Spherical and hyperbolic 2-spheres with cone singularities

Workshop “Hyperbolic geometry and dynamics”

May 16–20, 2016, Bogomolov lab, HSE, Moscow, Russia

Sasha Anan′in

Joint work in progress with Carlos H. Grossi, Jaejeong Lee, and João dos Reis jr.

ICMC, USP, São Carlos

May 17, 2016

2-sphere and holomorphic 1- and 2-balls

2-sphere and holomorphic 1- and 2-balls

Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -, - \rangle$ of signature $++$ or $+-$ or $+--$.

2-sphere and holomorphic 1- and 2-balls

Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $++$ or $+-$ or $+--$. Then

$$\mathrm{B}\,V := \big\{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle > 0\big\},\quad \mathrm{S}\,V := \big\{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle = 0\big\}$$

are the **round** 2-**sphere** or the **holomorphic** 1- or 2-**ball** and its **ideal boundary**, i.e., the **absolute** (empty in the case of the round 2-sphere).

2-sphere and holomorphic 1- and 2-balls

Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $++$ or $+-$ or $+--$. Then

$$\mathrm{B}\,V := \{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle > 0\}, \quad \mathrm{S}\,V := \{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle = 0\}$$

are the **round** 2-**sphere** or the **holomorphic** 1- or 2-**ball** and its **ideal boundary**, i.e., the **absolute** (empty in the case of the round 2-sphere).

At points $p \in \mathbb{P}_{\mathbb{C}} V \setminus \mathrm{S}\,V$, we have a **(pseudo-)hermitian metric** induced by the trace function on $\mathrm{Lin}_{\mathbb{C}}(V,V)$

$$\mathrm{T}_p\mathbb{P}_{\mathbb{C}} V = \mathrm{Lin}_{\mathbb{C}}(p, V/p) \simeq \mathrm{Lin}_{\mathbb{C}}(p, p^{\perp}) \leqslant \mathrm{Lin}_{\mathbb{C}}(V,V).$$

2-sphere and holomorphic 1- and 2-balls

Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $++$ or $+-$ or $+--$. Then

$$\mathrm{B}\,V := \big\{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle > 0\big\}, \quad \mathrm{S}\,V := \big\{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle = 0\big\}$$

are the **round** 2-**sphere** or the **holomorphic** 1- or 2-**ball** and its **ideal boundary**, i.e., the **absolute** (empty in the case of the round 2-sphere).

At points $p \in \mathbb{P}_{\mathbb{C}} V \setminus \mathrm{S}\,V$, we have a **(pseudo-)hermitian metric** induced by the trace function on $\mathrm{Lin}_{\mathbb{C}}(V,V)$

$$\mathrm{T}_p \mathbb{P}_{\mathbb{C}} V = \mathrm{Lin}_{\mathbb{C}}(p, V/p) \simeq \mathrm{Lin}_{\mathbb{C}}(p, p^{\perp}) \leqslant \mathrm{Lin}_{\mathbb{C}}(V,V).$$

The group $\mathrm{Aut}^{\mathrm{hol}}\,\mathrm{B}\,V = \mathrm{PU}\,V = \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V \leqslant \mathrm{Isom}\,\mathrm{B}\,V$ is an index 2 subgroup.

2-sphere and holomorphic 1- and 2-balls

Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $++$ or $+-$ or $+--$. Then

$$\mathrm{B}\,V := \{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle > 0\}, \quad \mathrm{S}\,V := \{p \in \mathbb{P}_{\mathbb{C}} V \mid \langle p,p\rangle = 0\}$$

are the **round 2-sphere** or the **holomorphic 1- or 2-ball** and its **ideal boundary**, i.e., the **absolute** (empty in the case of the round 2-sphere).

At points $p \in \mathbb{P}_{\mathbb{C}} V \setminus \mathrm{S}\,V$, we have a **(pseudo-)hermitian metric** induced by the trace function on $\mathrm{Lin}_{\mathbb{C}}(V,V)$

$$\mathrm{T}_p\mathbb{P}_{\mathbb{C}} V = \mathrm{Lin}_{\mathbb{C}}(p, V/p) \simeq \mathrm{Lin}_{\mathbb{C}}(p, p^{\perp}) \leqslant \mathrm{Lin}_{\mathbb{C}}(V,V).$$

The group $\mathrm{Aut}^{\mathrm{hol}}\,\mathrm{B}\,V = \mathrm{PU}\,V = \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V \leqslant \mathrm{Isom}\,\mathrm{B}\,V$ is an index 2 subgroup. (In the case $+-$, we get the Riemann sphere glued from two Poincaré discs.)

2-sphere and holomorphic 1- and 2-balls

Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $++$ or $+-$ or $+--$. Then

$$\mathrm{B}\,V := \{p \in \mathbb{P}_{\mathbb{C}}V \mid \langle p,p\rangle > 0\}, \quad \mathrm{S}\,V := \{p \in \mathbb{P}_{\mathbb{C}}V \mid \langle p,p\rangle = 0\}$$

are the **round** 2-**sphere** or the **holomorphic** 1- or 2-**ball** and its **ideal boundary**, i.e., the **absolute** (empty in the case of the round 2-sphere).

At points $p \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}\,V$, we have a **(pseudo-)hermitian metric** induced by the trace function on $\mathrm{Lin}_{\mathbb{C}}(V,V)$

$$\mathrm{T}_p\mathbb{P}_{\mathbb{C}}V = \mathrm{Lin}_{\mathbb{C}}(p, V/p) \simeq \mathrm{Lin}_{\mathbb{C}}(p, p^{\perp}) \leqslant \mathrm{Lin}_{\mathbb{C}}(V,V).$$

The group $\mathrm{Aut}^{\mathrm{hol}}\,\mathrm{B}\,V = \mathrm{PU}\,V = \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V \leqslant \mathrm{Isom}\,\mathrm{B}\,V$ is an index 2 subgroup. (In the case $+-$, we get the Riemann sphere glued from two Poincaré discs.)

The **geodesics** have the form $\mathbb{P}_{\mathbb{C}}W$, where $V \geqslant W$ is a 2-dimensional $\mathbb{R}$-linear subspace such that $0 \neq \langle W,W\rangle \subset \mathbb{R}$.

2-sphere and holomorphic 1- and 2-balls

Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $++$ or $+-$ or $+--$. Then

$$\mathrm{B}\,V := \big\{p \in \mathbb{P}_{\mathbb{C}}V \mid \langle p,p\rangle > 0\big\},\quad \mathrm{S}\,V := \big\{p \in \mathbb{P}_{\mathbb{C}}V \mid \langle p,p\rangle = 0\big\}$$

are the **round 2-sphere** or the **holomorphic 1-** or **2-ball** and its **ideal boundary**, i.e., the **absolute** (empty in the case of the round 2-sphere).

At points $p \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}\,V$, we have a **(pseudo-)hermitian metric** induced by the trace function on $\mathrm{Lin}_{\mathbb{C}}(V,V)$

$$\mathrm{T}_p\mathbb{P}_{\mathbb{C}}V = \mathrm{Lin}_{\mathbb{C}}(p, V/p) \simeq \mathrm{Lin}_{\mathbb{C}}(p, p^{\perp}) \leqslant \mathrm{Lin}_{\mathbb{C}}(V,V).$$

The group $\mathrm{Aut}^{\mathrm{hol}}\,\mathrm{B}\,V = \mathrm{PU}\,V = \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V \leqslant \mathrm{Isom}\,\mathrm{B}\,V$ is an index 2 subgroup. (In the case $+-$, we get the Riemann sphere glued from two Poincaré discs.)

The **geodesics** have the form $\mathbb{P}_{\mathbb{C}}W$, where $V \geqslant W$ is a 2-dimensional $\mathbb{R}$-linear subspace such that $0 \neq \langle W,W\rangle \subset \mathbb{R}$. The distance is a monotonic function of the **tance** $\mathrm{ta}(q_1,q_2) := \frac{\langle q_1,q_2\rangle\langle q_2,q_1\rangle}{\langle q_1,q_1\rangle\langle q_2,q_2\rangle}$.

Crash survey on compact 2-ball quotients

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $+--$.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -, - \rangle$ of signature $+ - -$. A lattice $L \leqslant \mathrm{SU}\, V$ is **arithmetic of the first type** iff

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -, - \rangle$ of signature $+--$. A lattice $L \leqslant \mathrm{SU}\, V$ is **arithmetic of the first type** iff $\langle -, - \rangle$ is defined over some quadratic imaginary extension F of a totally real number field $R \leqslant \mathbb{R}$,

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $+--$. A lattice $L \leqslant \mathrm{SU}\, V$ is **arithmetic of the first type** iff $\langle -,-\rangle$ is defined over some quadratic imaginary extension F of a totally real number field $R \leqslant \mathbb{R}$, the hermitian form $\langle -,-\rangle^g$ is definite for any embedding $g : F \hookrightarrow \mathbb{C}$ that is not the identity on R,

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $+--$. A lattice $L \leqslant \mathrm{SU}\, V$ is **arithmetic of the first type** iff $\langle -,-\rangle$ is defined over some quadratic imaginary extension F of a totally real number field $R \leqslant \mathbb{R}$, the hermitian form $\langle -,-\rangle^g$ is definite for any embedding $g : F \hookrightarrow \mathbb{C}$ that is not the identity on R, and $L \leqslant \mathrm{GL}\,\Lambda$, where Λ is a free $\mathcal{O}_F$-submodule in V such that $\mathbb{C} \otimes_{\mathcal{O}_F} \Lambda = V$ and $\mathcal{O}_F$ stands for the ring of all integers of F.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $+--$. A lattice $L \leqslant \mathrm{SU}\,V$ is **arithmetic of the first type** iff $\langle -,-\rangle$ is defined over some quadratic imaginary extension F of a totally real number field $R \leqslant \mathbb{R}$, the hermitian form $\langle -,-\rangle^g$ is definite for any embedding $g : F \hookrightarrow \mathbb{C}$ that is not the identity on R, and $L \leqslant \mathrm{GL}\,\Lambda$, where Λ is a free $\mathcal{O}_F$-submodule in V such that $\mathbb{C} \otimes_{\mathcal{O}_F} \Lambda = V$ and $\mathcal{O}_F$ stands for the ring of all integers of F. (In fact, we took Mostow-Vinberg criterion as a definition.) For cocompactness, one requires that L contains no unipotent elements.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $+--$. A lattice $L \leqslant \mathrm{SU}\, V$ is **arithmetic of the first type** iff $\langle -,-\rangle$ is defined over some quadratic imaginary extension F of a totally real number field $R \leqslant \mathbb{R}$, the hermitian form $\langle -,-\rangle^g$ is definite for any embedding $g : F \hookrightarrow \mathbb{C}$ that is not the identity on R, and $L \leqslant \mathrm{GL}\,\Lambda$, where Λ is a free $\mathcal{O}_F$-submodule in V such that $\mathbb{C} \otimes_{\mathcal{O}_F} \Lambda = V$ and $\mathcal{O}_F$ stands for the ring of all integers of F. (In fact, we took Mostow-Vinberg criterion as a definition.) For cocompactness, one requires that L contains no unipotent elements.

1.2. Arithmetic lattices of the second type are related to division algebras.

Crash survey on compact 2-ball quotients

1.0. Up to finite cover, such quotients are $\mathbb{C}$-surfaces of general type, hence, projective and algebraic. Clearly, each surface possesses plenty of smooth curves. Every quotient is an orbifold, known to be rigid. In what follows, we consider cocompact lattices up to commensurability.

1.1. Arithmetic lattices of the first type. Let V be a $\mathbb{C}$-linear space endowed with a hermitian form $\langle -,-\rangle$ of signature $+--$. A lattice $L \leqslant \mathrm{SU}\, V$ is **arithmetic of the first type** iff $\langle -,-\rangle$ is defined over some quadratic imaginary extension F of a totally real number field $R \leqslant \mathbb{R}$, the hermitian form $\langle -,-\rangle^g$ is definite for any embedding $g : F \hookrightarrow \mathbb{C}$ that is not the identity on R, and $L \leqslant \mathrm{GL}\,\Lambda$, where Λ is a free $\mathcal{O}_F$-submodule in V such that $\mathbb{C} \otimes_{\mathcal{O}_F} \Lambda = V$ and $\mathcal{O}_F$ stands for the ring of all integers of F. (In fact, we took Mostow-Vinberg criterion as a definition.) For cocompactness, one requires that L contains no unipotent elements.

1.2. Arithmetic lattices of the second type are related to division algebras. All we need to know about them is that they cannot contain a $\mathbb{C}$-fuchsian subgroup (defined later).

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples).

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one.

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \dots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed.

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \dots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$.

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \ldots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.)

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \dots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.) The vertices of P (complex numbers) define complex coordinates, another cut provides a $\mathbb{C}$-linear change of coordinates,

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \ldots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.) The vertices of P (complex numbers) define complex coordinates, another cut provides a $\mathbb{C}$-linear change of coordinates, and the area of a polygon can be calculated as $\langle p, p \rangle$, where the hermitian form $\langle -, - \rangle$ has signature $(1, n-1)$.

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \dots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.) The vertices of P (complex numbers) define complex coordinates, another cut provides a $\mathbb{C}$-linear change of coordinates, and the area of a polygon can be calculated as $\langle p, p \rangle$, where the hermitian form $\langle -, - \rangle$ has signature $(1, n-1)$. In this manner, $C(a_1, \dots, a_n)$ obtains the geometry of a holomorphic $(n-3)$-ball. It is smooth, but incomplete. The completion can be achieved by colliding each group of cone points with sum of curvatures $k_i := 2\pi - a_i$ less than 2π;

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \ldots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.) The vertices of P (complex numbers) define complex coordinates, another cut provides a $\mathbb{C}$-linear change of coordinates, and the area of a polygon can be calculated as $\langle p, p\rangle$, where the hermitian form $\langle -, -\rangle$ has signature $(1, n-1)$. In this manner, $C(a_1, \ldots, a_n)$ obtains the geometry of a holomorphic $(n-3)$-ball. It is smooth, but incomplete. The completion can be achieved by colliding each group of cone points with sum of curvatures $k_i := 2\pi - a_i$ less than 2π; the curvatures sum at the collision.

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \dots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.) The vertices of P (complex numbers) define complex coordinates, another cut provides a $\mathbb{C}$-linear change of coordinates, and the area of a polygon can be calculated as $\langle p, p\rangle$, where the hermitian form $\langle -, -\rangle$ has signature $(1, n-1)$. In this manner, $C(a_1, \dots, a_n)$ obtains the geometry of a holomorphic $(n-3)$-ball. It is smooth, but incomplete. The completion can be achieved by colliding each group of cone points with sum of curvatures $k_i := 2\pi - a_i$ less than 2π; the curvatures sum at the collision. (Let us drop the case $= 2\pi$ as it leads to a noncompact completion.)

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \dots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.) The vertices of P (complex numbers) define complex coordinates, another cut provides a $\mathbb{C}$-linear change of coordinates, and the area of a polygon can be calculated as $\langle p, p\rangle$, where the hermitian form $\langle -, -\rangle$ has signature $(1, n-1)$. In this manner, $C(a_1, \dots, a_n)$ obtains the geometry of a holomorphic $(n-3)$-ball. It is smooth, but incomplete. The completion can be achieved by colliding each group of cone points with sum of curvatures $k_i := 2\pi - a_i$ less than 2π; the curvatures sum at the collision. (Let us drop the case $= 2\pi$ as it leads to a noncompact completion.) We obtain a complete nonsmooth space with singularities stratified according to the mentioned groups of points (the inclusion of groups corresponds to that of strata).

1.3. Nonarithmetic Mostow-Deligne lattices (9 examples) and Deraux-Parker-Paupert lattices (5 examples). The second group of examples is quite similar to the first one. The latter was revised by Thurston as follows. Let $C(a_1, \dots, a_n)$ denote the space of all flat 2-spheres of area 1, considered up to orientation-preserving isometries, whose cone singularity angles $0 < a_i < 2\pi$ are prescribed. Each sphere can be cut to form a polygon P inside $\mathbb{C}$. (In the drawn picture, I am cheating a bit.) The vertices of P (complex numbers) define complex coordinates, another cut provides a $\mathbb{C}$-linear change of coordinates, and the area of a polygon can be calculated as $\langle p, p \rangle$, where the hermitian form $\langle -, - \rangle$ has signature $(1, n-1)$. In this manner, $C(a_1, \dots, a_n)$ obtains the geometry of a holomorphic $(n-3)$-ball. It is smooth, but incomplete. The completion can be achieved by colliding each group of cone points with sum of curvatures $k_i := 2\pi - a_i$ less than 2π; the curvatures sum at the collision. (Let us drop the case $= 2\pi$ as it leads to a noncompact completion.) We obtain a complete nonsmooth space with singularities stratified according to the mentioned groups of points (the inclusion of groups corresponds to that of strata). This space can be cut and then embedded into the holomorphic $(n-3)$-ball as a polyhedron, where the strata become faces.

Arguing as in Poincaré's polyhedron theorem

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$.

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$. The **orbifold condition** $k_i + k_j < 2\pi \Rightarrow \frac{2\pi}{2\pi - k_i - k_j} \in \mathbb{Z}$ **for all** $i \neq j$ is equivalent to the fact

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$. The **orbifold condition** $k_i + k_j < 2\pi \Rightarrow \frac{2\pi}{2\pi - k_i - k_j} \in \mathbb{Z}$ **for all** $i \neq j$ is equivalent to the fact that the space $C(a_1, \dots, a_n)$ is an orbifold holomorphic $(n-3)$-ball quotient.

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$. The **orbifold condition** $k_i + k_j < 2\pi \Rightarrow \frac{2\pi}{2\pi - k_i - k_j} \in \mathbb{Z}$ **for all** $i \neq j$ is equivalent to the fact that the space $C(a_1, \dots, a_n)$ is an orbifold holomorphic $(n-3)$-ball quotient. (In order to simplify the condition, we consider the space $C(a_1, \dots, a_n)$ of 2-spheres with **labelled cone points**, i.e., the isometries are required to preserve the labels.)

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$. The **orbifold condition** $k_i + k_j < 2\pi \Rightarrow \frac{2\pi}{2\pi - k_i - k_j} \in \mathbb{Z}$ **for all** $i \neq j$ is equivalent to the fact that the space $C(a_1, \dots, a_n)$ is an orbifold holomorphic $(n-3)$-ball quotient. (In order to simplify the condition, we consider the space $C(a_1, \dots, a_n)$ of 2-spheres with **labelled cone points**, i.e., the isometries are required to preserve the labels.)

Kalashnikov: disc bundles (noncompact)

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$. The **orbifold condition** $k_i + k_j < 2\pi \Rightarrow \frac{2\pi}{2\pi - k_i - k_j} \in \mathbb{Z}$ **for all** $i \neq j$ is equivalent to the fact that the space $C(a_1, \dots, a_n)$ is an orbifold holomorphic $(n-3)$-ball quotient. (In order to simplify the condition, we consider the space $C(a_1, \dots, a_n)$ of 2-spheres with **labelled cone points**, i.e., the isometries are required to preserve the labels.)

Kalashnikov: disc bundles (noncompact)

Here we discuss when a topological disc bundle $\pi : M \to S$ over a closed orientable surface S admits the geometry of the holomorphic 2-ball.

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$. The **orbifold condition** $k_i + k_j < 2\pi \Rightarrow \frac{2\pi}{2\pi - k_i - k_j} \in \mathbb{Z}$ **for all** $i \neq j$ is equivalent to the fact that the space $C(a_1, \dots, a_n)$ is an orbifold holomorphic $(n-3)$-ball quotient. (In order to simplify the condition, we consider the space $C(a_1, \dots, a_n)$ of 2-spheres with **labelled cone points**, i.e., the isometries are required to preserve the labels.)

Kalashnikov: disc bundles (noncompact)

Here we discuss when a topological disc bundle $\pi : M \to S$ over a closed orientable surface S admits the geometry of the holomorphic 2-ball. The topology of M is completely characterized by two numbers: the Euler characteristic χ of S and the Euler number e of the bundle (this is the intersection number of a couple of topological sections of the bundle).

Arguing as in Poincaré's polyhedron theorem (a bit sketchy; so, we are not sure we can follow his arguments), Thurston arrives at the

1.3.1. Orbifold condition (Thurston). Let $0 < k_i < 2\pi$ for all $1 \leqslant i \leqslant n$ be the cone point curvatures such that $\sum_i k_i = 4\pi$. The **orbifold condition** $k_i + k_j < 2\pi \Rightarrow \frac{2\pi}{2\pi - k_i - k_j} \in \mathbb{Z}$ **for all** $i \neq j$ is equivalent to the fact that the space $C(a_1, \dots, a_n)$ is an orbifold holomorphic $(n-3)$-ball quotient. (In order to simplify the condition, we consider the space $C(a_1, \dots, a_n)$ of 2-spheres with **labelled cone points**, i.e., the isometries are required to preserve the labels.)

Kalashnikov: disc bundles (noncompact)

Here we discuss when a topological disc bundle $\pi : M \to S$ over a closed orientable surface S admits the geometry of the holomorphic 2-ball. The topology of M is completely characterized by two numbers: the Euler characteristic χ of S and the Euler number e of the bundle (this is the intersection number of a couple of topological sections of the bundle). If the bundle admits a geometry of the holomorphic 2-ball, we get a representation $\pi_1 S = \pi_1 M \xrightarrow{\varrho} \mathrm{Isom}^{\mathrm{hol}}\, \mathrm{B}\, V = \mathrm{PU}(1,2)$.

For any representation $\pi_1 S \xrightarrow{\varrho} \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V$, there is a map $f : \tilde{S} \to \mathrm{B}\,V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi}\int_S \pi_* f^*\omega$, where ω stands for the Kahler form of $\mathrm{B}\,V$.

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\mathrm{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi} \int_S \pi_* f^* \omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\text{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi} \int_S \pi_* f^* \omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\mathrm{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi} \int_S \pi_* f^* \omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\mathrm{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi} \int_S \pi_* f^* \omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

For any representation $\pi_1 S \xrightarrow{\varrho} \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V$, there is a map $f : \tilde{S} \to \mathrm{B}\,V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi}\int_S \pi_* f^*\omega$, where ω stands for the Kahler form of $\mathrm{B}\,V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

2.2. Goldman-Kapovich-Leeb examples.

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\text{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi}\int_S \pi_* f^*\omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

2.2. Goldman-Kapovich-Leeb examples. They are sort of hybrids of the previous ones and satisfy $e = \chi + |\tau/2|$ and $\chi \leqslant e \leqslant \frac{1}{2}\chi$.

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\mathrm{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi} \int_S \pi_* f^* \omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

2.2. Goldman-Kapovich-Leeb examples. They are sort of hybrids of the previous ones and satisfy $e = \chi + |\tau/2|$ and $\chi \leqslant e \leqslant \frac{1}{2}\chi$.

2.3. Kalashnikov examples.

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\mathrm{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi}\int_S \pi_* f^* \omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

2.2. Goldman-Kapovich-Leeb examples. They are sort of hybrids of the previous ones and satisfy $e = \chi + |\tau/2|$ and $\chi \leqslant e \leqslant \frac{1}{2}\chi$.

2.3. Kalashnikov examples. These are AGG examples,

For any representation $\pi_1 S \xrightarrow{\varrho} \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V$, there is a map $f : \tilde{S} \to \mathrm{B}\,V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi}\int_S \pi_* f^*\omega$, where ω stands for the Kahler form of $\mathrm{B}\,V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

2.2. Goldman-Kapovich-Leeb examples. They are sort of hybrids of the previous ones and satisfy $e = \chi + |\tau/2|$ and $\chi \leqslant e \leqslant \frac{1}{2}\chi$.

2.3. Kalashnikov examples. These are AGG examples, examples of trivial bundles (Goldman problem),

For any representation $\pi_1 S \xrightarrow{\varrho} \mathrm{Isom}^{\mathrm{hol}}\,\mathrm{B}\,V$, there is a map $f : \tilde{S} \to \mathrm{B}\,V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi}\int_S \pi_* f^*\omega$, where ω stands for the Kahler form of $\mathrm{B}\,V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

2.2. Goldman-Kapovich-Leeb examples. They are sort of hybrids of the previous ones and satisfy $e = \chi + |\tau/2|$ and $\chi \leqslant e \leqslant \frac{1}{2}\chi$.

2.3. Kalashnikov examples. These are AGG examples, examples of trivial bundles (Goldman problem), and examples constructed by C. H. Grossi (yet unpublished).

For any representation $\pi_1 S \xrightarrow{\varrho} \operatorname{Isom}^{\mathrm{hol}} \mathrm{B}\, V$, there is a map $f : \tilde{S} \to \mathrm{B}\, V$ which is $\pi_1 S$-equivariant with respect to ϱ, where $\pi : \tilde{S} \to S$ is a universal covering. The **Toledo invariant** τ of ϱ (or of the bundle M) is given by $\tau := \frac{1}{2\pi} \int_S \pi_* f^* \omega$, where ω stands for the Kahler form of $\mathrm{B}\, V$.

All known bundles admitting the geometry of the holomorphic 2-ball satisfy the following variant of the Gromov-Lawson-Thurston conjecture.

Conjecture. An oriented disc bundle over a closed orientable surface is a quotient of the holomorphic 2-ball iff $|e/\chi| \leqslant 1$ and $\chi < 0$.

2.1. Simple disc bundles.

- $\mathbb{C}$-fuchsian bundles, characterized by $\chi = \tau$ (Goldman-Toledo rigidity); satisfy $e = \chi/2$.
- $\mathbb{R}$-fuchsian bundles: satisfy $e = \chi$ (tangent bundle of S) and $\tau = 0$.

2.2. Goldman-Kapovich-Leeb examples. They are sort of hybrids of the previous ones and satisfy $e = \chi + |\tau/2|$ and $\chi \leqslant e \leqslant \frac{1}{2}\chi$.

2.3. Kalashnikov examples. These are AGG examples, examples of trivial bundles (Goldman problem), and examples constructed by C. H. Grossi (yet unpublished). They all satisfy the relation $2(\chi + e) = 3\tau$.

Suppose we have a smooth $\mathbb{C}$-curve in a disc bundle M admitting the geometry of the holomorphic 2-ball homotopic to a section of the bundle. Then the relation $2(\chi + e) = 3\tau$ holds (it is nothing but the adjunction formula).

Suppose we have a smooth $\mathbb{C}$-curve in a disc bundle M admitting the geometry of the holomorphic 2-ball homotopic to a section of the bundle. Then the relation $2(\chi + e) = 3\tau$ holds (it is nothing but the adjunction formula).

[Typical talk on a Russian factory: ‘Oh, it does not really matter.

Suppose we have a smooth $\mathbb{C}$-curve in a disc bundle M admitting the geometry of the holomorphic 2-ball homotopic to a section of the bundle. Then the relation $2(\chi + e) = 3\tau$ holds (it is nothing but the adjunction formula).

[Typical talk on a Russian factory: 'Oh, it does not really matter. Whatever we try to build, we will end up with a Kalashnikov gun.']

Suppose we have a smooth $\mathbb{C}$-curve in a disc bundle M admitting the geometry of the holomorphic 2-ball homotopic to a section of the bundle. Then the relation $2(\chi + e) = 3\tau$ holds (it is nothing but the adjunction formula).

[Typical talk on a Russian factory: 'Oh, it does not really matter. Whatever we try to build, we will end up with a Kalashnikov gun.']

Conjecture. In all Kalashnikov examples there is a smooth (**Kalashnikov**) **$\mathbb{C}$-curve** homotopic to a section.

Later we will see why this conjecture can be interesting.

Suppose we have a smooth $\mathbb{C}$-curve in a disc bundle M admitting the geometry of the holomorphic 2-ball homotopic to a section of the bundle. Then the relation $2(\chi + e) = 3\tau$ holds (it is nothing but the adjunction formula).

[Typical talk on a Russian factory: 'Oh, it does not really matter. Whatever we try to build, we will end up with a Kalashnikov gun.']

Conjecture. In all Kalashnikov examples there is a smooth (**Kalashnikov**) **$\mathbb{C}$-curve** homotopic to a section.

Later we will see why this conjecture can be interesting.

2-spheres with cone singularities

Suppose we have a smooth $\mathbb{C}$-curve in a disc bundle M admitting the geometry of the holomorphic 2-ball homotopic to a section of the bundle. Then the relation $2(\chi + e) = 3\tau$ holds (it is nothing but the adjunction formula).

[Typical talk on a Russian factory: 'Oh, it does not really matter. Whatever we try to build, we will end up with a Kalashnikov gun.']

Conjecture. In all Kalashnikov examples there is a smooth (**Kalashnikov**) **$\mathbb{C}$-curve** homotopic to a section.

Later we will see why this conjecture can be interesting.

2-spheres with cone singularities

Let us return to cocompact nonarithmetic lattices in $\mathrm{PU}(1,2)$. Namely, let $C_h(a_1, \dots, a_n)$ denote the space of all hyperbolic 2-spheres whose cone singularity angles $0 < a_i < 2\pi$ are prescribed and satisfy the inequality $\sum_i k_i < 4\pi$, where $k_i := 2\pi - a_i$ (by Gauss-Bonnet, this is a neccessary and sufficient condition for $C_h(a_1, \dots, a_n)$ to be nonempty);

Suppose we have a smooth $\mathbb{C}$-curve in a disc bundle M admitting the geometry of the holomorphic 2-ball homotopic to a section of the bundle. Then the relation $2(\chi + e) = 3\tau$ holds (it is nothing but the adjunction formula).

[Typical talk on a Russian factory: 'Oh, it does not really matter. Whatever we try to build, we will end up with a Kalashnikov gun.']

Conjecture. In all Kalashnikov examples there is a smooth (**Kalashnikov**) **$\mathbb{C}$-curve** homotopic to a section.

Later we will see why this conjecture can be interesting.

2-spheres with cone singularities

Let us return to cocompact nonarithmetic lattices in $\mathrm{PU}(1,2)$. Namely, let $C_h(a_1, \dots, a_n)$ denote the space of all hyperbolic 2-spheres whose cone singularity angles $0 < a_i < 2\pi$ are prescribed and satisfy the inequality $\sum_i k_i < 4\pi$, where $k_i := 2\pi - a_i$ (by Gauss-Bonnet, this is a neccessary and sufficient condition for $C_h(a_1, \dots, a_n)$ to be nonempty); the singularities are labelled and the 2-spheres are considered up to orientation-preserving and label-preserving isometries.

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points.

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \dots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated.

In order to get a compact completion of $C_h(a_1, \ldots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \ldots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \dots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

3.0. Thurston orbifold condition and PPT.

Proposition. *The completion is morally the same as in the flat case. The orbifold condition is the same as in the flat case.*

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \dots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

3.0. Thurston orbifold condition and PPT.

Proposition. *The completion is morally the same as in the flat case. The orbifold condition is the same as in the flat case.*

Pseudo proof. Yet, we have no geometry on $C_h(a_1, \dots, a_n)$ to speak of completion

In order to get a compact completion of $C_h(a_1,\dots,a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1,\dots,a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

3.0. Thurston orbifold condition and PPT.

Proposition. *The completion is morally the same as in the flat case. The orbifold condition is the same as in the flat case.*

Pseudo proof. Yet, we have no geometry on $C_h(a_1,\dots,a_n)$ to speak of completion (but there are many providing the same topology on the completion).

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \dots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

3.0. Thurston orbifold condition and PPT.

Proposition. *The completion is morally the same as in the flat case. The orbifold condition is the same as in the flat case.*

Pseudo proof. Yet, we have no geometry on $C_h(a_1, \dots, a_n)$ to speak of completion (but there are many providing the same topology on the completion). For the orbifold condition, one should apply a version of Poincaré's polyhedron theorem ∎

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \dots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

3.0. Thurston orbifold condition and PPT.

Proposition. *The completion is morally the same as in the flat case. The orbifold condition is the same as in the flat case.*

Pseudo proof. Yet, we have no geometry on $C_h(a_1, \dots, a_n)$ to speak of completion (but there are many providing the same topology on the completion). For the orbifold condition, one should apply a version of Poincaré's polyhedron theorem $\blacksquare$

3.1. It would be handy if each sphere could be cut to form a polygon P inside the hyperbolic disc $\mathrm{B}\, V$.

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \dots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

3.0. Thurston orbifold condition and PPT.

Proposition. *The completion is morally the same as in the flat case. The orbifold condition is the same as in the flat case.*

Pseudo proof. Yet, we have no geometry on $C_h(a_1, \dots, a_n)$ to speak of completion (but there are many providing the same topology on the completion). For the orbifold condition, one should apply a version of Poincaré's polyhedron theorem ■

3.1. It would be handy if each sphere could be cut to form a polygon P inside the hyperbolic disc $\mathrm{B}\, V$. In general, this might be difficult or even impossible to achieve.

In order to get a compact completion of $C_h(a_1, \dots, a_n)$, we should also require that $\sum_j k_{i_j} \neq 2\pi$ for any subset of cone points. The case $C_s(a_1, \dots, a_n)$ of spherical 2-spheres with $\sum_i k_i > 4\pi$ can be dealt with in a similar way, except that it is more complicated. (Also, there are rumors that Dmitri Panov did something in this direction; so, it seems reasonable first to read his paper.)

3.0. Thurston orbifold condition and PPT.

Proposition. *The completion is morally the same as in the flat case. The orbifold condition is the same as in the flat case.*

Pseudo proof. Yet, we have no geometry on $C_h(a_1, \dots, a_n)$ to speak of completion (but there are many providing the same topology on the completion). For the orbifold condition, one should apply a version of Poincaré's polyhedron theorem ∎

3.1. It would be handy if each sphere could be cut to form a polygon P inside the hyperbolic disc $\mathrm{B}\,V$. In general, this might be difficult or even impossible to achieve. So, for simplicity, we require that $a_i \leqslant \pi$ for all i.

Let p be one of the cone points.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i. Of course, there are infinitely many polygons that provide the same 2-sphere after gluing.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i. Of course, there are infinitely many polygons that provide the same 2-sphere after gluing. For instance, one can cut a polygon P along the geodesic segment joining c_i and p_j and rotate a certain half of P by r_i getting a new polygon by means of such **bending** b_i.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i. Of course, there are infinitely many polygons that provide the same 2-sphere after gluing. For instance, one can cut a polygon P along the geodesic segment joining c_i and p_j and rotate a certain half of P by r_i getting a new polygon by means of such **bending** b_i. It is easy to see that the new polygon generates an isometric 2-sphere.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i. Of course, there are infinitely many polygons that provide the same 2-sphere after gluing. For instance, one can cut a polygon P along the geodesic segment joining c_i and p_j and rotate a certain half of P by r_i getting a new polygon by means of such **bending** b_i. It is easy to see that the new polygon generates an isometric 2-sphere.

Proposition. *Suppose that the 2-spheres made from polygons P and P' are isometric.*

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i. Of course, there are infinitely many polygons that provide the same 2-sphere after gluing. For instance, one can cut a polygon P along the geodesic segment joining c_i and p_j and rotate a certain half of P by r_i getting a new polygon by means of such **bending** b_i. It is easy to see that the new polygon generates an isometric 2-sphere.

Proposition. *Suppose that the 2-spheres made from polygons P and P' are isometric. Then P' can be obtained from P by finitely many bendings.*

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i. Of course, there are infinitely many polygons that provide the same 2-sphere after gluing. For instance, one can cut a polygon P along the geodesic segment joining c_i and p_j and rotate a certain half of P by r_i getting a new polygon by means of such **bending** b_i. It is easy to see that the new polygon generates an isometric 2-sphere.

Proposition. *Suppose that the 2-spheres made from polygons P and P' are isometric. Then P' can be obtained from P by finitely many bendings.*

Now we can describe the space $C_h(a_1,\dots,a_n)$ as $C_h(a_1,\dots,a_n)=C/G$, where C is the space of the polygons in question and G is the group acting on C and generated by the bendings.

Let p be one of the cone points. We join it by shortest geodesic segments with the remaining cone points and cut along the segments. It is easy to see that we get a convex (hence, embeddable into the hyperbolic disc $\mathrm{B}\,V$) polygon P. The polygon P has $2(n-1)$ vertices and is equipped with a gluing pattern. The p-vertices p_i correspond to the cone point p. The other vertices c_i correspond to the other cone points. The gluing pattern can be realized by means of counterclockwise rotations r_i by k_i about c_i. Of course, there are infinitely many polygons that provide the same 2-sphere after gluing. For instance, one can cut a polygon P along the geodesic segment joining c_i and p_j and rotate a certain half of P by r_i getting a new polygon by means of such **bending** b_i. It is easy to see that the new polygon generates an isometric 2-sphere.

Proposition. *Suppose that the 2-spheres made from polygons P and P' are isometric. Then P' can be obtained from P by finitely many bendings.*

Now we can describe the space $C_h(a_1,\dots,a_n)$ as $C_h(a_1,\dots,a_n)=C/G$, where C is the space of the polygons in question and G is the group acting on C and generated by the bendings. In the case $n=4$, for example, C is a component of a real surface in $\mathbb{R}^3(t_1,t_2,t_3)$ given by the equation

$$(\mathbf{3.1.1})\qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix}=0,$$

where $|u_i|=1$ and $\operatorname{Arg} u_i=k_i$ for all $0\leqslant i\leqslant 3$.

$$
\textbf{(3.1.1)} \qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix}=0,
$$

where $|u_i|=1$ and $\operatorname{Arg} u_i=k_i$ for all $0\leqslant i\leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i.

$$\textbf{(3.1.1)}\qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix}=0,$$

where $|u_i|=1$ and $\operatorname{Arg} u_i=k_i$ for all $0\leqslant i\leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i. A similar equation can be written for the most interesting case $n=5$.

$$\textbf{(3.1.1)}\qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix}=0,$$

where $|u_i|=1$ and $\operatorname{Arg}u_i=k_i$ for all $0\leqslant i\leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i. A similar equation can be written for the most interesting case $n=5$. One of the mentioned geometries comes from this equation.

$$\textbf{(3.1.1)} \qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix} = 0,$$

where $|u_i| = 1$ and $\operatorname{Arg} u_i = k_i$ for all $0 \leqslant i \leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i. A similar equation can be written for the most interesting case $n = 5$. One of the mentioned geometries comes from this equation. (This one is natural and hyperbolic. It can be deformed . . .)

$$\textbf{(3.1.1)}\qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix}=0,$$

where $|u_i|=1$ and $\operatorname{Arg} u_i=k_i$ for all $0\leqslant i\leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i. A similar equation can be written for the most interesting case $n=5$. One of the mentioned geometries comes from this equation. (This one is natural and hyperbolic. It can be deformed ...)

The space $C_h(a_1,\dots,a_n)$ is also related to an open part of (a sort of) a relative character variety living in $\operatorname{Hom}\big(H,\operatorname{PU}(1,2)\big)/\operatorname{PU}(1,2)$, where $H:=\langle x_1,\dots,x_n\mid x_n\dots x_1=1\rangle$ is a free group of rank $n-1$.

$$\textbf{(3.1.1)}\qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix}=0,$$

where $|u_i|=1$ and $\operatorname{Arg} u_i=k_i$ for all $0\leqslant i\leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i. A similar equation can be written for the most interesting case $n=5$. One of the mentioned geometries comes from this equation. (This one is natural and hyperbolic. It can be deformed ...)

The space $C_h(a_1,\dots,a_n)$ is also related to an open part of (a sort of) a relative character variety living in $\operatorname{Hom}\big(H,\mathrm{PU}(1,2)\big)/\,\mathrm{PU}(1,2)$, where $H:=\langle x_1,\dots,x_n\mid x_n\dots x_1=1\rangle$ is a free group of rank $n-1$. It is formed by the homomorphisms $[h]$ with the prescribed set of conjugacy classes of the images hx_i, elliptic in our case.

$$\textbf{(3.1.1)} \qquad \det \begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \big(t_2(\overline{u}_2-u_2)+u_2\big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \big(t_2(u_2-\overline{u}_2)+\overline{u}_2\big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix} = 0,$$

where $|u_i| = 1$ and $\operatorname{Arg} u_i = k_i$ for all $0 \leqslant i \leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i. A similar equation can be written for the most interesting case $n = 5$. One of the mentioned geometries comes from this equation. (This one is natural and hyperbolic. It can be deformed ...)

The space $C_h(a_1, \dots, a_n)$ is also related to an open part of (a sort of) a relative character variety living in $\operatorname{Hom}\big(H, \mathrm{PU}(1,2)\big)/\,\mathrm{PU}(1,2)$, where $H := \langle x_1, \dots, x_n \mid x_n \dots x_1 = 1\rangle$ is a free group of rank $n-1$. It is formed by the homomorphisms $[h]$ with the prescribed set of conjugacy classes of the images hx_i, elliptic in our case.

3.2. W. M. Goldman, *The modular group action on real* $\mathrm{SL}(2)$*-characters of a one-holed torus,* Geometry and Topology **7** (2003), 443–486;

$$(\mathbf{3.1.1})\qquad \det\begin{bmatrix} 1 & t_3(u_3-\overline{u}_3)+\overline{u}_3 & \Big(t_2(\overline{u}_2-u_2)+u_2\Big)\overline{u}_0 \\ t_3(\overline{u}_3-u_3)+u_3 & 1 & t_1(u_1-\overline{u}_1)+\overline{u}_1 \\ \Big(t_2(u_2-\overline{u}_2)+\overline{u}_2\Big)u_0 & t_1(\overline{u}_1-u_1)+u_1 & 1 \end{bmatrix}=0,$$

where $|u_i|=1$ and $\operatorname{Arg} u_i=k_i$ for all $0\leqslant i\leqslant 3$. In terms of these coordinates, the bending b_i is an involution that interchanges the roots of the equation (3.1.1), quadratic in t_i. A similar equation can be written for the most interesting case $n=5$. One of the mentioned geometries comes from this equation. (This one is natural and hyperbolic. It can be deformed ...)

The space $C_h(a_1,\dots,a_n)$ is also related to an open part of (a sort of) a relative character variety living in $\operatorname{Hom}\big(H,\mathrm{PU}(1,2)\big)/\mathrm{PU}(1,2)$, where $H:=\langle x_1,\dots,x_n\mid x_n\dots x_1=1\rangle$ is a free group of rank $n-1$. It is formed by the homomorphisms $[h]$ with the prescribed set of conjugacy classes of the images hx_i, elliptic in our case.

3.2. W. M. Goldman, *The modular group action on real* $\mathrm{SL}(2)$*-characters of a one-holed torus,* Geometry and Topology **7** (2003), 443–486;
W. M. Goldman, G. McShane, G. Stantchev, S. P. Tan, *Automorphisms of two-generator free groups and spaces of isometric actions on the hyperbolic plane,* arXiv: 1509.03790v2

3.3. Nonarithmetic cocompact lattices of the second type.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}} V$ that intersects $\mathrm{B}\, V$ and whose stabilizer S

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L)

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.)

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C=D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**. The remaining nonarithmetic compact holomorphic 2-ball quotients are of the **second type**.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**. The remaining nonarithmetic compact holomorphic 2-ball quotients are of the **second type**.

Remembering that every smooth compact holomorphic 2-ball quotient possesses plenty of smooth $\mathbb{C}$-curves,

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**. The remaining nonarithmetic compact holomorphic 2-ball quotients are of the **second type**.

Remembering that every smooth compact holomorphic 2-ball quotient possesses plenty of smooth $\mathbb{C}$-curves, we can imagine that a Kalashnikov $\mathbb{C}$-curve is an evidence of the existence of a quotient of the second type.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**. The remaining nonarithmetic compact holomorphic 2-ball quotients are of the **second type**.

Remembering that every smooth compact holomorphic 2-ball quotient possesses plenty of smooth $\mathbb{C}$-curves, we can imagine that a Kalashnikov $\mathbb{C}$-curve is an evidence of the existence of a quotient of the second type. As the discrete group $S \leqslant \mathrm{PU}(1,2)$ providing a Kalashnikov curve is explicitly known *a priori*,

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**. The remaining nonarithmetic compact holomorphic 2-ball quotients are of the **second type**.

Remembering that every smooth compact holomorphic 2-ball quotient possesses plenty of smooth $\mathbb{C}$-curves, we can imagine that a Kalashnikov $\mathbb{C}$-curve is an evidence of the existence of a quotient of the second type. As the discrete group $S \leqslant \mathrm{PU}(1,2)$ providing a Kalashnikov curve is explicitely known *a priori*, it must be easy to find the corresponding lattice.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}}V$ that intersects $\mathrm{B}\,V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**. The remaining nonarithmetic compact holomorphic 2-ball quotients are of the **second type**.

Remembering that every smooth compact holomorphic 2-ball quotient possesses plenty of smooth $\mathbb{C}$-curves, we can imagine that a Kalashnikov $\mathbb{C}$-curve is an evidence of the existence of a quotient of the second type. As the discrete group $S \leqslant \mathrm{PU}(1,2)$ providing a Kalashnikov curve is explicitly known *a priori*, it must be easy to find the corresponding lattice.

Conjecture. There are infinitely many nonarithmetic cocompact lattices of the second type.

3.3. Nonarithmetic cocompact lattices of the second type. All known examples of nonarithmetic smooth compact holomorphic 2-ball quotients possess a smooth **$\mathbb{C}$-fuchsian $\mathbb{C}$-curve**. Such a curve C comes from a projective line D in $\mathbb{P}_{\mathbb{C}} V$ that intersects $\mathrm{B}\, V$ and whose stabilizer S (called **$\mathbb{C}$-fuchsian subgroup** in L) in the corresponding lattice L provides $C = D/S$. (Everybody knows essentially three classes of such curves, but there exists a fourth one.) Let us say that such lattices are of the **first type**. The remaining nonarithmetic compact holomorphic 2-ball quotients are of the **second type**.

Remembering that every smooth compact holomorphic 2-ball quotient possesses plenty of smooth $\mathbb{C}$-curves, we can imagine that a Kalashnikov $\mathbb{C}$-curve is an evidence of the existence of a quotient of the second type. As the discrete group $S \leqslant \mathrm{PU}(1,2)$ providing a Kalashnikov curve is explicitly known *a priori*, it must be easy to find the corresponding lattice.

Conjecture. There are infinitely many nonarithmetic cocompact lattices of the second type.

Thank you for attention!