Locally conformally Kähler structures on homogeneous sapces

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A *locally conformally Kähler structure*, or shortly *l.c.K. structure* on a differentiable manifold M is a Hermitian structure h on M with its associated fundamental form Ω satisfying $d\Omega = \theta \wedge \Omega$ for some closed 1-form θ (which is so called Lee form). A differentiable manifold M is called a *locally conformally Kähler manifold*, or shortly *l.c.K. manifold* if M admits a l.c.K. structure. Note that l.c.K. structure Ω is globally conformally Kähler (or Kähler) if and only if θ is exact (or 0 respectively).

After the inaugural paper of I. Vaisman "On locally conformally almost Kähler maifolds (Israel J. Math. **24**, 1976), there have been extensive studies on l.c.K. manifolds. Among many examples of l.c.K. manifolds, Hopfs manifold and more recently Inoue manifolds have been well studied. These manifolds are very interesting and important since they are of homogenous type and admit complex analytic deformations, which may or may not preserve l.c.K. structures.

A homogeneous Hermitian manifold M with its homogeneous Hermitian structure h, defining a locally conformally Kähler structure Ω is called a *homogeneous locally conformally Kähler* (or *homogeneous l.c.K*) *manifold.* If a simply connected homogeneous l.c.K. manifold $M = G/H$, where G is a connected Lie group and H a closed subgroup of G, admits a free action of a discrete subgroup Γ of G from the left, then a double coset space $\Gamma\backslash G/H$ is called a *locally homogeneous l.c.K. manifold.* We will discuss explicitly homogeneous l.c.K. structures and locally homogeneous l.c.K. structures on Hopf surfaces and Inoue surfaces, and their deformations.

We will show as a main result a structure theorem of compact homogeneous l.c.K. manifolds, asserting that it has a structure of a holomorphic principal fiber bundle over a flag manifold with fiber a 1-dimensional complex torus. As an application of the theorem, we can show that only compact homogeneous l.c.K. manifolds of complex dimension 2 are Hopf surfaces of homogeneous type. We can also show that there exist no compact complex homogeneous l.c.K. manifolds; in particular neither complex Lie groups nor complex paralellizable manifolds admit their compatible l.c.K. structures.

We can also determine all compact locally homogeneous l.c.K. manifolds of complex dimension 2, which cover most of non-Kähler complex surfaces of real homogeneous type. Concerning l.c.K. structures on Lie groups, we can obtain classifications of certain Lie groups of general dimension: nilpotent and reductive Lie groups. There is a class of locally homogeneous l.c.K. solvmanifolds considered as generalized Inoue manifolds, recently constructed by K. Oeljeklaus and M. Toma. We need a further study of l.c.K. structures on solvmanifolds.

This talk is based on a joint work with Y. Kamishima "Locally conformally Kähler structures on homogeneous spaces" (arXiv:1101.3693), and partially a current work with D. Alekseevsky and V. Cortés, which is to be an extended version of the above work.