A NEW EQUATION ON THE LOW-DIMENSIONAL CALABI–YAU METRICS

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Abstract. In this paper we introduce a new equation on the compact Kähler manifolds. Solution of this equation corresponds to the Calabi–Yau metric. New equation differs from the Monge–Ampère equation considered by Calabi and Yau.

Let (M, ω) be the compact Kähler n-manifold. Calabi made the following conjecture proven by Yau: If $c_1(M) = 0$, then there exists a Riemannian metric with the holonomy group contained in $SU(n)$ [1, 2]. Explicitly, Calabi conjecture is stated as follows: There exists unique real function φ such that:

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F \omega^n, \qquad \int_M \varphi\, \omega^n = 0, \tag{1}$$

where F is a smooth real function on M such that $\int_M e^F \omega^n = \int_M \omega^n$. It is assumed that $\omega + i\partial\bar{\partial}\varphi > 0$.

Existence of the solution of (1) implies existence of the Ricci-flat metric on M.

Note that the Monge–Ampère equation describes a deformation of the Kähler form. We study the deformation of the holomorphic volume form on the Kähler manifold M in complex dimensions 2 and 3.

Theorem. *Let M be a compact Kähler n-manifold. Suppose $c_1(M) = 0$. Denote a holomorphic volume form by Ω. For $n = 2, 3$ there exists a solution of the following equation:*

$$(\Omega + dd^s\psi) \wedge (\bar{\Omega} + dd^s\bar{\psi}) = e^F \Omega \wedge \bar{\Omega}, \tag{2}$$

where ψ is a complex n-form such that $\tilde{\Omega} = \Omega + dd^s\psi$ is a stable primitive form; F is a smooth real function such that $\int_M e^F \Omega \wedge \bar{\Omega} = \int_M \Omega \wedge \bar{\Omega}$. Here $d^s = d^{c*}$ is the symplectic differential operator (see [3]).

Here $\tilde{\Omega}$ is said to be stable if there exists some complex structure J on M such that $\tilde{\Omega}$ is a holomorphic volume form with respect to J [4].

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