose with respect to the standard symplectic form

$$G := \begin{pmatrix} 0 & -1_{k+1} \\ 1_{k+1} & 0 \end{pmatrix}.$$

Since the localized maps α_x are injective for all $x \in Q_{2l+1}$, the cohomology sheaf of the monad

$$S_\bullet : \quad 0 \longrightarrow \mathcal{O}_{Q_{2l+1}}(-1)^k \xrightarrow{\alpha} \mathcal{O}^{2k+2l}_{Q_{2l+1}} \xrightarrow{\beta} \mathcal{O}_{Q_{2l+1}}(1)^k \longrightarrow 0$$

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Corollary:

Any instanton bundle E on Q_{2l+1} with quantum number k satisfies:

(i) $c_t(E) = \frac{1}{(1-e_1 t)^k (1+e_1 t)^k}$,

(ii) E has natural cohomology with respect to

$$\sigma = (\Sigma(-2l-1), \mathcal{O}_{Q_{2l+1}}(-2l), \cdots, \mathcal{O}_{Q_{2l+1}}(-1), \mathcal{O}_{Q_{2l+1}})$$

and $H^i(Q_{2l+1}, E \otimes \Sigma(-2l)) = 0$ for all $i \geq 0$,

(iii) E has trivial splitting type.

Conversely, any rank $2l$ vector bundle E on Q_{2l+1} verifying the conditions (i), (ii) and (iii) is an instanton bundle E on Q_{2l+1}.

Definition:

For a torsion free sheaf F on Q_{2l+1} we set

$$\mu(F) = \frac{c_1(F)}{rk(F)}.$$

The sheaf F is said to be **semistable** if

$$\mu(E) \leq \mu(F)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then F is **stable.**

Notation: Let E be a rank r vector bundle on Q_n. We set $E_{norm} := E(k_E)$ where k_E is the unique integer such that $c_1(E(k_E)) \in \{-r+1, \cdots, 0\}$.

Proposition: (Hoppe's criterion)

Let E be a rank r locally-free sheaf on Q_n. We have:

(a) If $H^0(X, (\Lambda^q E)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq r-1$, then E is stable.

(b) If $H^0(X, (\Lambda^q E)_{\mathrm{norm}}(-1)) = 0$ for $1 \leq q \leq r-1$, then E is semistable.

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Proposition:

Any instanton bundle E on Q_{2l+1} is simple and

$$H^0(Q_{2l+1}, E) = 0.$$

Sketch of the Proof: Let

$$S_\bullet : \mathcal{O}_{Q_{2l+1}}(-1)^k \xrightarrow{A} \mathcal{O}_{Q_{2l+1}}^{2k+2l} \xrightarrow{B^t} \mathcal{O}_{Q_{2l+1}}(1)^k$$

be the monad associated to E and the exact sequences:

$$0 \longrightarrow K = \ker(B^t) \longrightarrow \mathcal{O}_{Q_{2l+1}}^{2k+2l} \longrightarrow \mathcal{O}_{Q_{2l+1}}(1)^k \longrightarrow 0, \quad \text{and} \quad (3)$$

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Using Hoppe's criterion we prove that K is stable.

We have the exact sequence

$$0 \to E^* \otimes E \to K^* \otimes E \to E(1)^k \to 0$$

from which we deduce that $h^0(E^* \otimes E) \leq h^0(K^* \otimes E)$. Since $h^0 K^*(-1) = h^1 K^*(-1) = 0$, from the exact sequence

$$0 \to K^*(-1) \to K^* \otimes K \to K^* \otimes E \to 0$$

we get

$$h^0(K^* \otimes E) = h^0(K^* \otimes K) = 1$$

where the last equality follows from the fact that K is stable and hence simple.
Thus,

$$h^0(Q_{2l+1}, E \otimes E^*) = 1$$

or, equivalently, E is simple.

Since K is stable and $c_1(K) = -k < 0$, there exists $\lambda \geq 0$ such that

$$H^0(Q_{2l+1}, K_{norm}) = H^0(Q_{2l+1}, K \otimes \mathcal{O}_{Q_{2l+1}}(\lambda)) = 0$$

and we get

$$H^0(Q_{2l+1}, E) = H^0(Q_{2l+1}, K) = 0$$

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Definition:

A vector bundle E is said to be symplectic if there exists an isomorphism $\phi : E \to E^*$ such that $\phi^* = -\phi$.

Remark:

$$E \quad \text{is symplectic} \quad \text{iff} \quad H^0(\bigwedge^2 E) \neq 0.$$

Lemma:

Let E be a symplectic vector bundle on a projective variety X with $Pic(X) \cong \mathbb{Z}$ such that for q odd, $1 \leq q \leq \frac{rk(E)}{2}$ the following hold:
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Proposition:

Any instanton bundle on Q_3 and on Q_5 is stable.

Sketch of the Proof: Any rank two vector bundle on Q_n is stable if it is simple. Hence, any instanton bundle on Q_3 is stable. Let E be an instanton bundle on Q_5. If it is symplectic, it is stable by the Lemma.
Assume E is not symplectic. Since $(\bigwedge^q E)_{\text{norm}} = \bigwedge^q E$, $\bigwedge^3 E \cong E^*$, $\bigwedge^2 E \cong \bigwedge^2 E^*$, we have

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Is $MI_{Q_{2l+1}}(k)$ irreducible? smooth? rational? Which is its dimension?

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Questions:
Is $MI_{Q_{2l+1}}(k)$ irreducible? smooth? rational? Which is its dimension?

Theorem:

The moduli space $MI_{Q_{2l+1}}(1)$ is smooth and irreducible of dimension $2l^2 + 5l + 2$.

Sketch of the proof: We know that any $E \in MI_{Q_{2l+1}}(1)$ is stable. Hence,

$$\dim_{[E]} MI_{Q_{2l+1}}(1) = \dim T_{[E]} MI_{Q_{2l+1}}(1) = \dim Ext^1(E, E)$$

and if $Ext^2(E, E) = 0$, then the moduli space is smooth at E.

Claim: For any $E \in MI_{Q_{2l+1}}(1)$, $h^2(E \otimes E^*) = 0$.

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Claim: For any $E \in MI_{Q_{2l+1}}(1)$, $h^2(E \otimes E^*) = 0$.

Indeed, let $E \in MI_{Q_{2l+1}}(1)$ given by a monad $M_\bullet$ and consider its display

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{Q_{2l+1}}^{2+2l} \longrightarrow \mathcal{O}_{Q_{2l+1}}(1) \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow \mathcal{O}_{Q_{2l+1}}(-1) \longrightarrow K \longrightarrow E \longrightarrow 0. \tag{6}$$

From the exact sequence

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Dualize (6) and tensor by K to get

$$0 \longrightarrow E^* \otimes K \longrightarrow K \otimes K^* \longrightarrow K(1) \longrightarrow 0$$

and hence

$$H^1(K(1)) \longrightarrow H^2(E^* \otimes K) \longrightarrow H^2(K \otimes K^*).$$

From (5) we deduce that $H^2(K \otimes K^*) = 0$ and from the fact that $K(1) \cong \bigwedge^{2l} K^*$ we get $H^1(K(1)) = 0$. Thus

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By Riemann-Roch or using the display of the monad we obtain

$$\dim MI_{Q_{2l+1}}(1) = \dim Ext^1(E, E) = 2l^2 + 5l + 2.$$

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Finally we establish a dominant map

$$\Pi : MI_{Q_{2l+1}}(1) \to M$$

where M is a moduli space of stable vector bundles that contain the kernel bundles K.
From this we deduce the irreducibility of $MI_{Q_{2l+1}}(1)$

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Example: Let E_0 be an instanton bundle on Q_5 given as the cohomology of the monad

$$\mathcal{O}_{Q_5}(-1)^2 \to \mathcal{O}_{Q_5}^8 \to \mathcal{O}_{Q_5}(1)^2$$

defined by the matrices

$$A_0 = \begin{pmatrix} 0 & f & e & d & 0 & -c & -b & -a \\ e & d & 0 & 2f & -b & -a & 0 & -2c \end{pmatrix}$$

and

$$B^t = \begin{pmatrix} a & b & c & 0 & d & e & f & 0 \\ 0 & a & b & c & 0 & d & e & f \end{pmatrix}$$

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and

$$B^t = \begin{pmatrix} a & b & c & 0 & d & e & f & 0 \\ 0 & a & b & c & 0 & d & e & f \end{pmatrix}.$$

Using Macaulay we get

$$h^2(E_0 \otimes E_0^*) = 0$$

Thus $MI_{Q_5}(2)$ is smooth at $[E_0]$ of dimension $ext^1(E_0, E_0) = 45$.
On the other hand,

$$h^2(E \otimes E^*) = 2 \quad \text{and} \quad ext^1(E, E) = 47 > 45.$$

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Conjecture:

For any $k \geq 1$, the moduli space of instanton bundles $MI_{Q_3}(k)$ is smooth, irreducible of dimension $12k - 3$.