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The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds¹

By YUM-TONG SIU

In 1960 Calabi and Vesentini [3] proved that compact quotients of bounded symmetric domains are rigid in the sense that they do not admit any nontrivial infinitesimal holomorphic deformation. In 1970 Mostow discovered the phenomenon of strong rigidity [7]. He proved that the fundamental group of a compact locally symmetric Riemannian manifold of nonpositive curvature determines the manifold up to an isometry and a choice of normalizing constants if the manifold admits no closed one or two dimensional geodesic submanifolds which are locally direct factors. In particular, two compact quotients of the ball of complex dimension ≥ 2 with isomorphic fundamental groups are either biholomorphic or conjugate biholomorphic. Yau conjectured that this phenomenon of strong rigidity should hold also for compact Kähler manifolds of complex dimension ≥ 2 with negative sectional curvature. That is, two compact Kähler manifolds of complex dimension ≥ 2 with negative sectional curvature are biholomorphic or conjugate biholomorphic if they are of the same homotopy type. In this paper we prove that Yau's conjecture is true when the curvature tensor of one of the two compact Kähler manifolds is strongly negative in the sense defined in Section 2 with no curvature assumption on the other manifold. The strong negativity of the curvature tensor is a condition stronger than the negativity of the sectional curvature. This strong negative curvature condition is satisfied by quotients of the ball and also by the compact Kähler surface recently constructed by Mostow and Siu [8] which has negative sectional curvature and whose universal covering is not biholomorphic to the ball. Until now there is no known example of a compact Kähler manifold of negative sectional curvature which does not admit a Kähler metric with strongly negative curvature tensor.

Our result is proved by showing that a harmonic map of compact Kähler manifolds is either holomorphic or conjugate holomorphic if the rank over

R of the differential of the map is ≥ 4 at some point and if the curvature tensor of the image manifold is strongly negative. This result on the complex-analyticity of harmonic maps is obtained by a Bochner type argument. The usual technique of proving properties of a harmonic map f is to obtain a Bochner type formula by considering the Laplacian of the pointwise square norm of df . For the complex manifold case the pointwise square norm of df is replaced by the pointwise square norm of $\bar{\partial}f$ in this technique (see [9]). The metric tensor of the image manifold and the inverse matrix of the metric tensor of the domain manifold appear in the pointwise square norm of $\bar{\partial}f$. Hence in this Bochner type formula an expression involving the difference of the curvature tensors of the domain manifold and the image manifold appears. This prevents one from drawing any conclusion when both manifolds have negative curvature. In our proof we overcome this difficulty by replacing the pointwise square norm of $\bar{\partial}f$ by the contraction of $\bar{\partial}f \wedge \overline{\bar{\partial}f}$ with the metric tensor of the image manifold and by replacing the Laplacian by $\bar{\partial}\bar{\partial}$. This method enables us to get rid of the curvature tensor of the domain manifold in the Bochner type formula. Our result on the complex-analyticity of harmonic maps can be applied to the problem of representing homology classes by complex-analytic subvarieties. Unfortunately it is still far from being able to prove the Hodge conjecture even for the case of the compact quotients of the ball.

Our method of proving the complex-analyticity of harmonic maps and strong rigidity can be applied to a much wider class of compact Kähler manifolds than the class of those with strongly negative curvature tensor. As examples, we apply our method to compact quotients of the four types of classical bounded symmetric domains and obtain the following result. Any harmonic map from a compact Kähler manifold to a compact quotient of an irreducible classical bounded symmetric domain of dimension ≥ 2 is either holomorphic or conjugate holomorphic if the map is a submersion at some point. In particular, any compact Kähler manifold which is of the same homotopy type as a compact quotient of an irreducible classical bounded symmetric domain of dimension ≥ 2 is either biholomorphic or conjugate biholomorphic to it. This rigidity result is stronger than the corresponding strong rigidity theorem of Mostow [7], because here only one manifold is assumed to be locally symmetric whereas in Mostow's theorem both manifolds have to be assumed locally symmetric.

These results were announced in [10].

I would like to thank S.-T. Yau for introducing me to harmonic maps and his conjecture and for many conversations in connection with his con-

jecture. E. Bedford drew my attention to the inequality that the second elementary symmetric function of a finite number of real numbers is dominated by the square of their sum. He showed me how to use such an inequality and the complex Bernstein formula [2, p. 378] to obtain, in the pseudoconvex case, a new proof of Bochner's theorem on extending holomorphic functions from boundaries. This simple inequality is used in our proof of the complex-analyticity of harmonic maps and its use is inspired by Bedford's proof of the special case of Bochner's theorem. I would like to express my indebtedness to him.

Table of Contents

Section	
1. Harmonic maps	75
2. Curvature conditions and statement of results	76
3. A Bochner type identity	79
4. Complex-analyticity of harmonic maps	81
5. Bounded symmetric domains	90
6. Adequate negativity of the curvature of D_{mn}^I	93
7. Adequate negativity of the curvature of D_n^{II}	98
8. Adequate negativity of the curvature of D_n^{III}	102
9. Adequate negativity of the curvature of D_n^{IV}	107
10. Strong rigidity and other applications	110

1. Harmonic maps

Let $f: N \rightarrow M$ be a map of Riemannian manifolds whose metrics are respectively

$$ds_M^2 = \sum_{\alpha, \beta} g_{\alpha\beta} dx^\alpha dx^\beta ,$$

$$ds_N^2 = \sum_{i, j} h_{ij} dy^i dy^j .$$

The energy $E(f)$ of f is defined to be

$$\frac{1}{2} \int_N \text{trace}_{ds_N^2} (f^* ds_M^2) ,$$

that is,

$$E(f) = \frac{1}{2} \int \sum_{i, j, \alpha, \beta} g_{\alpha\beta} h^{ij} \frac{\partial f^\alpha}{\partial y^i} \frac{\partial f^\beta}{\partial y^j}$$

in terms of local coordinate charts. The Euler-Lagrange equation for the energy functional E is

$$\Delta_N f^\alpha + \sum_{i, j, \beta, \gamma} {}^M \Gamma_{\beta\gamma}^\alpha \frac{\partial f^\beta}{\partial y^i} \frac{\partial f^\gamma}{\partial y^j} h^{ij} = 0$$

for all α , where Δ_N is the Laplace-Beltrami operator of N and ${}^M \Gamma_{\beta\gamma}^\alpha$ is the

Christoffel symbol of M . The map f is said to be *harmonic* if f satisfies the Euler-Lagrange equation for the energy functional.

Eells-Sampson [4] proved that when M and N are both compact and M has nonpositive sectional curvature, every continuous map from N to M is homotopic to a harmonic map. Hartman [6] proved that the harmonic map is unique in each homotopy class if M has strictly negative curvature. Since the Euler-Lagrange equation for the energy functional is a second-order quasilinear elliptic system of partial differential equations, it follows that any harmonic map between smooth Riemannian manifolds is smooth. The reader is referred to [4] for a survey of the theory of harmonic maps.

2. Curvature conditions and statement of results

Suppose M is a complex manifold with Kähler metric

$$ds_M^2 = 2 \operatorname{Re} \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} dz^\alpha dz^{\bar{\beta}}.$$

The curvature tensor is given by

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \partial_\gamma \partial_{\bar{\delta}} g_{\alpha\bar{\beta}} - \sum_{\lambda, \mu} g^{\lambda\mu} \partial_\alpha g_{\gamma\bar{\mu}} \partial_{\bar{\beta}} g_{\lambda\bar{\delta}}.$$

The sectional curvature at the 2-plane spanned by the two tangent vectors

$$p = 2 \operatorname{Re} \sum_{\alpha} \xi^\alpha \frac{\partial}{\partial z_\alpha},$$

$$q = 2 \operatorname{Re} \sum_{\alpha} \eta^\alpha \frac{\partial}{\partial z_\alpha}$$

is given by

$$- \|p \wedge q\|^{-2} \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta) (\bar{\xi}^\delta \bar{\eta}^\gamma - \eta^\delta \bar{\xi}^\gamma)$$

where

$$\begin{aligned} \|p \wedge q\|^2 = & \sum_{\alpha, \beta, \gamma, \delta} g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}} [(\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta) (\bar{\xi}^\delta \bar{\eta}^\gamma - \eta^\delta \bar{\xi}^\gamma) \\ & + (\xi^\alpha \bar{\eta}^\gamma - \eta^\alpha \bar{\xi}^\gamma) (\bar{\xi}^\beta \bar{\eta}^\delta - \eta^\beta \bar{\xi}^\delta)]. \end{aligned}$$

Nonpositivity of the sectional curvature is equivalent to

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta) (\bar{\xi}^\delta \bar{\eta}^\gamma - \eta^\delta \bar{\xi}^\gamma) \geq 0$$

for all complex numbers ξ^α, η^α . The sectional curvature is negative if, in addition in the above inequality, equality holds if and only if

$$\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta = 0 \quad \text{for all } \alpha, \beta,$$

because the vanishing of all $\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta$ is equivalent to the vanishing of $\|p \wedge q\|^2$ as one can easily verify by diagonalizing the matrix $(g_{\alpha\bar{\beta}})$ and making use of the identity

$$|\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta|^2 = |\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta|^2 - (\xi^\alpha \bar{\eta}^\alpha - \eta^\alpha \bar{\xi}^\alpha)(\bar{\xi}^\beta \bar{\eta}^\beta - \eta^\beta \bar{\xi}^\beta).$$

We now introduce some notions of negative curvature stronger than the negativity of sectional curvature.

Definition. The curvature $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is said to be *strongly negative* (respectively *strongly seminegative*) if

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (A^\alpha \bar{B}^\beta - C^\alpha \bar{D}^\beta)(\overline{A^\delta \bar{B}^\delta - C^\delta \bar{D}^\delta})$$

is positive (respectively nonnegative) for arbitrary complex numbers $A^\alpha, B^\alpha, C^\alpha, D^\alpha$ when $A^\alpha \bar{B}^\beta - C^\alpha \bar{D}^\beta \neq 0$ for at least one pair of indices (α, β) .

Definition. The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is said to be *very strongly negative* (respectively *very strongly seminegative*) if

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha\bar{\beta}} \bar{\xi}^{\delta\bar{\gamma}}$$

is positive (respectively nonnegative) for arbitrary complex numbers $\xi^{\alpha\bar{\beta}}$ when $\xi^{\alpha\bar{\beta}} \neq 0$ for at least one pair of indices (α, β) .

Clearly very strong negativity of the curvature tensor implies strong negativity of the curvature tensor. Strong negativity of the curvature tensor implies negativity of the sectional curvature.

THEOREM 1. *Suppose M and N are compact Kähler manifolds and the curvature tensor of M is strongly negative. Suppose $f: N \rightarrow M$ is a harmonic map and the rank over $\mathbf{R}$ of the differential df of f is at least 4 at some point of N . Then f is either holomorphic or conjugate holomorphic.*

In Theorem 1 the curvature condition on M can be weakened to the following. The curvature tensor of M is strongly seminegative everywhere and is strongly negative at $f(P)$ for some $P \in N$ with $\text{rank}_{\mathbf{R}} df \geq 4$ at P .

The following strong rigidity theorem is a consequence of Theorem 1.

THEOREM 2. *Let M be a compact Kähler manifold of complex dimension at least two whose curvature tensor is strongly negative. Then a compact Kähler manifold of the same homotopy type as M must be either biholomorphic or conjugate biholomorphic to M .*

Though most of the bounded symmetric domains do not have strongly negative curvature tensor, our method can also yield the strong rigidity of compact quotients of classical bounded symmetric domains and the complex-analyticity of harmonic maps into them. The precise statements are contained in the following two theorems.

THEOREM 3. *Suppose N is a compact Kähler manifold and M is a compact quotient of a bounded symmetric domain of type $I_{m,n}$ ($mn \geq 2$), II_n*

($n \geq 3$), III_n ($n \geq 2$), or IV_n ($n \geq 3$) (whose precise descriptions are given in § 5). Suppose $f: N \rightarrow M$ is a harmonic map which is a submersion at some point of N . Then f is either holomorphic or conjugate holomorphic.

THEOREM 4. *Let M be a compact quotient of a bounded symmetric domain of type $\text{I}_{m,n}$ ($mn \geq 2$), II_n ($n \geq 3$), III_n ($n \geq 2$), or IV_n ($n \geq 3$). Then any compact Kähler manifold of the same homotopy type as M must be either biholomorphic or conjugate biholomorphic to M .*

The main part of this paper will devoted to the proofs of these theorems.

As examples of compact Kähler manifolds with very strongly negative curvature tensor, we show that any compact quotient of the ball has very strongly negative curvature tensor.

PROPOSITION 1. *The curvature tensor of the invariant metric of the ball is very strongly negative.*

Proof. The invariant metric $2 \operatorname{Re} \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ of the ball B of $\mathbb{C}^n$ is given by

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} (-\log(1 - |z|^2)) .$$

Since

$$-\log(1 - |z|^2) = |z|^2 + \frac{|z|^4}{2} + \frac{|z|^6}{3} + \dots ,$$

it follows that at the origin all the components of the curvature tensor

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \partial_\alpha \partial_{\bar{\beta}} \partial_\gamma \partial_{\bar{\delta}} (-\log(1 - |z|^2)) = \partial_\alpha \partial_{\bar{\beta}} \partial_\gamma \partial_{\bar{\delta}} \left(\frac{|z|^4}{2} \right)$$

are zero except the following

$$\begin{aligned} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} &= 2 , \\ R_{\alpha\bar{\alpha}\beta\bar{\beta}} &= R_{\alpha\bar{\beta}\beta\bar{\alpha}} = 1 \quad \text{for } \alpha \neq \beta . \end{aligned}$$

Hence at the origin

$$\begin{aligned} \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha\bar{\beta}} \xi^{\gamma\bar{\delta}} &= \sum_\alpha (\sum_{\gamma, \delta} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \xi^{\alpha\bar{\alpha}} \xi^{\gamma\bar{\gamma}}) + \sum_{\alpha \neq \beta} (\sum_{\gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha\bar{\beta}} \xi^{\gamma\bar{\delta}}) \\ &= \sum_\alpha (\sum_\gamma R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \xi^{\alpha\bar{\alpha}} \xi^{\gamma\bar{\gamma}}) + \sum_{\alpha \neq \beta} R_{\alpha\bar{\beta}\beta\bar{\alpha}} \xi^{\alpha\bar{\beta}} \xi^{\beta\bar{\alpha}} \\ &= 2 \sum_\alpha |\xi^{\alpha\bar{\alpha}}|^2 + \sum_{\alpha \neq \gamma} \xi^{\alpha\bar{\alpha}} \xi^{\gamma\bar{\gamma}} + \sum_{\alpha \neq \beta} |\xi^{\alpha\bar{\beta}}|^2 \\ &= \sum_\alpha |\xi^{\alpha\bar{\alpha}}|^2 + \sum_{\alpha, \gamma} \xi^{\alpha\bar{\alpha}} \xi^{\gamma\bar{\gamma}} + \sum_{\alpha \neq \beta} |\xi^{\alpha\bar{\beta}}|^2 \\ &= \sum_\alpha |\xi^{\alpha\bar{\alpha}}|^2 + |\sum_\alpha \xi^{\alpha\bar{\alpha}}|^2 + \sum_{\alpha \neq \beta} |\xi^{\alpha\bar{\beta}}|^2 \end{aligned}$$

which is ≥ 0 and is zero if and only if all $\xi^{\alpha\bar{\beta}} = 0$.

Q.E.D

Compact quotients of the ball are not the only examples in complex dimension ≥ 2 of compact Kähler manifolds with very strongly negative curvature tensor. Recently Mostow-Siu [8] constructed a compact Kähler

surface with very strongly negative curvature tensor whose universal covering is not biholomorphic to the ball of complex dimension two.

3. A Bochner type identity

Suppose M is a Kähler manifold with metric tensor

$$g = 2 \operatorname{Re} \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$$

and suppose N is a complex manifold and $f: N \rightarrow M$ is a smooth (i.e., C^∞) map. Let T_M denote the real tangent bundle of M when M is regarded as a real manifold. The complex structure of M gives a decomposition of $T_M \otimes \mathbb{C}$ into tangent vectors of type $(1, 0)$ and type $(0, 1)$,

$$T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1}.$$

The differential $df: T_N \rightarrow T_M$ of f given rise to a map

$$df \otimes \mathbb{C}: T_N \otimes \mathbb{C} \longrightarrow T_M \otimes \mathbb{C}.$$

Composing this with the projection map

$$\Pi_{1,0}: T_M \otimes \mathbb{C} \longrightarrow T_M^{1,0}$$

we obtain

$$\Pi_{1,0} \circ (df \otimes \mathbb{C}): T_N \otimes \mathbb{C} \longrightarrow T_M^{1,0}.$$

This is equivalent to a bundle map

$$T_N \otimes \mathbb{C} \longrightarrow f^* T_M^{1,0}$$

of $\mathbb{C}$ -vector bundles over N . Composing this bundle map with the inclusion map

$$T_N^{0,1} \hookrightarrow T_N \otimes \mathbb{C},$$

we obtain a bundle map from $T_N^{0,1} \rightarrow f^* T_M^{1,0}$ which we denote by $\bar{\partial}f$. Hence $\bar{\partial}f$ is a smooth section of the $\mathbb{C}$ -vector bundle

$$\operatorname{Hom}_{\mathbb{C}}(T_N^{0,1}, f^* T_M^{1,0}) = (T_N^{1,0})^* \otimes f^* T_M^{1,0}$$

over N . In other words $\bar{\partial}f$ is an $f^* T_M^{1,0}$ -valued $(0, 1)$ -form on N .

Let (w^i) be a local holomorphic coordinate chart of N . Then in terms of the local coordinate charts (z^α) and (w^i) of M and N , $\bar{\partial}f$ is simply represented by $(\partial_i f^\alpha)$, where $\partial_i = \partial/\partial w^i$. In Sections 1-5 we will use the notation $\partial_i = \partial/\partial \bar{w}^i$ and also the notations $\partial_i = \partial/\partial w^i$, $\partial_\alpha = \partial/\partial z^\alpha$, and $\partial_{\bar{\alpha}} = \partial/\partial \bar{z}^\alpha$. In these notations we may substitute another lower case italic (respectively Greek) for i (respectively α).

Likewise we define ∂f , $\bar{\partial}f$, and $\bar{\partial}f$. ∂f is an $f^* T_M^{1,0}$ -valued $(1, 0)$ -form on N represented by $(\partial_i f^\alpha)$. $\bar{\partial}f$ is an $f^* T_M^{0,1}$ -valued $(1, 0)$ -form on N represented by $(\partial_i \bar{f}^\alpha)$. $\bar{\partial}f$ is an $f^* T_M^{0,1}$ -valued $(0, 1)$ -form on N represented by $(\partial_i \bar{f}^\alpha)$. It

is clear that $\partial\bar{f}$ is the complex conjugate of $\bar{\partial}f$ and $\bar{\partial}\bar{f}$ is the complex conjugate of ∂f .

Let ∇ be the Riemannian connection of M defined by its Kähler metric. It induces a connection $f^*\nabla$ on the vector bundle $f^*T^{1,0}$. This connection together with the ∂ operator of N enable us to define, for any $f^*T^{1,0}$ -valued $(0, 1)$ -form ω on N , the ∂ exterior derivative of ω , which we denote by $D\omega$. It is an $f^*T^{1,0}$ -valued $(1, 1)$ -form on N . In terms of local coordinates, if $\omega = (\omega_j^\alpha)$, then

$$D\omega = (\sum_{i,j} \eta_{i\bar{j}}^\alpha dw^i \wedge d\bar{w}^j)$$

with

$$\eta_{i\bar{j}}^\alpha = \partial_i \omega_j^\alpha + \sum_{\beta, \gamma} {}^M\Gamma_{\beta\gamma}^\alpha \omega_j^\beta \partial_i f^\gamma$$

where ${}^M\Gamma_{\beta\gamma}^\alpha$ is the Christoffel symbol of M (evaluated at $f(Q)$ when the equation is considered at the point Q of N).

Likewise we define, for any $f^*T^{0,1}$ -valued $(1, 0)$ -form ω' on N , the $\bar{\partial}$ exterior derivative $\bar{D}\omega'$ of ω' which is an $f^*T^{0,1}$ -valued $(1, 1)$ -form on N .

Let A_r^s be the bundle over M of (complex-valued) tensors of contravariant order r and covariant order s . Let σ be a section of A_r^s on M and let τ be an $f^*A_r^s$ -valued p -form on N . We denote by $\langle \sigma, \tau \rangle$ the p -form on N obtained by the contraction which contracts elements of A_r^s with elements of A_r^s to form scalars.

Let $R = (R_{\alpha\bar{\beta}\gamma\bar{\delta}})$ denote the curvature tensor of M . We are now ready to state our Bochner type identity.

PROPOSITION 2.

$$\partial\bar{\partial}\langle g, \bar{\partial}f \wedge \partial\bar{f} \rangle = \langle R, \bar{\partial}f \wedge \partial\bar{f} \wedge \partial f \wedge \bar{\partial}\bar{f} \rangle - \langle g, D\bar{\partial}f \wedge \bar{D}\partial\bar{f} \rangle.$$

In local coordinates

$$\begin{aligned} \partial\bar{\partial} \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} \bar{\partial}f^\alpha \wedge \partial\bar{f}^\beta &= \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \bar{\partial}f^\alpha \wedge \partial\bar{f}^\beta \wedge \partial f^\gamma \wedge \bar{\partial}\bar{f}^\delta \\ &\quad - \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} D\bar{\partial}f^\alpha \wedge \bar{D}\partial\bar{f}^\beta, \end{aligned}$$

where

$$\begin{aligned} D\bar{\partial}f^\alpha &= \partial\bar{\partial}f^\alpha + \sum_{\beta, \gamma} {}^M\Gamma_{\beta\gamma}^\alpha \partial f^\beta \wedge \bar{\partial}f^\gamma, \\ \bar{D}\partial\bar{f}^\beta &= \bar{\partial}\partial\bar{f}^\beta + \sum_{\alpha, \gamma} {}^M\Gamma_{\alpha\gamma}^\beta \bar{\partial}f^\alpha \wedge \partial\bar{f}^\gamma. \end{aligned}$$

Proof. Fix a point Q of N where we want to verify the equation. Let $P = f(Q)$. We choose a holomorphic local coordinate system (z^α) of M at P such that $dg_{\alpha\bar{\beta}} = 0$ at P . Then at P all the Christoffel symbols ${}^M\Gamma_{\beta\gamma}^\alpha$ vanish. It follows that at P ,

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \partial_\gamma \partial_{\bar{\delta}} g_{\alpha\bar{\beta}}$$

and at Q

$$\begin{aligned} D\bar{\partial}f^\alpha &= \partial\bar{\partial}f^\alpha, \\ \bar{D}\partial f^\alpha &= \bar{\partial}\partial f^\alpha. \end{aligned}$$

Using $\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}} = 0$ at P and recalling that in the equation $g_{\alpha\bar{\beta}}$ stands for $g_{\alpha\bar{\beta}} \circ f$, we have at Q

$$\begin{aligned} \partial\bar{\partial}\sum_{\alpha,\beta}g_{\alpha\bar{\beta}}\bar{\partial}f^\alpha\wedge\partial f^\beta &= \sum_{\alpha,\beta,\gamma,\delta}\partial_\delta\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta \\ &+ \sum_{\alpha,\beta,\gamma,\delta}\partial_\delta\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta \\ &+ \sum_{\alpha,\beta,\gamma,\delta}\partial_\delta\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta \\ &+ \sum_{\alpha,\beta,\gamma,\delta}\partial_\delta\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta \\ &- \sum_{\alpha,\beta}g_{\alpha\bar{\beta}}\partial\bar{\partial}f^\alpha\wedge\bar{\partial}\partial f^\beta. \end{aligned}$$

Since $\partial_\delta\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}$ is symmetric in α, γ (due to the fact that the metric $2\operatorname{Re}\sum g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ is Kähler) and since $\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta$ is skew-symmetric in α, γ , it follows that the first term on the right-hand side of the preceding equation is zero. Likewise, it follows from the symmetry of $\partial_\delta\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}$ in α, γ and the skew-symmetry of $\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta$ in α, γ that the second term on the right-hand side is zero. And it follows from the symmetry of $\partial_\delta\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}$ in β, δ and the skew-symmetry of $\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta$ that the last term on the right-hand side is zero. Hence

$$\begin{aligned} \partial\bar{\partial}\sum_{\alpha,\beta}g_{\alpha\bar{\beta}}\bar{\partial}f^\alpha\wedge\partial f^\beta &= \sum_{\alpha,\beta,\gamma,\delta}R_{\alpha\bar{\beta}\delta\bar{\gamma}}\partial f^\delta\wedge\bar{\partial}f^\gamma\wedge\bar{\partial}f^\alpha\wedge\partial f^\beta \\ &- \sum_{\alpha,\beta}g_{\alpha\bar{\beta}}D\bar{\partial}f^\alpha\wedge\bar{D}\partial f^\beta. \end{aligned} \quad \text{Q.E.D.}$$

4. Complex-analyticity of harmonic maps

PROPOSITION 3. *Let M, N be compact Kähler manifolds and $f: N \rightarrow M$ be a harmonic map. Let $m = \dim_{\mathbb{C}} M$, $n = \dim_{\mathbb{C}} N$, and f^α ($1 \leq \alpha \leq m$) be the components of f with respect to a local coordinate system of M . Let w^i ($1 \leq i \leq n$) be a local coordinate system of N and let*

$$\xi_{i\bar{j}}^{\alpha\bar{\beta}} = (\partial_i f^\alpha)(\bar{\partial}_j \bar{f}^\beta) - (\partial_j f^\alpha)(\bar{\partial}_i \bar{f}^\beta).$$

If the curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of M is strongly seminegative, then for $1 \leq i, j \leq n$,

$$\sum_{\alpha,\beta,\gamma,\delta}R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi_{i\bar{j}}^{\alpha\bar{\beta}}\xi_{j\bar{i}}^{\gamma\bar{\delta}} = 0.$$

Proof. Let $g = 2\operatorname{Re}\sum_{\alpha,\beta}g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ and $h = 2\operatorname{Re}\sum_{i,j}h_{i\bar{j}}dw^i d\bar{w}^j$ be respectively the Kähler metrics of M and N . Denote the Kähler form $\sqrt{-1}\sum_{i,j}h_{i\bar{j}}dw^i d\bar{w}^j$ of N by ω . We use the notations of Section 3. First we show that

$$(4.1) \quad \langle g, D\bar{\partial}f \wedge \bar{D}\partial\bar{f} \rangle \wedge \omega^{n-2} = \chi\omega^n$$

for some nonnegative function χ on N .

Fix a point Q of N and let $P = f(Q)$. To prove (4.1) at Q , we choose local holomorphic coordinate systems (z^α) at P and (w^i) at Q such that with respect to these coordinate systems

$$\begin{aligned} g_{\alpha\bar{\beta}} &= \delta_{\alpha\bar{\beta}} && \text{(the Kronecker delta) at } P, \\ dg_{\alpha\bar{\beta}} &= 0 && \text{at } P, \\ h_{i\bar{j}} &= \delta_{i\bar{j}} && \text{at } Q, \\ dh_{i\bar{j}} &= 0 && \text{at } Q. \end{aligned}$$

Let u^α and v^α be respectively the real and imaginary parts of f^α . Let $\lambda_1^{(\alpha)}, \dots, \lambda_n^{(\alpha)}$ (respectively $\mu_1^{(\alpha)}, \dots, \mu_n^{(\alpha)}$) be the eigenvalues of the complex Hessian of u^α (respectively v^α) at Q with respect to the coordinates (w^i) . We have

$$\begin{aligned} \langle g, D\bar{\partial}f \wedge \bar{D}\partial\bar{f} \rangle \wedge \omega^{n-2} &= \sum_\alpha \partial\bar{\partial}f^\alpha \wedge \bar{\partial}\partial\bar{f}^\alpha \wedge \omega^{n-2} \\ &= \sum_\alpha (\partial\bar{\partial}u^\alpha \wedge \bar{\partial}\partial u^\alpha + \partial\bar{\partial}v^\alpha \wedge \bar{\partial}\partial v^\alpha) \wedge \omega^{n-2} \\ &= \frac{-1}{n(n-1)} \sum_\alpha \sum_{i \neq j} (\lambda_i^{(\alpha)} \lambda_j^{(\alpha)} + \mu_i^{(\alpha)} \mu_j^{(\alpha)}) \omega^n \end{aligned}$$

where the last identity is obtained by diagonalizing the complex Hessians of u^α and v^α (nonsimultaneously). Now (4.1) follows from the identity

$$\sum_{i \neq j} \lambda_i \lambda_j = (\sum_i \lambda_i)^2 - \sum_i \lambda_i^2$$

and the fact that the harmonicity of f implies

$$\sum_i \lambda_i^{(\alpha)} = \sum_i \mu_i^{(\alpha)} = 0.$$

This proof of (4.1) yields also the statement that $\chi = 0$ at Q if and only if $D\bar{\partial}f^\alpha = 0$ at Q for all α .

We now express

$$\langle R, \bar{\partial}f \wedge \partial\bar{f} \wedge \partial f \wedge \bar{\partial}\bar{f} \rangle \wedge \omega^{n-2}$$

in terms of local coordinates. Fix $Q \in N$. Choose an arbitrary local holomorphic coordinate system (w^i) at Q such that $h_{i\bar{j}} = \delta_{i\bar{j}}$ at Q . Then at Q

$$\begin{aligned} &\langle R, \bar{\partial}f \wedge \partial\bar{f} \wedge \partial f \wedge \bar{\partial}\bar{f} \rangle \wedge \omega^{n-2} \\ &= \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \bar{\partial}f^\alpha \wedge \partial\bar{f}^\beta \wedge \partial f^\gamma \wedge \bar{\partial}\bar{f}^\delta \wedge (n-2)! (\sqrt{-1})^{n-2} \\ &\quad \times \sum_{1 \leq i < j \leq n} \wedge_{1 \leq k \leq n, k \neq i, j} (dw^k \wedge d\bar{w}^k) \\ &= (n-2)! (\sqrt{-1})^{n-2} \sum_{\alpha, \beta, \gamma, \delta, 1 \leq i < j \leq n} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \big(-(\partial_i f^\alpha)(\partial_i \bar{f}^\beta)(\partial_j f^\gamma)(\partial_j \bar{f}^\delta) \\ &\quad + (\partial_i f^\alpha)(\partial_j \bar{f}^\beta)(\partial_i f^\gamma)(\partial_j \bar{f}^\delta) + (\partial_j f^\alpha)(\partial_i \bar{f}^\beta)(\partial_j f^\gamma)(\partial_i \bar{f}^\delta) \\ &\quad - (\partial_j f^\alpha)(\partial_j \bar{f}^\beta)(\partial_i f^\gamma)(\partial_i \bar{f}^\delta) \big) \wedge \big(\wedge_{1 \leq k \leq n} (dw^k \wedge d\bar{w}^k) \big) \end{aligned}$$

$$\begin{aligned}
 &= \frac{-1}{n(n-1)} \sum_{\alpha, \beta, \gamma, \delta, 1 \leq i < j \leq n} R_{\alpha \bar{\beta} \gamma \bar{\delta}} ((\partial_i f^\alpha)(\overline{\partial_j f^\delta}) - (\partial_j f^\alpha)(\overline{\partial_i f^\delta})) \\
 &\quad \times ((\overline{\partial_i f^\beta})(\partial_j f^\gamma) - (\overline{\partial_j f^\beta})(\partial_i f^\gamma)) \omega^n.
 \end{aligned}$$

Since $R_{\alpha \bar{\beta} \gamma \bar{\delta}}$ is symmetric in β, δ , it follows that

$$(4.2) \quad \langle R, \bar{\partial} f \wedge \partial \bar{f} \wedge \partial f \wedge \bar{\partial} \bar{f} \rangle \wedge \omega^{n-2} = \frac{4}{n(n-1)} \sum_{\alpha, \beta, \gamma, \delta, 1 \leq i < j \leq n} R_{\alpha \bar{\beta} \gamma \bar{\delta}} \xi_{ij}^{\alpha \bar{\beta}} \bar{\xi}_{ij}^{\gamma \bar{\delta}} \omega^n$$

at any point Q of N when $h_{ij} = \delta_{ij}$ at Q .

The strong seminegativity of the curvature tensor $R_{\alpha \bar{\beta} \gamma \bar{\delta}}$ implies that

$$(4.3) \quad \langle R, \bar{\partial} f \wedge \partial \bar{f} \wedge \partial f \wedge \bar{\partial} \bar{f} \rangle \wedge \omega^{n-2} = \sigma \omega^n$$

for some nonpositive function σ on N .

By Proposition 2,

$$\begin{aligned}
 \partial \bar{\partial} \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge \omega^{n-2} &= \langle R, \bar{\partial} f \wedge \partial \bar{f} \wedge \partial f \wedge \bar{\partial} \bar{f} \rangle \wedge \omega^{n-2} \\
 &\quad - \langle g, D \bar{\partial} f \wedge \bar{D} \partial f \rangle \wedge \omega^{n-2}.
 \end{aligned}$$

Since the left-hand side is an exact form, it follows that

$$\int_N \langle R, \bar{\partial} f \wedge \partial \bar{f} \wedge \partial f \wedge \bar{\partial} \bar{f} \rangle \wedge \omega^{n-2} - \langle g, D \bar{\partial} f \wedge \bar{D} \partial f \rangle \wedge \omega^{n-2} = 0.$$

From (4.1) and (4.3) we conclude that χ and σ are identically zero. By (4.2),

$$(4.4) \quad \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \bar{\beta} \gamma \bar{\delta}} \xi_{ij}^{\alpha \bar{\beta}} \bar{\xi}_{ij}^{\gamma \bar{\delta}} = 0.$$

To finish the proof, we take an arbitrary local holomorphic coordinate system (ζ^i) at Q . Let

$$\eta_{ij}^{\alpha \bar{\beta}} = \frac{\partial f^\alpha}{\partial \zeta^i} \frac{\overline{\partial f^\beta}}{\partial \bar{\zeta}^j} - \frac{\partial f^\alpha}{\partial \zeta^j} \frac{\overline{\partial f^\beta}}{\partial \bar{\zeta}^i}$$

evaluated at Q . We have to show that

$$(4.5) \quad \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \bar{\beta} \gamma \bar{\delta}} \eta_{ij}^{\alpha \bar{\beta}} \bar{\eta}_{ij}^{\gamma \bar{\delta}} = 0.$$

Fix $1 \leq i, j \leq n$. Since $\eta_{ij}^{\alpha \bar{\beta}} = 0$ when $i = j$, we can assume that $i \neq j$. At Q let

$$\begin{aligned}
 \frac{\partial}{\partial \zeta^i} &= \sum_k a_k \frac{\partial}{\partial w^k}, \\
 \frac{\partial}{\partial \zeta^j} &= \sum_k b_k \frac{\partial}{\partial w^k}.
 \end{aligned}$$

There exist uniquely two unit orthogonal n -vectors $(A_1^{(\nu)}, \dots, A_n^{(\nu)})$, $\nu = 1, 2$, such that $a_k = c A_k^{(1)}$ and $b_k = c' A_k^{(1)} + c'' A_k^{(2)}$, $1 \leq k \leq n$, for some complex numbers c, c', c'' . Construct a unitary matrix $(A_k^{(\nu)})$, $1 \leq \nu, k \leq n$ whose first two rows are these two unit orthogonal n -vectors. Define a local holomorphic coordinate system (τ^k) at Q so that

$$\frac{\partial}{\partial \tau^\nu} = \sum_k A_k^{(\nu)} \frac{\partial}{\partial w^k}$$

at Q . Let

$$\theta^{\alpha\bar{\beta}} = \frac{\partial f^\alpha}{\partial \tau^1} \frac{\overline{\partial f^\beta}}{\partial \tau^2} - \frac{\partial f^\alpha}{\partial \tau^1} \frac{\overline{\partial f^\beta}}{\partial \tau^2}$$

evaluated at Q . By replacing the coordinate system (w^k) by the coordinate system (τ^k) , corresponding to (4.4) we obtain the following:

$$(4.6) \quad \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \theta^{\alpha\bar{\beta}} \overline{\theta^{\gamma\bar{\delta}}} = 0.$$

Now

$$\begin{aligned} \eta_{ij}^{\alpha\bar{\beta}} &= \sum_{k,l} \bar{a}_k \bar{b}_l \xi_{kl}^{\alpha\bar{\beta}} \\ &= \bar{c} \bar{c}' \sum_{k,l} \overline{A_k^{(1)}} \overline{A_l^{(1)}} \xi_{kl}^{\alpha\bar{\beta}} + \bar{c} \bar{c}'' \sum_{k,l} \overline{A_k^{(1)}} \overline{A_l^{(2)}} \xi_{kl}^{\alpha\bar{\beta}} \\ &= \bar{c} \bar{c}'' \theta^{\alpha\bar{\beta}}, \end{aligned}$$

where the vanishing of $\sum_{k,l} \overline{A_k^{(1)}} \overline{A_l^{(1)}} \xi_{kl}^{\alpha\bar{\beta}}$ is due to the skew-symmetry of $\xi_{kl}^{\alpha\bar{\beta}}$ in k, l . Hence (4.5) follows from (4.6). Q.E.D.

In order to avoid repeating part of the argument in the case of classical bounded symmetric domains, instead of proving directly Theorem 1, we prove a more general result. To state this more general result, we have to introduce the following definition.

Definition. The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of a Kähler manifold of complex dimension m is called *negative of order k* if it is strongly seminegative and it enjoys the following property. If $A = (A_i^\alpha)$, $B = (B_i^\alpha)$ are any two $m \times k$ matrices ($1 \leq \alpha \leq m$, $1 \leq i \leq k$) with

$$\text{rank} \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} = 2k$$

and if

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi_{ij}^{\alpha\bar{\beta}} \overline{\xi_{ij}^{\gamma\bar{\delta}}} = 0$$

for all $1 \leq i, j \leq k$, where

$$\xi_{ij}^{\alpha\bar{\beta}} = A_i^\alpha \bar{B}_j^\beta - A_j^\alpha \bar{B}_i^\beta,$$

then either $A = 0$ or $B = 0$.

The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is called *adequately negative* if it is negative of order m .

The above definition is motivated by the following equivalent definition which is clumsier to state but which renders more transparent the motivation behind the definition.

Definition. The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of a Kähler manifold M is said to be *negative of order k at a point P of M* if it is strongly seminegative at

P and if it enjoys the following property. If $f: U \rightarrow M$ is a smooth map from an open neighborhood U of 0 in $\mathbb{C}^k$ to M with $f(0) = P$ and $\text{rank}_{\mathbf{R}} df = 2k$ at 0 and if at P

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi_{ij}^{\alpha\bar{\beta}} \overline{\xi_{ij}^{\gamma\bar{\delta}}} = 0$$

for $1 \leq i, j \leq k$, where

$$\xi_{ij}^{\alpha\bar{\beta}} = (\partial_i f^\alpha)(0) \overline{(\partial_j f^\beta)(0)} - (\partial_j f^\alpha)(0) \overline{(\partial_i f^\beta)(0)}$$

(the coordinates of $\mathbb{C}^k$ being w^i , $1 \leq i \leq k$), then either $\partial f = 0$ at 0 or $\bar{\partial} f = 0$ at 0.

The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is said to be *adequately negative* at P if the above holds with the condition $\text{rank}_{\mathbf{R}} df = 2k$ at 0 replaced by the condition that f is locally diffeomorphic at 0.

We will use only the latter definition, because there the indices have transparent meanings and are easier to keep track of.

THEOREM 5. *Let $k \geq 2$. Suppose M and N are compact Kähler manifolds and the curvature tensor of M is negative of order k . Suppose $f: N \rightarrow M$ is a harmonic map and the rank over $\mathbf{R}$ of the differential df of f is at least $2k$ at some point of N . Then f is either holomorphic or conjugate holomorphic.*

Before we prove Theorem 5, we have to prove first the following very simple lemma in linear algebra.

LEMMA 1. *Suppose V is a vector space of dimension n over $\mathbb{C}$ and W is an $\mathbf{R}$ -vector subspace of V and the dimension of W over $\mathbf{R}$ is $\leq 2n - 2k$. Let E be the set of all bases of V over $\mathbb{C}$ and let F be the subset of E consisting of all bases $(e_1, \dots, e_n)$ of V over $\mathbb{C}$ such that $(\sum_{v=1}^k \mathbb{C}e_{i_v}) \cap W = 0$ for all $1 \leq i_1 < \dots < i_k \leq n$. Then F is a dense open subset of E .*

Proof. For $1 \leq i_1 < \dots < i_k \leq n$ let $F_{i_1 \dots i_k}$ be the subset of E consisting of all $(e_1, \dots, e_n)$ with $(\sum_{v=1}^k \mathbb{C}e_{i_v}) \cap W = 0$. It suffices to show that each $F_{i_1 \dots i_k}$ is a dense open subset of E . Let H be the set of all $\mathbb{C}$ -vector subspaces L of complex dimension k in V with $L \cap W = 0$. It suffices to show that H is a dense open subset of the Grassmannian $G_k(V)$ of all $\mathbb{C}$ -vector subspaces of complex dimension k in V . Clearly H is open in $G_k(V)$.

Let $K = W \cap \sqrt{-1}W$ be the maximum $\mathbb{C}$ -vector subspace of V contained in W . We can choose a basis $e_1, \dots, e_n$ of V over $\mathbb{C}$ such that $e_1, \dots, e_p$ is a basis of K over $\mathbb{C}$ and

$$e_1, \dots, e_p, \sqrt{-1}e_1, \dots, \sqrt{-1}e_p, e_{p+1}, \dots, e_n$$

is a basis of W over $\mathbf{R}$. We have $l + p = \dim_{\mathbf{R}} W \leq 2n - 2k$. First we show

that H is nonempty. Let q be the largest integer $\leq (l - p/2)$. Let Q be the $\mathbb{C}$ -vector subspace of V spanned by

$$e_{p+1} + \sqrt{-1}e_{p+2}, e_{p+3} + \sqrt{-1}e_{p+4}, \dots, e_{p+2q-1} + \sqrt{-1}e_{p+2q}, e_{l+1}, \dots, e_n$$

over $\mathbb{C}$. Clearly $Q \cap W = 0$. We claim that $\dim_{\mathbb{C}} Q \geq k$. Since $2q \geq l - p - 1$ and $2n - 2k \geq l + p$, it follows that $2n - 2k + 2q \geq 2l - 1$, which implies $n - k + q \geq l$, because n, k, q, l are integers. Hence $\dim_{\mathbb{C}} Q = q + (n - l) \geq k$. Choose a $\mathbb{C}$ -vector subspace L' of complex dimension k in Q . Then $L' \in H$.

Take $L \in G_k(V)$ and let $g_1, \dots, g_k$ be a basis of L over $\mathbb{C}$. Let

$$g_i = \sum_{j=1}^n a_{ji} e_j \quad (1 \leq i \leq k).$$

Let $A = (a_{ji})_{p < j \leq l, 1 \leq i \leq k}$ and $B = (a_{ji})_{l < j \leq n, 1 \leq i \leq k}$. Then $L \cap W \neq 0$ if and only if for some nonzero column c of k complex numbers $\text{Im } Ac = 0$ and $Bc = 0$, i.e.,

$$\begin{cases} Ac - \bar{A}\bar{c} = 0 \\ Bc = 0 \\ \bar{B}\bar{c} = 0 \end{cases}$$

which is equivalent to

$$\text{rank} \begin{pmatrix} A & -\bar{A} \\ B & 0 \\ 0 & \bar{B} \end{pmatrix} < 2k,$$

because, if

$$\begin{cases} Ac - \bar{A}d = 0 \\ Bc = 0 \\ \bar{B}d = 0 \end{cases}$$

for some c, d not both zero, then by suitably adding the equations to and subtracting the equations from their complex conjugates one obtains

$$\begin{cases} A\xi - \bar{A}\bar{\xi} = 0 \\ B\xi = 0 \\ \bar{B}\bar{\xi} = 0 \end{cases}$$

for $\xi = c + \bar{d}$, $\sqrt{-1}(c - \bar{d})$, one of which is nonzero. By replacing L by L' we obtain g'_i, a'_{ji}, A' , and B' . For $0 \leq t \leq 1$, let

$$\begin{aligned} g_i(t) &= (1 - t)g_i + tg'_i, \\ a_{ji}(t) &= (1 - t)a_{ji} + ta'_{ji}, \\ A(t) &= (1 - t)A + tA', \\ B(t) &= (1 - t)B + tB'. \end{aligned}$$

The following two inequalities

$$\begin{aligned} \text{rank}(a_{ji}(t))_{1 \leq j \leq n, 1 \leq i \leq k} &\geq k, \\ \text{rank} \begin{pmatrix} A(t) & -\overline{A(t)} \\ B(t) & 0 \\ 0 & \overline{B(t)} \end{pmatrix} &\geq 2k \end{aligned}$$

hold at $t = 1$ and hence hold for all $t \in [0, 1] - J$, where J is some finite set. For $t \in [0, 1] - J$ let $L(t)$ be the $\mathbb{C}$ -vector subspace of V spanned by $g_1(t), \dots, g_k(t)$ over $\mathbb{C}$. Then $L(t) \in H$ and $L(t)$ approaches L in $G_k(V)$ as t approaches 0. Hence H is dense in $G_k(V)$. Q.E.D.

Proof of Theorem 5. Now $\text{rank}_{\mathbb{R}} df \geq 2k$ at some point of N and hence at every point of some nonempty connected open subset U of N . We first prove that

$$(4.7) \quad \text{either } \partial f \equiv 0 \text{ on } U \text{ or } \bar{\partial} f \equiv 0 \text{ on } U.$$

It suffices to show that for every point Q of U

$$(4.8) \quad \text{either } \partial f = 0 \text{ at } Q \text{ or } \bar{\partial} f = 0 \text{ at } Q,$$

because the nowhere-vanishing of df on U implies that the two closed subsets $U \cap \{\partial f = 0\}$ and $U \cap \{\bar{\partial} f = 0\}$ are disjoint and since by (4.8) their union is the connected set U , one of them is equal to U .

Since the curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is strongly seminegative, it follows from Proposition 3 that for $1 \leq i, j \leq n$ and $1 \leq \alpha, \beta \leq m$

$$(4.9) \quad \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \zeta_i^{\alpha\bar{\beta}} \bar{\zeta}_j^{\gamma\bar{\delta}} = 0 \quad \text{at } P$$

where

$$\zeta_i^{\alpha\bar{\beta}} = (\partial_i f^\alpha)(Q) \overline{(\partial_j f^\beta)(Q)} - (\partial_j f^\alpha)(Q) \overline{(\partial_i f^\beta)(Q)}$$

(w^i , $1 \leq i \leq m$, being local coordinates of N and f^α , $1 \leq \alpha \leq n$, being components of f with respect to a local coordinate system of M).

Let $P = f(Q)$ and let $T_{M,P}$ (respectively $T_{N,Q}$) be the tangent space of M at P (respectively N at Q) when M (respectively N) is regarded as a real manifold. Let K be the kernel of $df: T_{N,Q} \rightarrow T_{M,P}$. The complex structure of M makes $T_{M,P}$ a vector space over $\mathbb{C}$. Since $\text{rank}_{\mathbb{R}} df \geq 2k$, it follows that $\dim_{\mathbb{R}} K \leq 2n - 2k$. By Lemma 1, there exists a basis $g_1, \dots, g_n$ of $T_{N,Q}$ over $\mathbb{C}$ such that for $1 \leq i_1 < \dots < i_k \leq n$ the intersection of K with the $\mathbb{C}$ -vector subspace of $T_{N,Q}$ spanned by $g_{i_1}, \dots, g_{i_k}$ is the zero vector subspace. Choose a holomorphic coordinate system (w^i) of an open neighborhood W of Q in U such that $g_i = 2 \text{Re}(\partial/\partial w^i)$. For $1 \leq i_1 < \dots < i_k \leq n$ let $\Phi_{i_1, \dots, i_k}$ be the restriction of f to

$$W \cap \{w_\gamma = w_\gamma(Q) \text{ for } 1 \leq \gamma \leq n, \gamma \neq i_1, \dots, i_k\}.$$

Then $\text{rank}_{\mathbf{R}} d\Phi_{i_1, \dots, i_k} = 2k$ at Q . Since $R_{\alpha\bar{\beta}\gamma\delta}$ is negative of order k at every point of M , it follows from (4.9) that for $1 \leq i_1 < \dots < i_k \leq n$ either $\partial_{\bar{j}} f^\alpha = 0$ at Q for $1 \leq \alpha \leq m$ and $j = i_1, \dots, i_k$ or $\partial_{\bar{j}} f^\alpha = 0$ at Q for $1 \leq \alpha \leq m$ and $j = i_1, \dots, i_k$. Now (4.8) follows from $k \geq 2$, because, if $\partial_{\bar{j}} f^\alpha \neq 0$ at Q for some $1 \leq \alpha \leq m$ and some $1 \leq j \leq n$, and $\partial_l f^\beta \neq 0$ at Q for some $1 \leq \beta \leq m$ and some $1 \leq l \leq n$, then we can select $1 \leq i_1 < \dots < i_k \leq n$ such that both j and l belong to the set $\{i_1, \dots, i_k\}$. The theorem now follows from (4.8) and the following proposition. Q.E.D.

PROPOSITION 4. *Suppose M, N are compact Kähler manifolds and $f: N \rightarrow M$ is a harmonic map. Let U be a nonempty open subset of N . If f is holomorphic (respectively conjugate holomorphic) on U , then f is holomorphic (respectively conjugate holomorphic) on N .*

Proof. Since the proofs of the holomorphic case and the conjugate holomorphic case are similar, we prove only the holomorphic case.

Let Ω be the largest connected open subset of N containing U such that $\bar{\partial}f$ vanishes identically on Ω . It suffices to show that Ω is closed in N . Suppose the contrary. Ω has a boundary point Q . Let W be a connected open neighborhood of Q in N such that

i) there exists a holomorphic coordinate system (w^i) on some open neighborhood of the closure of W and

ii) there exists a holomorphic coordinate system (z^α) on some open neighborhood of the closure of $f(W)$.

The harmonicity of f is given by the equation

$$\Delta_N f^\alpha + \sum_{i,j,\beta,\gamma} {}^M \Gamma_{\beta\gamma}^\alpha (\partial_i f^\beta) (\partial_{\bar{j}} f^\gamma) h^{\bar{j}i} = 0,$$

where $(h^{\bar{j}i})$ is the inverse matrix of the matrix $(h_{i\bar{j}})$ of the Kähler metric of N , Δ_N is the Laplace-Beltrami operator of N , and ${}^M \Gamma_{\beta\gamma}^\alpha$ is the Christoffel symbol of M . Applying $\partial_{\bar{k}}$ to this equation and recalling $\Delta_N = 2 \sum_{i,j} h^{\bar{j}i} \partial_i \partial_{\bar{j}}$, we obtain

$$\begin{aligned} \Delta_N (\partial_{\bar{k}} f^\alpha) + \sum_{i,j} (2 \partial_{\bar{k}} h^{\bar{j}i}) \partial_i \partial_{\bar{j}} f^\alpha + \sum_j (\sum_{i,\beta,\gamma} \partial_{\bar{k}} ({}^M \Gamma_{\beta\gamma}^\alpha (\partial_i f^\beta) h^{\bar{j}i})) \partial_{\bar{j}} f^\gamma \\ + \sum_j (\sum_{i,\beta,\gamma} {}^M \Gamma_{\beta\gamma}^\alpha (\partial_i f^\beta) h^{\bar{j}i}) \partial_{\bar{k}} \partial_{\bar{j}} f^\gamma = 0. \end{aligned}$$

Hence for some positive number C

$$|\Delta_N (\partial_{\bar{k}} f^\alpha)| \leq C (\sum_{i,j} |\partial_i \partial_{\bar{j}} f^\alpha| + \sum_{j,\gamma} |\partial_{\bar{j}} f^\gamma| + \sum_{j,\gamma} |\partial_{\bar{k}} \partial_{\bar{j}} f^\gamma|)$$

on W . Let u_k^α and v_k^α be respectively the real and imaginary parts of $\partial_{\bar{k}} f^\alpha$. Since Δ_N is a real operator (i.e., it maps real functions to real functions), it follows that for some positive number C'

$$\begin{aligned} |\Delta_N u_k^\alpha|^2 &\leq C' \sum_{\beta,l} (|\text{grad } u_l^\beta|^2 + |\text{grad } v_l^\beta|^2 + |u_l^\beta|^2 + |v_l^\beta|^2), \\ |\Delta_N v_k^\alpha|^2 &\leq C' \sum_{\beta,l} (|\text{grad } u_l^\beta|^2 + |\text{grad } v_l^\beta|^2 + |u_l^\beta|^2 + |v_l^\beta|^2) \end{aligned}$$

on W , where grad is the gradient with respect to the coordinate system whose coordinate functions are the real and imaginary parts of w^i . By applying Aronszajn's unique continuation theorem [1, p. 248] to the system of functions u_k^α, v_k^α ($1 \leq \alpha \leq m, 1 \leq k \leq n$) and to the elliptic operator Δ_N , we conclude from the identical vanishing of u_k^α, v_k^α on $W \cap \Omega$ that u_k^α, v_k^α vanish identically on W . This contradicts the fact that Q is a boundary point of Ω . Hence $\Omega = N$ and $\bar{\partial}f \equiv 0$ on N . Q.E.D.

Theorem 1 follows from Theorem 5 and the following lemma.

LEMMA 2. *If the curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of a Kähler manifold M is strongly negative, then it is negative of order 2.*

Proof. Let U be an open neighborhood of 0 in $\mathbb{C}^2$ with coordinates w^1, w^2 . Suppose $f: U \rightarrow M$ is a smooth map which is a local immersion at 0 such that for $1 \leq i, j \leq 2$

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi_i^{\alpha\bar{\beta}} \bar{\xi}_j^{\gamma\bar{\delta}} = 0$$

at $P = f(0)$, where

$$\xi_i^{\alpha\bar{\beta}} = (\partial_i f^\alpha)(0) \overline{(\partial_j f^\beta)(0)} - (\partial_j f^\alpha)(0) \overline{(\partial_i f^\beta)(0)}.$$

We have to show that either $\partial f = 0$ at 0 or $\bar{\partial}f = 0$ at 0.

Let $m = \dim_{\mathbb{C}} M$. Let $T_{M,P}$ be the tangent space of M at P when M is regarded as a real manifold. The complex structure of M makes $T_{M,P}$ a vector space over $\mathbb{C}$. Since f is a local immersion at 0, the image L of df is an $\mathbb{R}$ -vector subspace of real dimension 4 in $T_{M,P}$. By Lemma 1, we can find a basis $e_1, \dots, e_n$ of T_M over $\mathbb{C}$ such that for $1 \leq \alpha < \beta \leq m$ the intersection of L with the $\mathbb{C}$ -vector subspace of $T_{M,P}$ spanned by $e_\gamma, 1 \leq \gamma \leq n, \gamma \neq \alpha, \beta$, is the zero vector subspace. Choose a local coordinate system (z^α) of M at P such that $e_\alpha = 2 \operatorname{Re}(\partial/\partial z^\alpha)$ at P . For $1 \leq \alpha < \beta \leq m$ the map

$$\Phi_{\alpha\beta}: (w^1, w^2) \longmapsto (f^\alpha(w^1, w^2), f^\beta(w^1, w^2))$$

is locally diffeomorphic at 0.

Since $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is strongly negative, it follows that $\xi_i^{\alpha\bar{\beta}} = 0$ for $1 < \alpha, \beta \leq m$ and $1 \leq i, j \leq 2$. Assume $\partial f \neq 0$ at 0. We want to prove that $\bar{\partial}f = 0$ at 0. Since $\partial f^\alpha \neq 0$ at 0 for some $1 \leq \alpha \leq m$, without loss of generality we can assume that $\partial f^1 \neq 0$ at 0. Let p be the number of $\mathbb{C}$ -linearly independent $(1, 0)$ -forms among $\partial f^1, \dots, \partial f^m$ at 0. We distinguish between two cases.

Case 1. $p = 2$. Without loss of generality we can assume that ∂f^1 and ∂f^2 are $\mathbb{C}$ -linearly independent at 0. For $1 \leq \alpha \leq m$ and $\beta = 1, 2$, it follows from $\xi_{12}^{\alpha\bar{\beta}} = 0$ that $\bar{\partial}f^\alpha = c_{\alpha\beta} \overline{\partial f^\beta}$ at 0 for some $c_{\alpha\beta} \in \mathbb{C}$. Hence, if $\bar{\partial}f^\alpha \neq 0$ at 0 for some $1 \leq \alpha \leq n$, then $\partial f^1 = (c_{\alpha 2}/c_{\alpha 1}) \partial f^2$ at 0, contradicting the $\mathbb{C}$ -linear independence of ∂f^1 and ∂f^2 at 0.

Case 2. $p = 1$. For $1 < \alpha \leq m$ there exist complex numbers r_α such that $\partial f^\alpha = r_\alpha \partial f^1$ at 0. For $1 \leq \alpha \leq m$ it follows from $\xi_{1\bar{1}}^{\alpha\bar{1}} = 0$ that $\bar{\partial} f^\alpha = c_\alpha \bar{\partial} f^1$ at 0 for some $c_\alpha \in \mathbb{C}$. For $1 \leq \alpha < \beta \leq m$, $df^\alpha \wedge d\bar{f}^\alpha \wedge df^\beta \wedge d\bar{f}^\beta$ is a linear combination of exterior products of ∂f^1 and $\bar{\partial} f^1$ at 0 and hence vanishes at 0, contradicting the fact that $\Phi_{\alpha\beta}$ is locally diffeomorphic at 0. Q.E.D.

5. Bounded symmetric domains

We recall the four types of classical bounded symmetric domains which we denote by $D_{m,n}^I$, D_n^{II} , $D_{m,n}^{III}$, D_n^{IV} and compute their curvature tensors.

Type I_{m,n}. The domain $D_{m,n}^I$ is an open subset of $\mathbb{C}^{mn}$ and is the set of all $m \times n$ matrices $Z = (z_{\alpha\beta})$ with complex entries such that $I_n - {}^t Z \bar{Z}$ is positive definite, where I_n is the identity matrix of order n and ${}^t \bar{Z}$ is the transpose of the complex conjugate of Z . An invariant Kähler metric has the potential function

$$\begin{aligned} \Phi &= \log \det(I_n - {}^t Z \bar{Z})^{-1} \\ &= \sum |z_{\alpha\beta}|^2 + \frac{1}{2} \sum \bar{z}_{\alpha\gamma} z_{\alpha\delta} \bar{z}_{\beta\delta} z_{\beta\gamma} + \text{higher order terms.} \end{aligned}$$

At $Z = 0$ the coordinates $(Z_{\alpha\beta})$ are normal coordinates in the sense that the first order derivatives of the coefficients of the metric tensor with respect to these coordinates vanish at $Z = 0$. Hence the curvature tensor at $Z = 0$ is given by

$$R_{\alpha\gamma \bar{\beta}\bar{\rho} \lambda\sigma, \bar{\mu}\bar{\tau}} = \frac{\partial^4 \Phi}{\partial z_{\alpha\gamma} \partial \bar{z}_{\beta\bar{\rho}} \partial z_{\lambda\sigma} \partial \bar{z}_{\mu\bar{\tau}}} = \delta_{\alpha\beta} \delta_{\lambda\mu} \delta_{\gamma\tau} \delta_{\rho\sigma} + \delta_{\alpha\mu} \delta_{\beta\lambda} \delta_{\gamma\rho} \delta_{\sigma\tau}.$$

It follows that at $Z = 0$

$$\begin{aligned} (5.1) \quad \sum R_{\alpha\gamma, \bar{\beta}\bar{\rho}, \lambda\sigma, \bar{\mu}\bar{\tau}} \xi^{\alpha\gamma, \bar{\beta}\bar{\rho}} \bar{\xi}^{\mu\tau, \bar{\lambda}\bar{\sigma}} \\ &= \sum \delta_{\alpha\beta} \delta_{\lambda\mu} \delta_{\gamma\tau} \delta_{\rho\sigma} \xi^{\alpha\gamma, \bar{\beta}\bar{\rho}} \bar{\xi}^{\mu\tau, \bar{\lambda}\bar{\sigma}} + \sum \delta_{\alpha\mu} \delta_{\beta\lambda} \delta_{\gamma\rho} \delta_{\sigma\tau} \xi^{\alpha\gamma, \bar{\beta}\bar{\rho}} \bar{\xi}^{\mu\tau, \bar{\lambda}\bar{\sigma}} \\ &= \sum_{\alpha, \lambda, \gamma, \rho} \xi^{\alpha\gamma, \bar{\alpha}\bar{\rho}} \bar{\xi}^{\lambda\gamma, \bar{\lambda}\bar{\rho}} + \sum_{\alpha, \beta, \gamma, \sigma} \xi^{\alpha\gamma, \bar{\beta}\bar{\gamma}} \bar{\xi}^{\alpha\sigma, \bar{\beta}\bar{\sigma}} \\ &= \sum_{\gamma, \rho} \left| \sum_{\alpha} \xi^{\alpha\gamma, \bar{\alpha}\bar{\rho}} \right|^2 + \sum_{\alpha, \beta} \left| \sum_{\gamma} \xi^{\alpha\gamma, \bar{\beta}\bar{\gamma}} \right|^2. \end{aligned}$$

Type II_n. The domain D_n^{II} is an open subset of $\mathbb{C}^{n(n-1)/2}$ and is the set of all skew-symmetric $n \times n$ matrices $Z = (z_{\alpha\beta})$ with complex entries such that $I_n - {}^t Z \bar{Z}$ is positive definite. It is a complex submanifold of $D_{n,n}^I$. The invariant Kähler metric of $D_{n,n}^I$ induces an invariant Kähler metric of D_n^{II} . At $Z = 0$, the coordinates $(z_{\alpha\beta})$ for $\alpha < \beta$ are normal coordinates. Hence the second fundamental form of D_n^{II} in $D_{n,n}^I$ vanishes at $Z = 0$ and the curvature tensor of D_n^{II} at $Z = 0$ is the restriction of the curvature of $D_{n,n}^I$. It follows that at $Z = 0$

$$\begin{aligned} (5.2) \quad \sum_{\alpha < \gamma \beta < \rho \lambda < \sigma \mu < \tau} R_{\alpha\gamma \bar{\beta}\bar{\rho} \lambda\sigma, \bar{\mu}\bar{\tau}} \xi^{\alpha\gamma, \bar{\beta}\bar{\rho}} \bar{\xi}^{\mu\tau, \bar{\lambda}\bar{\sigma}} \\ &= \sum_{\gamma, \rho=1}^n \left| \sum_{\alpha=1}^n \xi^{\alpha\gamma, \bar{\alpha}\bar{\rho}} \right|^2 + \sum_{\alpha, \beta=1}^n \left| \sum_{\gamma=1}^n \xi^{\alpha\gamma, \bar{\beta}\bar{\gamma}} \right|^2 \end{aligned}$$

where $\xi^{\alpha\gamma, \bar{\beta}\rho}$ is skew-symmetric in α, γ and skew-symmetric in β, ρ .

Type III_n. The domain D_n^{III} is an open subset of $\mathbb{C}^{n(n+1)/2}$ and is the set of all symmetric $n \times n$ matrices $Z = (z_{\alpha\beta})$ with complex entries such that $I_n - {}^t\bar{Z}Z$ is positive definite. It is a complex submanifold of D_n^I . The invariant Kähler metric of D_n^I induces an invariant Kähler metric of D_n^{III} . At $Z = 0$, the coordinates $(z_{\alpha\beta})$ for $\alpha \leq \beta$ are normal coordinates. Hence the second fundamental form of D_n^{III} in D_n^I vanishes at $Z = 0$ and the curvature tensor of D_n^{III} at $Z = 0$ is the restriction of the curvature tensor of D_n^I . It follows that at $Z = 0$

$$(5.3) \quad \begin{aligned} & \sum_{\alpha \leq \gamma, \beta \leq \rho, \lambda \leq \sigma, \mu \leq \tau} R_{\alpha\gamma, \bar{\beta}\rho, \lambda\sigma, \bar{\mu}\tau} \xi^{\alpha\gamma, \bar{\beta}\rho} \xi^{\mu\tau, \bar{\lambda}\sigma} \\ &= \sum_{\gamma, \rho=1}^n |\sum_{\alpha=1}^n \xi^{\alpha\gamma, \bar{\alpha}\rho}|^2 + \sum_{\alpha, \beta=1}^n |\sum_{\gamma=1}^n \xi^{\alpha\gamma, \bar{\beta}\gamma}|^2 \end{aligned}$$

where $\xi^{\alpha\gamma, \bar{\beta}\rho}$ is symmetric in α, γ and symmetric in β, ρ .

Type IV_n. The domain D_n^{IV} is an open subset of $\mathbb{C}^n$ and is the set of all $z = (z_1, \dots, z_n)$ in $\mathbb{C}^n$ satisfying

$$\begin{cases} 1 + |\sum_{\alpha} z_{\alpha}^2|^2 - 2\sum_{\alpha} |z_{\alpha}|^2 > 0 \\ \sum_{\alpha} |z_{\alpha}|^2 < 1. \end{cases}$$

An invariant Kähler metric is given by the potential function

$$\begin{aligned} \Phi &= -\log(1 - 2\sum_{\alpha} |z_{\alpha}|^2 + |\sum_{\alpha} z_{\alpha}^2|^2) \\ &= 2\sum_{\alpha} |z_{\alpha}|^2 - |\sum_{\alpha} z_{\alpha}^2|^2 + 2(\sum_{\alpha} |z_{\alpha}|^2)^2 + \text{higher order terms.} \end{aligned}$$

At $z = 0$ the coordinates z_{α} ($1 \leq \alpha \leq n$) are normal coordinates. Hence the curvature tensor is given by

$$R_{\alpha\bar{\rho}\bar{\beta}\sigma} = \frac{\partial^4 \Phi}{\partial z_{\alpha} \partial \bar{z}_{\rho} \partial z_{\beta} \partial \bar{z}_{\sigma}} = 4(\delta_{\alpha\rho} \delta_{\beta\sigma} + \delta_{\alpha\sigma} \delta_{\beta\rho} - \delta_{\alpha\beta} \delta_{\rho\sigma}).$$

It follows that at $z = 0$

$$(5.4) \quad \begin{aligned} & \sum R_{\alpha\bar{\rho}\bar{\beta}\sigma} \xi^{\alpha\bar{\rho}} \xi^{\sigma\bar{\beta}} \\ &= 4\sum \delta_{\alpha\rho} \delta_{\beta\sigma} \xi^{\alpha\bar{\rho}} \xi^{\sigma\bar{\beta}} + 4\sum \delta_{\alpha\sigma} \delta_{\beta\rho} \xi^{\alpha\bar{\rho}} \xi^{\sigma\bar{\beta}} - 4\sum \delta_{\alpha\beta} \delta_{\rho\sigma} \xi^{\alpha\bar{\rho}} \xi^{\sigma\bar{\beta}} \\ &= 4\sum_{\alpha, \beta} \xi^{\alpha\bar{\alpha}} \xi^{\beta\bar{\beta}} + 4\sum_{\alpha, \beta} \xi^{\alpha\bar{\beta}} \xi^{\alpha\bar{\beta}} - 4\sum_{\alpha, \rho} \xi^{\alpha\bar{\rho}} \xi^{\bar{\rho}\alpha} \\ &= 4|\sum_{\alpha} \xi^{\alpha\bar{\alpha}}|^2 + 2\sum_{\alpha \neq \beta} |\xi^{\alpha\bar{\beta}} - \xi^{\beta\bar{\alpha}}|^2. \end{aligned}$$

The curvature tensors of these four types of classical bounded symmetric domains are strongly seminegative, but are not strongly negative. In the following four Sections 6-9, we will show that these curvature tensors are adequately negative so that Theorem 3 follows from Theorem 5. For the proof of the adequate negativity of the curvature tensors, we will need the following very simple lemma in linear algebra.

LEMMA 3. Suppose φ is a smooth map from an open neighborhood of 0 in $\mathbb{C}^m$ to an open neighborhood of 0 in $\mathbb{C}^n$ with $\varphi(0) = 0$. Let $w_1, \dots, w_m$ (respectively $z_1, \dots, z_n$) be the coordinates of $\mathbb{C}^m$ (respectively $\mathbb{C}^n$).

a) If the rank of the $n \times m$ matrix $(\partial z_\alpha / \partial w_i)$ at 0 is n , then there exists a nonsingular linear transformation

$$(5.5) \quad w_i = \sum_{j=1}^m b_{ij} w'_j \quad (1 \leq i \leq m)$$

such that at 0,

$$\frac{\partial z_\alpha}{\partial w'_i} = \delta_{\alpha i} \quad \text{for } 1 \leq \alpha \leq n, 1 \leq i \leq m.$$

b) If the rank of the $n \times m$ matrix $(\partial z_\alpha / \partial w_i)$ at 0 is $p < n$, then there exist a nonsingular linear transformation (5.5) and a unitary transformation

$$z'_\alpha = \sum_{\beta=1}^n a_{\alpha\beta} z_\beta \quad (1 \leq \alpha \leq n)$$

such that at 0,

$$\frac{\partial z'_\alpha}{\partial w'_i} = \delta_{\alpha i} \quad \text{for } 1 \leq \alpha \leq p, 1 \leq i \leq m$$

and

$$\frac{\partial z'_\alpha}{\partial w'_i} = 0 \quad \text{for } p < \alpha \leq n, 1 \leq i \leq m.$$

c) If the rank of the $n \times m$ matrix $(\partial z_\alpha / \partial w_i)$ at 0 is $p < n$ and $\partial z_1 / \partial w_i \neq 0$ at 0 for some $1 \leq i \leq m$, then there exist a nonsingular linear transformation (5.5) and a unitary transformation of the form

$$\begin{aligned} z'_1 &= z_1, \\ z'_\alpha &= \sum_{\beta=2}^n a_{\alpha\beta} z_\beta \quad (2 \leq \alpha \leq n) \end{aligned}$$

such that at 0

$$\frac{\partial z'_\alpha}{\partial w'_i} = \delta_{\alpha i} \quad \text{for } 1 \leq \alpha \leq p, 1 \leq i \leq m$$

and

$$\frac{\partial z'_\alpha}{\partial w'_i} = 0 \quad \text{for } p < \alpha \leq n, 2 \leq i \leq m.$$

Proof. Let $\Phi: \mathbb{C}^m \rightarrow \mathbb{C}^n$ be the $\mathbb{C}$ -linear map defined by the $n \times m$ matrix $(\partial z_\alpha / \partial w_i)$ at 0. Let $e_1, \dots, e_n$ (respectively $u_1, \dots, u_m$) be the standard unit basis vectors of $\mathbb{C}^n$ (respectively $\mathbb{C}^m$).

a) Take a basis $v_{n+1}, \dots, v_m$ of $\text{Ker } \Phi$. Let v_i be a vector in $\mathbb{C}^m$ with $\Phi(v_i) = e_i$ ($1 \leq i \leq n$). Let

$$v_i = \sum_{j=1}^m b_{ij} u_j \quad (1 \leq i \leq m).$$

Then (b_{ij}) satisfies the requirement.

b) Take a unitary basis $d_1, \dots, d_n$ of $\mathbb{C}^n$ such that $d_\alpha \in \text{Im } \Phi$ for $1 \leq \alpha \leq p$. Take a basis $v_{p+1}, \dots, v_m$ of $\text{Ker } \Phi$. Let v_i be a vector in $\mathbb{C}^m$ with $\Phi(v_i) = e_i$ ($1 \leq i \leq p$). Let

$$v_i = \sum_{j=1}^m b_{ij} u_j \quad (1 \leq i \leq m)$$

and

$$d_\alpha = \sum_{\beta=1}^n \alpha_{\alpha\beta} e_\beta \quad (1 \leq \alpha \leq n).$$

Then (b_{ij}) and $(a_{\alpha\beta})$ satisfy the requirement.

c) Let $\pi: \mathbb{C}^n \rightarrow \mathbb{C}e_1$ denote the orthogonal projection. Since $\partial z_i / \partial w_i \neq 0$ at 0 for some $1 \leq i \leq m$, $\pi \circ \Phi$ is surjective. Let $K = \text{Ker}(\pi \circ \Phi)$. Let $\Psi: K \rightarrow \text{Ker } \pi = \bigoplus_{\alpha=2}^n \mathbb{C}e_\alpha$ be induced by Φ . Then $\text{rank } \Psi = p - 1$. Take a unitary basis $d_2, \dots, d_n$ in $\bigoplus_{\alpha=2}^n \mathbb{C}e_\alpha$ such that $d_\alpha \in \text{Im } \Psi$ for $2 \leq \alpha \leq p$. Let $d_1 = e_1$. Let π' denote the orthogonal projection from $\mathbb{C}^n$ onto the linear subspace V spanned by $d_1, \dots, d_p$. Then $\pi' \circ \Phi: \mathbb{C}^m \rightarrow V$ is surjective. Choose a basis $v_{p+1}, \dots, v_m$ of $\text{Ker}(\pi' \circ \Phi)$. Let v_i be a vector in $\mathbb{C}^m$ with $(\pi' \circ \Phi)(v_i) = d_i$ for $1 \leq i \leq p$. Let

$$v_i = \sum_{j=1}^m b_{ij} u_j \quad (1 \leq i \leq m)$$

and

$$d_\alpha = \sum_{\beta=2}^n a_{\alpha\beta} e_\beta \quad (2 \leq \alpha \leq p).$$

Then (b_{ij}) and $(a_{\alpha\beta})$ satisfy the requirements, because $\Phi(v_i) = d_i$ for $2 \leq i \leq p$ which follows from the fact that $\Phi(v_i) - d_i$ belongs both to $\text{Im } \Phi$ and to the linear subspace spanned by $d_{p+1}, \dots, d_n$. Q.E.D.

6. Adequate negativity of the curvature of D_{mn}^I

PROPOSITION 5. For $mn \geq 2$ the curvature tensor $R_{\alpha\gamma, \bar{\beta}\bar{\rho}, \lambda\sigma, \bar{\mu}\bar{\tau}}$ ($1 \leq \alpha, \beta, \lambda, \mu \leq m, 1 \leq \gamma, \rho, \sigma, \tau \leq n$) of D_{mn}^I is adequately negative.

Proof. Let U be an open neighborhood of 0 in $\mathbb{C}^{mn}$. Let $(w_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be the coordinates of $\mathbb{C}^{mn}$. Denote $\partial / \partial \overline{w_{ij}}, \partial / \partial \overline{w_{ij}}$ by $\partial_{ij}, \partial_{\bar{i}\bar{j}}$ respectively. Let $f: U \rightarrow D_{mn}^I$ be a smooth map which maps the origin to the zero matrix and which is locally diffeomorphic at 0. Let $f^{\alpha\beta}$ ($1 \leq \alpha \leq m, 1 \leq \beta \leq n$) be the components and let

$$\xi_{ij, \bar{k}\bar{l}}^{\alpha\beta, \bar{\gamma}\bar{\delta}} = (\partial_{\bar{i}\bar{j}} f^{\alpha\beta})(0) \overline{(\partial_{\bar{k}\bar{l}} f^{\gamma\delta})(0)} - (\partial_{\bar{k}\bar{l}} f^{\alpha\beta})(0) \overline{(\partial_{\bar{i}\bar{j}} f^{\gamma\delta})(0)}.$$

Assume that

$$(6.1) \quad \sum_{\alpha, \beta, \gamma, \rho, \lambda, \mu, \sigma, \tau} R_{\alpha\gamma, \bar{\beta}\bar{\rho}, \lambda\sigma, \bar{\mu}\bar{\tau}} \xi_{ij, \bar{k}\bar{l}}^{\alpha\gamma, \bar{\beta}\bar{\delta}} \overline{\xi_{ij, \bar{k}\bar{l}}^{\mu\tau, \bar{\lambda}\bar{\sigma}}} = 0$$

at $Z = 0$ for all $(i, j), (k, l)$. We have to prove that

$$(6.2) \quad \begin{cases} \text{either } \partial_{\bar{i}\bar{j}} f^{\alpha\beta} = 0 \text{ at } 0 \text{ for all } (i, j) \text{ and } (\alpha, \beta) \\ \text{or } \partial_{ij} f^{\alpha\beta} = 0 \text{ at } 0 \text{ for all } (i, j) \text{ and } (\alpha, \beta). \end{cases}$$

The following three conventions will be used in this proof and also in the proofs of Propositions 6 and 7. Moreover, in the proofs of Propositions 6 and 7 the notations ∂_{ij} , $\partial_{\bar{i}\bar{j}}$, $f^{\alpha\beta}$, and $\xi_{\bar{i}\bar{j}, \bar{k}\bar{l}}^{\alpha\beta, \bar{\gamma}\bar{\sigma}}$ will carry analogous meanings.

i) For notational simplicity we will denote $(\partial_{ij}f^{\alpha\beta})(0)$, $(\partial_{\bar{i}\bar{j}}f^{\alpha\beta})(0)$, $(\partial_{ij}f^{\bar{\alpha}\bar{\beta}})(0)$, $(\partial_{\bar{i}\bar{j}}f^{\bar{\alpha}\bar{\beta}})(0)$ simply by $\partial_{ij}f^{\alpha\beta}$, $\partial_{\bar{i}\bar{j}}f^{\alpha\beta}$, $\partial_{ij}f^{\bar{\alpha}\bar{\beta}}$, $\partial_{\bar{i}\bar{j}}f^{\bar{\alpha}\bar{\beta}}$ respectively.

ii) After we apply linear transformations to the coordinates (w_{ij}) and $(z_{\alpha\beta})$, we will use the same symbols for the new coordinates.

iii) We use the lexicographical ordering for the double indices (i, j) . By $(i, j) > (k, l)$ we mean either $i = k$ and $j > l$ or $i > k$. By $(i, j) \geq (k, l)$ we mean $(i, j) > (k, l)$ or $(i, j) = (k, l)$.

From (6.1) and (5.1) it follows that

$$(6.3) \quad \begin{aligned} \sum_{\alpha} \xi_{\bar{i}\bar{j}, \bar{k}\bar{l}}^{\alpha\gamma, \bar{\alpha}\bar{\rho}} &= 0 & \text{for all } (\gamma, \rho), \\ \sum_{\gamma} \xi_{\bar{i}\bar{j}, \bar{k}\bar{l}}^{\alpha\gamma, \bar{\beta}\bar{\tau}} &= 0 & \text{for all } (\alpha, \beta). \end{aligned}$$

We will prove (6.2) from the equations (6.3). The equations (6.3) are invariant under the following transformations of $D_{m,n}^1$.

$$(6.4) \quad \begin{aligned} Z &\longmapsto \bar{Z}, \\ Z &\longmapsto {}^tZ, \\ Z &\longmapsto AZB, \end{aligned}$$

where A and B are fixed unitary matrices respectively of orders m and n .

To prove the proposition from (6.3) we can assume without loss of generality (after the transformation $Z \rightarrow {}^tZ$ if necessary) that $m \geq 2$.

Assume that $\partial_{ij}f^{\alpha\beta} \neq 0$ for some (i, j) and some (α, β) . We want to prove that $\partial_{\bar{i}\bar{j}}f^{\alpha\beta} = 0$ for all (i, j) and (α, β) . By applying a unitary transformation to the rows of Z (i.e., a transformation of the form (6.4) with $B = I_n$), we can assume without loss of generality that $\partial_{ij}f^{1\beta} \neq 0$ for some β and some (i, j) . Let the rank of the $n \times (mn)$ matrix

$$(\partial_{ij}f^{1\beta})_{1 \leq \beta \leq n, 1 \leq i \leq m, 1 \leq j \leq n}$$

be p . By applying Lemma 3b) to the smooth map

$$(f^{11}, \dots, f^{1n}): U \longrightarrow \mathbf{C}^n,$$

we conclude that, after we apply a linear transformation to the coordinates (w_{ij}) and apply a unitary transformation to the columns of Z (i.e., a transformation of the form (6.4) with $A = I_m$), we can assume without loss of generality that for $1 \leq \beta \leq n$,

$$(6.5) \quad \begin{cases} \partial_{ij}f^{1\beta} = \delta_{j\beta} & \text{for } 1 \leq j \leq p \\ \partial_{ij}f^{1\beta} = 0 & \text{for } (i, j) > (1, p). \end{cases}$$

For $1 \leq i \leq p$ and $(k, l) > (1, p)$, consider

$$\sum_{\beta=1}^n \xi_{i\bar{i}}^{1\beta} \bar{\xi}_{k\bar{l}}^{1\beta} = 0 ;$$

that is,

$$\sum_{\beta=1}^n (\partial_{i\bar{i}} f^{1\beta} \overline{\partial_{kl} f^{1\beta}} - \partial_{k\bar{l}} f^{1\beta} \overline{\partial_{i\bar{i}} f^{1\beta}}) = 0 .$$

Since $\partial_{kl} f^{1\beta} = 0$ and $\partial_{i\bar{i}} f^{1\beta} = \delta_{i\beta}$, it follows that

$$(6.6) \quad \partial_{k\bar{l}} f^{1i} = 0 \quad \text{for } 1 \leq i \leq p \text{ and } (k, l) > (1, p) .$$

We claim that the rank of the $(n - p) \times (mn - p)$ matrix

$$P := (\partial_{k\bar{l}} f^{1\gamma})_{p < \gamma \leq n, (k, l) > (1, p)}$$

is $n - p$. Suppose the contrary. Then at 0,

$$\bar{\partial} f^{1, p+1} \wedge \cdots \wedge \bar{\partial} f^{1n} = 0 \quad \text{mod}(d\bar{w}_{11}, \dots, d\bar{w}_{1p}) .$$

By (6.5), $\partial f^{1\beta} = 0$ for $p < \beta \leq n$. Hence at 0,

$$(6.7) \quad df^{1, p+1} \wedge \cdots \wedge df^{1n} = 0 \quad \text{mod}(d\bar{w}_{11}, \dots, d\bar{w}_{1p}) .$$

By (6.5) and (6.6), $df^{1\beta}$, $1 \leq \beta \leq p$, is a linear combination of $dw_{11}, \dots, dw_{1p}$, $d\bar{w}_{11}, \dots, d\bar{w}_{1p}$ at 0. It follows that at 0,

$$\wedge_{\beta=1}^p (df^{1\beta} \wedge df^{1\bar{\beta}}) = c \wedge_{\beta=1}^p (dw_{1\beta} \wedge d\bar{w}_{1\beta})$$

for some constant c . This together with (6.7) implies that at 0,

$$\wedge_{\beta=1}^n (df^{1\beta} \wedge df^{1\bar{\beta}}) = 0 ,$$

contradicting the assumption that f is locally diffeomorphic at 0. Hence the rank of P is $n - p$. By applying Lemma 3 a) to the map $(\bar{f}^{1, p+1}, \dots, \bar{f}^{1n})$ whose domain variables are (w_{kl}) , $(k, l) > (1, p)$ (the other variables being fixed), we conclude that after a linear coordinate change in the variables (w_{kl}) , $(k, l) > (1, p)$, we have for $p < \gamma \leq n$,

$$(6.8) \quad \begin{cases} \partial_{i\bar{j}} f^{1\gamma} = \delta_{j\gamma} & \text{for } p < j \leq n \\ \partial_{k\bar{l}} f^{1\gamma} = 0 & \text{for } 2 \leq k \leq m, 1 \leq l \leq n . \end{cases}$$

Since this last coordinate change involves only w_{kl} with $(k, l) > (1, p)$, the validity of (6.5) is not affected.

Take any $1 < \alpha \leq m$. Let $1 < k \leq m$, $1 \leq l \leq n$, and $p < i \leq n$. Consider

$$\sum_{\beta=1}^n \xi_{i\bar{i}}^{1\beta} \bar{\xi}_{k\bar{l}}^{\alpha\beta} = 0 ;$$

that is,

$$\sum_{\beta=1}^n (\partial_{i\bar{i}} f^{1\beta} \overline{\partial_{kl} f^{\alpha\beta}} - \partial_{k\bar{l}} f^{1\beta} \overline{\partial_{i\bar{i}} f^{\alpha\beta}}) = 0 .$$

Since $\partial_{k\bar{l}} f^{1\beta} = 0$ and $\partial_{i\bar{i}} f^{1\beta} = \delta_{i\beta}$ by (6.6) and (6.8), it follows that

$$(6.9) \quad \partial_{kl} f^{\alpha i} = 0 \quad \text{for } 1 < \alpha \leq m, 1 < k \leq m, 1 \leq l \leq n, \text{ and } p < i \leq n .$$

Consider now

$$\sum_{\beta=1}^n \xi_{kl, \bar{l}\bar{j}}^{\alpha\beta, \bar{l}\bar{j}} = 0 ,$$

where $1 < \alpha \leq m$, $1 < k \leq m$, $1 \leq l \leq m$, and $1 \leq j \leq p$. That is,

$$\sum_{\beta=1}^n (\partial_{kl} f^{\alpha\beta} \overline{\partial_{lj} f^{1\beta}} - \partial_{lj} f^{\alpha\beta} \overline{\partial_{kl} f^{1\beta}}) = 0 .$$

Since $\partial_{kl} f^{1\beta} = 0$ and $\partial_{lj} f^{1\beta} = \delta_{j\beta}$ by (6.5), it follows that

$$(6.10) \quad \partial_{kl} f^{\alpha j} = 0 \text{ for } 1 < \alpha \leq m, 1 < k \leq m, 1 \leq l \leq n, \text{ and } 1 \leq j \leq p .$$

We claim that, after a linear coordinate change in (w_{kl}) for $k > 1$ the following hold for $1 \leq i \leq m$:

$$(6.11)_i \quad \begin{cases} \partial_{il} f^{i\beta} = \delta_{l\beta} & \text{for } 1 \leq l \leq p \text{ and } 1 \leq \beta \leq n . \\ \partial_{kl} f^{i\beta} = 0 & \text{for } (k, l) > (i, p) \text{ and } 1 \leq \beta \leq n . \\ \partial_{il} f^{i\beta} = \delta_{l\beta} & \text{for } p < l, \beta \leq n . \\ \partial_{kl} f^{i\beta} = 0 & \text{for } k > i, 1 \leq l, \beta \leq n . \end{cases}$$

We prove this by induction on i . When $i = 1$, (6.11)_i follows from (6.5) and (6.8). Suppose (6.11)_i holds for $1 \leq i < j$ and $j \leq m$ and we want to show that (6.11)_j holds. From (6.11)_i, $1 \leq i < j$, it follows that

$$(6.12) \quad \begin{cases} df^{\alpha\beta}, 1 \leq \alpha < j, 1 \leq \beta \leq n, \text{ is a linear combination of} \\ dw_{kl}, d\bar{w}_{kl}, 1 \leq k < j, 1 \leq l \leq n, \text{ at } 0. \end{cases}$$

From (6.10) it follows that

$$(6.13) \quad \begin{cases} df^{j\beta}, 1 \leq \beta \leq p, \text{ is a linear combination of } dw_{kl}, 1 \leq k \leq m, \\ 1 \leq l \leq n, \text{ and } d\bar{w}_{lr}, 1 \leq r \leq n \text{ at } 0. \end{cases}$$

The $p \times (m - j + 1)n$ matrix

$$(\partial_{kl} f^{j\beta})_{1 \leq \beta \leq p, j \leq l \leq m, 1 \leq l \leq n}$$

must have rank p , otherwise from (6.12) and (6.13) it follows that

$$(\wedge_{1 \leq \alpha < j, 1 \leq \beta \leq n} (df^{\alpha\beta} \wedge df^{\alpha\bar{\beta}})) \wedge (\wedge_{1 \leq \beta \leq p} (df^{j\beta} \wedge df^{j\bar{\beta}}))$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying Lemma 3 a) to the map $(f^{j1}, \dots, f^{jp})$ whose domain variables are (w_{kl}) , $j \leq k \leq m$, $1 \leq l \leq n$ (the other variables being fixed), we conclude that, after a linear coordinate change in (w_{kl}) , $j \leq k \leq m$, $1 \leq l \leq n$, we have the first two equations of (6.11)_j. This implies that

$$(6.14) \quad \begin{cases} df^{j\beta}, 1 \leq \beta \leq p, \text{ is a linear combination of } dw_{kl}, d\bar{w}_{kl}, \\ (j, p) \geq (k, l), \text{ at } 0. \end{cases}$$

From (6.9) it follows that

$$(6.15) \quad \begin{cases} df^{j\beta}, p < \beta \leq n, \text{ is a linear combination of } d\bar{w}_{kl}, 1 \leq k \leq m, \\ 1 \leq l \leq n, \text{ and } dw_{lr}, 1 \leq r \leq n, \text{ at } 0. \end{cases}$$

The rank of the $(n - p) \times ((m - j + 1)n - p)$ matrix

$$(\partial_{\bar{k}l} f^{j\beta})_{p < \beta \leq n, (k,l) > (j,p)}$$

must be $n - p$, otherwise it follows from (6.12), (6.14), and (6.15) that

$$\bigwedge_{1 \leq \alpha \leq j, 1 \leq \beta \leq n} (df^{\alpha\beta} \wedge d\bar{f}^{\alpha\beta})$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying Lemma 3 a) to the map $(\bar{f}^{j,p+1}, \dots, \bar{f}^{jn})$ whose domain variables are (w_{kl}) with $(k, l) > (j, p)$ (the other variables being fixed), we conclude that, after a linear coordinate change in (w_{kl}) with $(k, l) > (j, p)$, we have the last two equations of (6.11)_j.

We distinguish between two cases.

Case 1. $p = n$. By the first two equations of (6.11)_i we have for all (α, β)

$$\begin{cases} \partial_{\alpha\beta} f^{\alpha\beta} = 1 \\ \partial_{kl} f^{\alpha\beta} = 0 \end{cases} \quad \text{for } (k, l) > (\alpha, \beta).$$

Hence the $(mn) \times (mn)$ matrix

$$(\partial_{ij} f^{\alpha\beta})_{1 \leq i, \alpha \leq m, 1 \leq j, \beta \leq n}$$

is nonsingular. By Lemma 3 a) we can apply a linear change of coordinates to (w_{ij}) , $1 \leq i \leq m$, $1 \leq j \leq n$ so that after this linear coordinate change we have

$$(6.16) \quad \partial_{ij} f^{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta};$$

but statements derived above concerning $\partial_{i\bar{j}} f^{\alpha\beta}$ may no longer remain valid, i.e., (6.6), (6.8), (6.10), and the last two equations of (6.11)_i may no longer hold.

We are now ready to show that $\partial_{i\bar{j}} f^{\alpha\beta} = 0$ for all α, β, i, j . Fix arbitrarily (α, β) and (i, j) . Consider first the case $i \neq \alpha$. We have

$$\sum_{\gamma=1}^n \xi_{i\bar{j}, \alpha\bar{\beta}}^{\alpha\gamma, \alpha\bar{\gamma}} = 0,$$

i.e.,

$$\sum_{\gamma=1}^n (\partial_{i\bar{j}} f^{\alpha\gamma} \overline{\partial_{\alpha\beta} f^{\alpha\gamma}} - \partial_{\alpha\bar{\beta}} f^{\alpha\gamma} \overline{\partial_{ij} f^{\alpha\gamma}}) = 0.$$

By (6.16), $\partial_{ij} f^{\alpha\gamma} = 0$ and $\partial_{\alpha\beta} f^{\alpha\gamma} = \delta_{\beta\gamma}$. Hence $\partial_{i\bar{j}} f^{\alpha\beta} = 0$. Consider now the remaining case $i = \alpha$. Since $m \geq 2$, there exists $1 \leq k \leq m$ with $k \neq \alpha$. We have

$$\sum_{\gamma=1}^n \xi_{\alpha\bar{j}, k\bar{\beta}}^{\alpha\gamma, k\bar{\gamma}} = 0,$$

that is,

$$\sum_{\gamma=1}^n (\partial_{\alpha\bar{j}} f^{\alpha\gamma} \overline{\partial_{k\beta} f^{k\gamma}} - \partial_{k\bar{\beta}} f^{\alpha\gamma} \overline{\partial_{\alpha j} f^{k\gamma}}) = 0.$$

By (6.16), $\partial_{\alpha j} f^{k\gamma} = 0$ and $\partial_{k\beta} f^{k\gamma} = \delta_{\beta\gamma}$. Hence $\partial_{\alpha\bar{j}} f^{\alpha\beta} = 0$.

Case 2. $p < n$. From the last two equations of (6.11)_i, we have

$$\begin{cases} \partial_{i\bar{n}} f^{in} = 1 \\ \partial_{i\bar{n}} f^{jn} = 0 \end{cases} \quad \text{for } i > j.$$

It follows that the rank of the $m \times (mn)$ matrix

$$(\partial_{i\bar{j}} f^{\alpha n})_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq \alpha \leq m}$$

is m . Since $p < n$, we must have $n \geq 2$. After we apply the transformations

$$Z \longmapsto A' \bar{Z}$$

where A is the matrix obtained from I_m by interchanging the first and last rows, we reduce this case to Case 1 and we therefore conclude, by the arguments of Case 1, that $\partial_{ij} f^{\alpha\beta} = 0$ for all $1 \leq i, \alpha \leq m$ and $1 \leq j, \beta \leq n$, which is a contradiction. Hence Case 2 cannot occur and $\partial_{i\bar{j}} f^{\alpha\beta} = 0$ for all $1 \leq i, \alpha \leq m$ and $1 \leq j, \beta \leq n$. Q.E.D.

7. Adequate negativity of the curvature of D_n^{II}

PROPOSITION 6. For $n \geq 3$ the curvature tensor $R_{\alpha\gamma, \bar{\beta}\bar{\rho}, \lambda\sigma, \bar{\mu}\bar{\tau}}$ ($1 \leq \alpha < \gamma \leq n$, $1 \leq \beta < \rho \leq n$, $1 \leq \lambda < \sigma \leq n$, $1 \leq \mu < \tau \leq n$) of D_n^{II} is adequately negative.

Proof. Let U be an open neighborhood of 0 in $\mathbb{C}^{(1/2)n(n-1)}$. Let $(w_{ij})_{1 \leq i < j \leq n}$ be the coordinates of $\mathbb{C}^{(1/2)n(n-1)}$. Let $f: U \rightarrow D_n^{\text{II}}$ be a smooth map which maps the origin to the zero matrix and which is locally diffeomorphic at 0. The components $f^{\alpha\beta}$ of the map f satisfy $f^{\alpha\beta} = -f^{\beta\alpha}$ for $1 \leq \alpha, \beta \leq n$. Assume

$$(7.1) \quad \sum_{\alpha < \gamma, \beta < \rho, \lambda < \sigma, \mu < \tau} R_{\alpha\gamma, \bar{\beta}\bar{\rho}, \lambda\sigma, \bar{\mu}\bar{\tau}} \xi_{i\bar{j}}^{\alpha\gamma, \bar{\beta}\bar{\rho}} \overline{\xi_{i\bar{j}, k\bar{l}}^{\mu\tau, \bar{\lambda}\bar{\sigma}}} = 0$$

at $Z = 0$. We have to prove that

$$(7.2) \quad \begin{cases} \text{either } \bar{\partial} f^{\alpha\beta} = 0 & \text{at } 0 \text{ for all } 1 \leq \alpha < \beta \leq n \\ \text{or } \partial f^{\alpha\beta} = 0 & \text{at } 0 \text{ for all } 1 \leq \alpha < \beta \leq n. \end{cases}$$

It follows from (7.1) and (5.2) that

$$(7.3) \quad \sum_{\gamma=1}^n \xi_{i\bar{j}, k\bar{l}}^{\alpha\gamma, \bar{\beta}\bar{\gamma}} = 0 \quad \text{for } 1 \leq \alpha, \beta \leq n.$$

We will prove (7.2) from the equations (7.3). The equations (7.3) are invariant under the following transformations of D_n^{II} :

$$(7.4) \quad \begin{aligned} Z &\longmapsto \bar{Z}, \\ Z &\longmapsto {}^t A Z A \end{aligned}$$

where A is a fixed unitary matrix of order n . Since f is locally diffeomorphic at 0, we have either

$$\partial_{ij} f^{1\beta} \neq 0 \quad \text{for some } i < j \text{ and some } \beta$$

or

$$\partial_{i\bar{j}} f^{1\beta} \neq 0 \quad \text{for some } i < j \text{ and some } \beta$$

(otherwise $df^{1\beta} = 0$ at 0 for all β , contradicting $\bigwedge_{\alpha < \beta} (df^{\alpha\beta} \wedge df^{\bar{\alpha}\bar{\beta}}) \neq 0$ at 0). Since the equations (7.3) are invariant under the transformation $Z \rightarrow \bar{Z}$, we can assume without loss of generality that $\partial_{ij} f^{1\beta} \neq 0$ for some $i < j$ and some β . Let the rank of the $(n-1) \times (1/2)n(n-1)$ matrix

$$(\partial_{ij} f^{1\beta})_{1 < \beta \leq n, 1 \leq i < j \leq n}$$

be $p-1$. By applying Lemma 3 b) to the smooth map

$$(f^{12}, \dots, f^{1n}): U \longrightarrow \mathbb{C}^{n-1},$$

we conclude that, after we apply a linear transformation to the coordinates (w_{ij}) and apply a transformation to Z of the form (7.4) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

where B is a unitary matrix of order $n-1$, we can assume without loss of generality that for $1 < \beta \leq n$

$$(7.5) \quad \begin{cases} \partial_{ij} f^{1\beta} = \delta_{j\beta} & \text{for } 1 < j \leq p \\ \partial_{ij} f^{1\beta} = 0 & \text{for } (i, j) > (1, p) \end{cases}.$$

For $1 < i \leq p$ and $(k, l) > (1, p)$, consider

$$\sum_{\beta=1}^n \xi_{1\bar{i}, \bar{k}\bar{l}}^{1\beta, 1\bar{\beta}} = 0.$$

Since $\partial_{ki} f^{1\beta} = 0$ and $\partial_{li} f^{1\beta} = \delta_{i\beta}$, it follows that

$$(7.6) \quad \partial_{\bar{k}\bar{l}} f^{1i} = 0 \quad \text{for } 1 < i \leq p \text{ and } (k, l) > (1, p).$$

The rank of the $(n-p) \times ((1/2)n(n-1) - p + 1)$ matrix

$$(\partial_{\bar{k}\bar{l}} f^{1\gamma})_{p < \gamma \leq n, (k, l) > (1, p)}$$

must be $n-p$, otherwise it follows from (7.5) and (7.6) that

$$\bigwedge_{\beta=1}^n (df^{1\beta} \wedge df^{\bar{1}\bar{\beta}})$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying Lemma 3 a) to the map $(f^{1, p+1}, \dots, f^{1n})$ whose domain variables are $(w_{k,l})$ with $(k, l) > (1, p)$ (the other variables being fixed), we conclude that, after a linear change in the coordinates (w_{kl}) , $(k, l) > (1, p)$, we have for $p < \gamma \leq n$,

$$(7.7) \quad \begin{cases} \partial_{i\bar{j}} f^{1\gamma} = \delta_{j\gamma} & \text{for } p < j \leq n \\ \partial_{\bar{k}\bar{l}} f^{1\gamma} = 0 & \text{for } 2 \leq k < l \leq n \end{cases}.$$

This does not affect the validity of (7.5).

For $1 < \alpha \leq n$, $1 < k < l \leq n$, and $p < i \leq n$, consider

$$\sum_{\beta=1}^n \zeta_{i\bar{l}, k\bar{l}}^{1\beta, \alpha\bar{\beta}} = 0.$$

Since $\partial_{k\bar{l}} f^{1\beta} = 0$ by (7.7) and $\partial_{i\bar{l}} f^{1\beta} = \delta_{i\beta}$ by (7.6) and (7.7), it follows that

$$(7.8) \quad \partial_{k\bar{l}} f^{\alpha i} = 0 \quad \text{for } 1 < \alpha \leq n, 1 < k < l \leq n, \text{ and } p < i \leq n.$$

For $1 < \alpha \leq n$, $1 < k < l \leq n$, and $1 < j \leq p$, consider

$$\sum_{\beta=1}^n \zeta_{k\bar{l}, i\bar{j}}^{\alpha\beta, \overline{1\beta}} = 0.$$

Since $\partial_{k\bar{l}} f^{1\beta} = 0$ and $\partial_{i\bar{j}} f^{1\beta} = \delta_{j\beta}$ by (7.5), it follows that

$$(7.9) \quad \partial_{k\bar{l}} f^{\alpha j} = 0 \quad \text{for } 1 < \alpha \leq n, 1 < k < l \leq n, \text{ and } 1 < j \leq p.$$

We claim that, after a linear coordinate change in (w_{kl}) for $1 < k < l \leq n$ the following holds for $1 \leq i \leq n$:

$$(7.10)_i \quad \begin{cases} \partial_{i\bar{l}} f^{i\beta} = \delta_{i\beta} & \text{for } i < l \leq p \text{ and } i < \beta \leq n \\ \partial_{k\bar{l}} f^{i\beta} = 0 & \text{for } (k, l) > (i, p) \\ \partial_{i\bar{l}} f^{i\beta} = \delta_{i\beta} & \text{for } \max(i, p) < l \leq n \text{ and } i < \beta \leq n \\ \partial_{k\bar{l}} f^{i\beta} = 0 & \text{for } i < k < l \leq n \text{ and } i < \beta \leq n. \end{cases}$$

We prove it by induction on i . When $i = 1$, $(7.10)_i$ follows from (7.5) and (7.7). Suppose $(7.10)_i$ holds for $1 \leq i < j$ and $j \leq n$ and we want to show that $(7.10)_j$ holds. From $(7.10)_i$, $1 \leq i < j$, it follows that

$$(7.11) \quad \begin{cases} df^{\alpha\beta}, 1 \leq \alpha < j, \alpha < \beta \leq n, \text{ is a linear combination of} \\ dw_{kl}, d\bar{w}_{kl}, 1 \leq k < j, k < l \leq n, \text{ at } 0. \end{cases}$$

From (7.9) it follows that

$$(7.12) \quad \begin{cases} df^{j\beta}, j < \beta \leq p, \text{ is a linear combination of } dw_{kl}, 1 \leq k < l \leq n, \\ \text{and } d\bar{w}_{kl}, 1 < r \leq n. \end{cases}$$

When $p > j$, the $(p - j) \times (1/2)(n - j + 1)(n - j)$ matrix

$$(\partial_{kl} f^{j\beta})_{j < \beta \leq p, j \leq k < l \leq n}$$

must have rank $p - j$, otherwise from (7.11) and (7.12) it follows that

$$(\wedge_{1 \leq \alpha < j, \alpha < \beta \leq n} (df^{\alpha\beta} \wedge df^{\alpha\bar{\beta}})) \wedge (\wedge_{j < \beta \leq p} (df^{j\beta} \wedge df^{j\bar{\beta}}))$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying Lemma 3 a) to the map $(f^{j, j+1}, \dots, f^{jn})$ whose domain variables are (w_{kl}) , $j \leq k < l \leq n$ (the other variables being fixed), we conclude that, after a linear coordinate change in (w_{kl}) , $j \leq k < l \leq n$, we have the first two equations of $(7.10)_j$. This implies that

$$(7.13) \quad \begin{aligned} df^{j\beta}, j < \beta \leq p, \text{ is a linear combination of } dw_{kl}, d\bar{w}_{kl}, \\ (j, p) \geq (k, l) \text{ at } 0. \end{aligned}$$

From (7.8) it follows that

$$(7.14) \quad df^{j\bar{\beta}}, \max(j, p) < \beta \leq n, \text{ is a linear combination of } d\bar{w}_{kl}, 1 \leq k < l \leq n, \text{ and } dw_{1r}, 1 < r \leq n, \text{ at } 0.$$

The rank of the $(n - \max(j, p)) \times (1/2(n - j + 1)(n - j) - \max(0, p - j))$ matrix

$$(\partial_{\bar{k}l} f^{j\bar{\beta}})_{\max(j, p) < \beta \leq n, (k, l) > (j, p)}$$

must be $n - \max(j, p)$, otherwise it follows from (7.11), (7.13), and (7.14) that

$$\bigwedge_{1 \leq \alpha \leq j, \alpha < \beta \leq n} (df^{\alpha\bar{\beta}} \wedge df^{\alpha\bar{\beta}})$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying Lemma 3 a) to the map $(f^{j, \max(j, p)+1}, \dots, f^{j, n})$ whose domain variables are (w_{kl}) with $(k, l) > (j, p)$ (the other variables being fixed), we conclude that, after a linear coordinate change in (w_{kl}) with $(k, l) > (j, p)$, we have the last two equations of (7.10)_j.

We distinguish between the following two cases.

Case 1. $p = n$. By the first two equations of (7.10)_i the $((1/2)n(n - 1)) \times ((1/2)n(n - 1))$ matrix

$$(\partial_{ij} f^{\alpha\bar{\beta}})_{1 \leq i < j \leq n, 1 \leq \alpha < \beta \leq n}$$

is nonsingular. By Lemma 3 a) we can apply a linear coordinate change to (w_{ij}) , $1 \leq i < j \leq n$, so that after this coordinate change we have

$$(7.15) \quad \partial_{ij} f^{\alpha\bar{\beta}} = \delta_{i\alpha} \delta_{j\beta} \quad \text{for } 1 \leq i < j \leq n \text{ and } 1 \leq \alpha < \beta \leq n,$$

at the expense of possibly sacrificing the validity of (7.6), (7.7), (7.9), and the last two equations of (7.10)_i.

We want to show that $\partial_{i\bar{j}} f^{\alpha\bar{\beta}} = 0$ for $1 \leq i < j \leq n$ and $1 \leq \alpha < \beta \leq n$. For notational convenience, we define, for $1 \leq i < j \leq n$, $\partial_{ji} = -\partial_{ij}$ and $\partial_{\bar{j}i} = -\partial_{i\bar{j}}$. First we show that

$$(7.16) \quad \partial_{\alpha\bar{j}} f^{\alpha\bar{\beta}} = 0 \quad \text{for } 1 \leq \alpha, \beta, j \leq n \text{ with } j \neq \alpha \text{ and } \beta \neq \alpha.$$

Since $n \geq 3$, we can choose $1 \leq k \leq n$ with $k \neq \alpha$ and $k \neq \beta$. Consider

$$\sum_{\gamma=1}^n \xi_{\alpha\bar{j}, k\bar{\beta}}^{\alpha\bar{\gamma}, k\bar{\gamma}} = 0;$$

that is,

$$(7.17) \quad \sum_{\gamma=1}^n (\partial_{\alpha\bar{j}} f^{\alpha\bar{\gamma}} \overline{\partial_{k\bar{\beta}} f^{k\bar{\gamma}}} - \partial_{k\bar{\beta}} f^{\alpha\bar{\gamma}} \overline{\partial_{\alpha\bar{j}} f^{k\bar{\gamma}}}) = 0.$$

Since $\alpha \neq k$, it follows from (7.15) that $\partial_{\alpha\bar{j}} f^{k\bar{\gamma}} = 0$ if $\gamma \neq \alpha$. On the other hand, if $\gamma = \alpha$, then $f^{\alpha\bar{\gamma}} \equiv 0$. Hence in any case $\partial_{k\bar{\beta}} f^{\alpha\bar{\gamma}} \overline{\partial_{\alpha\bar{j}} f^{k\bar{\gamma}}} = 0$. Since $\partial_{k\bar{\beta}} f^{k\bar{\gamma}} = \delta_{\beta\bar{\gamma}}$ by (7.15), it follows from (7.17) that $\partial_{\alpha\bar{j}} f^{\alpha\bar{\beta}} = 0$.

Now fix $1 \leq \alpha < \beta \leq n$ and $1 \leq i, j \leq n$ with $i \neq j$. We want to prove

that $\partial_{i\bar{j}}f^{\alpha\beta} = 0$. Because of (7.16), we can assume without loss of generality that α, β, i, j are distinct. Consider

$$\sum_{\gamma=1}^n \xi_{i\bar{j}}^{\alpha\bar{\gamma}} \overline{\xi_{j\bar{\gamma}}^{\alpha\gamma}} = 0 ;$$

that is,

$$(7.18) \quad \sum_{\gamma=1}^n (\partial_{i\bar{j}}f^{\alpha\gamma} \overline{\partial_{\alpha\bar{\beta}}f^{\alpha\gamma}} - \partial_{\alpha\bar{\beta}}f^{\alpha\gamma} \overline{\partial_{i\bar{j}}f^{\alpha\gamma}}) = 0 .$$

It follows from (7.16) (and the fact that $f^{\alpha\alpha} \equiv 0$) that $\partial_{\alpha\bar{\beta}}f^{\alpha\gamma} = 0$. Since $\partial_{\alpha\bar{\beta}}f^{\alpha\gamma} = \delta_{\beta\bar{\gamma}}$ by (7.15), we have $\partial_{i\bar{j}}f^{\alpha\beta} = 0$ from (7.18).

Case 2. $p < n$. From the last two equations of (7.10)_i, we have

$$\partial_{j\bar{n}}f^{in} = 1 \quad \text{for } 1 \leq i \leq j < n .$$

Since $f^{ni} = -f^{in}$, it follows that the rank of the $(n-1) \times (1/2)n(n-1)$ matrix

$$(\partial_{i\bar{j}}f^{n\alpha})_{1 \leq i < j \leq n, 1 \leq \alpha < n}$$

is $n-1$. After we apply the transformation

$$Z \longmapsto {}^t A \bar{Z} A$$

where A is the matrix obtained from I_n by interchanging the first and last columns, we reduce this case to Case 1 and we therefore conclude, by the arguments of Case 1, that $\partial_{i\bar{j}}f^{\alpha\beta} = 0$ for all $1 \leq i < j \leq n$ and $1 \leq \alpha < \beta \leq n$, which is a contradiction. Hence Case 2 cannot occur and $\partial_{i\bar{j}}f^{\alpha\beta} = 0$ for all $1 \leq i < j \leq n$ and $1 \leq \alpha < \beta \leq n$. Q.E.D.

8. Adequate negativity of the curvature of D_n^{III}

PROPOSITION 7. *For $n \geq 2$ the curvature tensor $R_{\alpha\bar{\gamma}, \bar{\beta}\bar{\rho}} \lambda_{\sigma} \bar{\mu}_{\tau}$ ($1 \leq \alpha \leq \gamma \leq n$, $1 \leq \beta \leq \rho \leq n$, $1 \leq \lambda \leq \sigma \leq n$, $1 \leq \mu \leq \tau \leq n$) of D_n^{III} is adequately negative.*

Proof. Let U be an open neighborhood of 0 in $\mathbb{C}^{(1/2)n(n+1)}$. Let $(w_{ij})_{1 \leq i \leq j \leq n}$ be the coordinates of $\mathbb{C}^{(1/2)n(n+1)}$. Let $f: U \rightarrow D_n^{\text{III}}$ be a smooth map which maps the origin to the zero matrix and which is locally diffeomorphic at 0. The components $f^{\alpha\beta}$ of f satisfy $f^{\alpha\beta} = f^{\beta\alpha}$ for $1 \leq \alpha, \beta \leq n$. Assume

$$(8.1) \quad \sum_{\alpha \leq \gamma, \beta \leq \rho, \lambda \leq \sigma, \mu \leq \tau} R_{\alpha\bar{\gamma}, \bar{\beta}\bar{\rho}} \lambda_{\sigma} \bar{\mu}_{\tau} \xi_{i\bar{j}}^{\alpha\bar{\gamma}} \overline{\xi_{j\bar{\tau}}^{\mu\bar{\sigma}}} \xi_{i\bar{j}}^{\beta\bar{\rho}} \overline{\xi_{j\bar{k}}^{\mu\bar{\sigma}}} = 0$$

at $Z = 0$. We have to prove that $\beta\rho$

$$(8.2) \quad \begin{aligned} &\text{either } \bar{\partial}f^{\alpha\beta} = 0 \text{ at } 0 \text{ for } 1 \leq \alpha \leq \beta \leq n \\ &\text{or } \partial f^{\alpha\beta} = 0 \text{ at } 0 \text{ for } 1 \leq \alpha \leq \beta \leq n . \end{aligned}$$

From (8.1) and (5.3) it follows that

$$(8.3) \quad \sum_{\gamma=1}^n \xi_{i\bar{j}}^{\alpha\bar{\gamma}} \overline{\xi_{j\bar{k}}^{\beta\bar{\gamma}}} = 0 \quad \text{for } 1 \leq \alpha, \beta \leq n .$$

We will prove (8.2) from the equations (8.3). The equations (8.3) are invariant

under the following transformations of D_n^{III} :

$$(8.4) \quad \begin{aligned} Z &\longmapsto \bar{Z}, \\ Z &\longmapsto {}^tAZA, \end{aligned}$$

where A is a fixed unitary matrix of order n .

Since f is locally diffeomorphic at 0, we have either

$$\partial_{ij}f^{11} \neq 0 \quad \text{for some } i \leq j$$

or

$$\partial_{i\bar{j}}f^{11} \neq 0 \quad \text{for some } i \leq j$$

(otherwise $df^{11} = 0$ at 0, contradicting $\bigwedge_{\alpha \leq \beta} (df^{\alpha\beta} \wedge d\bar{f}^{\alpha\bar{\beta}}) \neq 0$ at 0). Since the equations (8.3) are invariant under the transformation $Z \rightarrow \bar{Z}$, we can assume without loss of generality that $\partial_{ij}f^{11} \neq 0$ for some $i < j$. Let the rank of the $n \times (1/2)n(n+1)$ matrix

$$(\partial_{ij}f^{1\beta})_{1 \leq \beta \leq n, 1 \leq i \leq j \leq n}$$

be p . By applying Lemma 3c) to the smooth map

$$(f^{11}, \dots, f^{1n}): U \longrightarrow \mathbb{C}^n,$$

we conclude that, after we apply a linear transformation to the coordinates (w_{ij}) and apply a transformation to Z of the form (8.4) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

where B is a unitary matrix of order $n-1$, we can assume without loss of generality that

$$(8.5) \quad \begin{cases} \partial_{1j}f^{1\beta} = \delta_{j\beta} & \text{for } 1 \leq j \leq p, 1 \leq \beta \leq p \\ \partial_{1j}f^{1\beta} = 0 & \text{for } 1 < j \leq p, p < \beta \leq n \\ \partial_{ij}f^{1\beta} = 0 & \text{for } (i, j) > (1, p), 1 \leq \beta \leq n. \end{cases}$$

For $1 < i \leq p$ and $(k, l) > (1, p)$, consider

$$\sum_{\beta=1}^n \zeta_{1\bar{i}, k\bar{l}}^{1\beta, 1\bar{\beta}} = 0;$$

that is,

$$\sum_{\beta=1}^n (\partial_{1\bar{i}}f^{1\beta} \overline{\partial_{kl}f^{1\beta}} - \partial_{k\bar{l}}f^{1\beta} \overline{\partial_{1i}f^{1\beta}}) = 0.$$

Since $\partial_{1i}f^{1\beta} = \delta_{i\beta}$ and $\partial_{kl}f^{1\beta} = 0$ by (8.5), it follows that

$$(8.6) \quad \partial_{k\bar{l}}f^{1i} = 0, \quad 1 < i \leq p \quad \text{and} \quad (k, l) > (1, p).$$

Let q be the rank of the $(n-p) \times ((1/2)n(n+1) - p)$ matrix

$$P = (\partial_{k\bar{l}}f^{1i})_{p < i \leq n, (k, l) > (1, p)}.$$

Select $p < i_1 < \dots < i_q \leq n$ such that the rank of the $q \times ((1/2)n(n+1) - p)$

matrix

$$(\partial_{\bar{k}l} f^{1i\nu})_{1 \leq \nu \leq q, (k,l) > (1,p)}$$

is q . By applying Lemma 3a) to the map $(f^{1i_1}, \dots, f^{1i_q})$ whose domain variables are $(w_{kl}), (k, l) > (1, p)$ (the other variables being fixed), we conclude that, after a linear change in the variables $(w_{kl}), (k, l) > (1, p)$, we can assume without loss of generality that

$$(8.7) \quad \begin{cases} \partial_{\overline{1, j+p}} f^{1i\nu} = \delta_{j\nu} & \text{for } 1 \leq \nu, j \leq q \\ \partial_{\bar{k}l} f^{1i\nu} = 0 & \text{for } (k, l) > (1, p+q), 1 \leq \nu \leq q. \end{cases}$$

We must have

$$(8.8) \quad \partial_{\bar{k}l} f^{1i} = 0 \quad \text{for } (k, l) > (1, p+q), p < i \leq n,$$

otherwise the rank of the matrix P is $> q$. For $(k, l) > (1, p+q)$, consider

$$\sum_{\gamma=1}^n \xi_{\bar{k}l, 11}^{1\bar{\gamma}, 1\bar{\gamma}} = 0;$$

this is,

$$\sum_{\gamma=1}^n (\partial_{\bar{k}l} f^{1\gamma} \overline{\partial_{11} f^{1\gamma}} - \partial_{11} f^{1\gamma} \overline{\partial_{\bar{k}l} f^{1\gamma}}) = 0.$$

Since $\partial_{kl} f^{1\gamma} = 0$ and $\partial_{11} f^{1\gamma} = \delta_{1\gamma}$ for $1 \leq \gamma \leq p$ by (8.5), it follows from (8.8) that

$$(8.9) \quad \partial_{\bar{k}l} f^{11} = 0 \quad \text{for } (k, l) > (1, p+q).$$

When $p+q < n$, this implies together with (8.5), (8.6), (8.7), and (8.8) that at 0,

$$df^{11} \wedge \dots \wedge df^{1n} \wedge df^{1\bar{1}} \wedge \dots \wedge df^{1\bar{n}}$$

is a linear combination of exterior products of $dw_{il}, d\bar{w}_{i\bar{l}}$ for $1 \leq l \leq p+q$ and hence must be zero, which contradicts the fact that f is locally diffeomorphic to 0. Therefore we must have $q = n - p$. Thus $i_\nu = p + \nu$ and (8.7), (8.8), (8.9) read

$$(8.10) \quad \begin{cases} \partial_{\bar{l}l} f^{1\beta} = \delta_{l\beta} & \text{for } p < l, \beta \leq n \\ \partial_{\bar{k}l} f^{1\beta} = 0 & \text{for } 1 < k \leq l \leq n, p < \beta \leq n \\ \partial_{\bar{k}l} f^{11} = 0 & \text{for } 1 < k \leq l \leq n. \end{cases}$$

Take any $1 < \alpha \leq n$. Let $1 < k \leq l \leq n$ and $p < i \leq n$. Consider

$$\sum_{\beta=1}^n \xi_{1\bar{i}, \bar{k}l}^{1\beta, \alpha\bar{\beta}} = 0;$$

that is,

$$\sum_{\beta=1}^n (\partial_{1\bar{i}} f^{1\beta} \overline{\partial_{kl} f^{\alpha\bar{\beta}}} - \partial_{kl} f^{1\beta} \overline{\partial_{1\bar{i}} f^{\alpha\bar{\beta}}}) = 0.$$

Since $\partial_{\bar{k}l} f^{1\beta} = 0$ by (8.6) and (8.10) and since $\partial_{1\bar{i}} f^{1\beta} = \delta_{i\beta}$ for $1 < \beta \leq n$ by (8.6) and (8.10), it follows that

$$\partial_{1\bar{i}} f^{11} \overline{\partial_{kl} f^{\alpha\bar{1}}} + \partial_{kl} f^{\alpha i} = 0.$$

Because $f^{\alpha 1} = f^{1\alpha}$, by (8.5) we have $\partial_{kl}f^{\alpha 1} = 0$. Hence

$$(8.11) \quad \partial_{kl}f^{\alpha i} = 0 \quad \text{for } 1 < \alpha \leq n, p < i \leq n, 1 < k \leq l \leq n.$$

Consider now

$$\sum_{\beta=1}^n \xi_{kl, 1\bar{j}}^{\alpha\beta, 1\bar{\beta}} = 0,$$

where $1 < \alpha \leq n, 1 < k \leq l \leq n$, and $1 < j \leq p$. That is,

$$\sum_{\beta=1}^n (\partial_{kl}f^{\alpha\beta} \overline{\partial_{1j}f^{1\beta}} - \partial_{1j}f^{\alpha\beta} \overline{\partial_{kl}f^{1\beta}}) = 0.$$

Since $\partial_{kl}f^{1\beta} = 0$ and $\partial_{1j}f^{1\beta} = \delta_{j\beta}$ by (8.5), it follows that

$$(8.12) \quad \partial_{kl}f^{\alpha j} = 0 \quad \text{for } 1 < \alpha \leq n, 1 < k \leq l \leq n, \text{ and } 1 < j \leq p.$$

We are going to prove by induction on i for $1 < i \leq n$ that after a linear coordinate change in (w_{kl}) for $1 < k \leq l \leq n$, the following hold:

$$(8.13)_i \quad \begin{cases} \partial_{il}f^{i\beta} = \delta_{l\beta} & \text{for } i \leq \beta \leq n \text{ and } i \leq l \leq p \\ \partial_{kl}f^{i\beta} = 0 & \text{for } i \leq \beta \leq n \text{ and } (k, l) > (i, p) \\ \partial_{il}f^{i\beta} = \delta_{l\beta} & \text{for } \max(i, p+1) \leq \beta, l \leq n \\ \partial_{kl}f^{i\beta} = 0 & \text{for } i < k \leq l \leq n \text{ and } i \leq \beta \leq n. \end{cases}$$

To avoid a repetitious initial step of the induction process, we agree to mean by $(8.13)_i$ the vacuous statement and prove $(8.13)_i$ by induction on i for $1 \leq i \leq n$. Now we prove $(8.13)_{i-1} \Rightarrow (8.13)_i$ for $1 < i \leq n$. From (8.5), (8.6), (8.10), it follows that

$$(8.14) \quad df^{11}, \dots, df^{1n} \text{ is a linear combination of } dw_{11}, \dots, dw_{1n}, \\ d\bar{w}_{11}, \dots, d\bar{w}_{1n} \text{ at } 0.$$

From $(8.13)_\nu$ ($1 < \nu < i$) it follows that for $1 < \nu < i$

$$(8.15)_\nu \quad df^{\nu\nu}, \dots, df^{\nu n} \text{ is a linear combination of } dw_{kl}, d\bar{w}_{kl}, \\ 1 \leq k \leq \nu, k \leq l \leq n.$$

From (8.11) and (8.12) it follows that

$$(8.16) \quad df^{ii}, \dots, df^{ip} \text{ is a linear combination of } dw_{kl}, 1 \leq k \leq l \leq n, \\ \text{and } d\bar{w}_{1j}, 1 \leq j \leq n, \text{ at } 0,$$

and

$$(8.17) \quad df^{i\beta}, \max(i, p+1) \leq \beta \leq n, \text{ can be expressed in terms of } \\ d\bar{w}_{kl}, 1 \leq k \leq l \leq n, \text{ and } dw_{1j}, 1 \leq j \leq n \text{ at } 0.$$

When $p \geq i$, the $(p-i+1) \times (1/2)(n-i+1)(n-i+2)$ matrix

$$Q := (\partial_{kl}f^{i\beta})_{i \leq \beta \leq p, i \leq k < l \leq n}$$

must have rank $p-i+1$; otherwise from (8.14), (8.15) $_\nu$, $1 < \nu < i$, and (8.16) it follows that

$$(\wedge_{1 \leq \alpha < i, \alpha \leq \beta \leq n} (df^{\alpha\beta} \wedge df^{\bar{\alpha}\bar{\beta}})) \wedge (\wedge_{\beta=i}^p (df^{i\beta} \wedge df^{\bar{i}\bar{\beta}}))$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying Lemma 3a) to the map $(f^{ii}, \dots, f^{ip})$ whose domain variables are w_{kl} , $i \leq k < l \leq n$ (the other variables being fixed), we conclude that, after a linear coordinate change in w_{kl} , $i \leq k < l \leq n$, the first two equations of $(8.13)_i$ are satisfied. This implies that

$$(8.18) \quad df^{ii}, \dots, df^{ip} \text{ is a linear combination of } dw_{kl}, d\bar{w}_{kl}, \\ (i, p) \geq (k, l).$$

The $(n - \max(i, p + 1) + 1) \times ((1/2)(n - i + 1)(n - i + 2) - \max(0, p - i + 1))$ matrix

$$L: = (\partial_{\bar{k}l} f^{i\beta})_{\max(i, p+1) \leq \beta \leq n, (k, l) \geq (i, \max(i, p+1))}$$

must have rank $n - \max(i, p + 1) + 1$; otherwise from (8.14), (8.15) _{ν} , $1 < \nu < i$, (8.18), and (8.17) it follows that

$$\wedge_{1 \leq \alpha \leq i, \alpha \leq \beta \leq n} (df^{\alpha\beta} \wedge df^{\bar{\alpha}\bar{\beta}})$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying Lemma 3a) to the map $(f^{i, \max(i, p+1)}, \dots, f^{in})$ whose domain variables are w_{kl} , $(k, l) \geq (i, \max(i, p + 1))$ (the other variables being fixed), we conclude that, after a linear coordinate change in w_{kl} , $(k, l) \geq (\max(i, p + 1))$, the last two equations of $(8.13)_i$ are satisfied.

We distinguish between two cases.

Case 1. $p = n$. By (8.5) and $(8.13)_i$, $1 < i \leq n$, the $(1/2)n(n + 1) \times (1/2)n(n + 1)$ matrix

$$(\partial_{ij} f^{\alpha\beta})_{1 \leq i \leq j \leq n, 1 \leq \alpha \leq \beta \leq n}$$

is nonsingular. By Lemma 3a) we can apply a linear change of coordinates to (w_{ij}) , $1 \leq i \leq j \leq n$ so that after this linear coordinate change we have

$$(8.19) \quad \partial_{ij} f^{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta} \quad \text{for } 1 \leq i \leq j \leq n \text{ and } 1 \leq \alpha \leq \beta \leq n,$$

at the expense of possibly sacrificing the validity of (8.6), (8.10), (8.12), and the last two equations of $(8.13)_i$.

We want to show that $\partial_{i\bar{j}} f^{\alpha\beta} = 0$ for $1 \leq i \leq j \leq n$ and $1 \leq \alpha \leq \beta \leq n$. For notational convenience, we define, for $1 \leq i \leq j \leq n$, $\partial_{ji} = \partial_{ij}$ and $\partial_{\bar{j}\bar{i}} = \partial_{\bar{i}\bar{j}}$.

Take $1 \leq \alpha, \beta, j, k, l \leq n$ with $\beta \neq k, l$. Consider

$$\sum_{\gamma=1}^n \xi_{kl, \bar{\beta}\bar{j}}^{\alpha\gamma} = 0;$$

i.e.,

$$\sum_{\gamma=1}^n (\partial_{kl} f^{\alpha\gamma} \overline{\partial_{\beta j} f^{\bar{\beta}\bar{j}}}) - \partial_{\bar{\beta}\bar{j}} f^{\alpha\gamma} \overline{\partial_{kl} f^{\bar{\beta}\bar{j}}} = 0.$$

It follows from $\beta \neq k, l$ and (8.19) that $\partial_{kl} f^{\beta\gamma} = 0$. From (8.19) we also have

$\partial_{\beta j} f^{\beta \bar{j}} = \delta_{j\bar{j}}$. Hence

$$(8.20) \quad \partial_{\bar{k}l} f^{\alpha j} = 0 \quad \text{for } 1 \leq \alpha, j, k, l \leq n,$$

if there exists $1 \leq \beta \leq n$ with $\beta \neq k, l$. When $n \geq 3$, for any given $1 \leq k, l \leq n$, we can always find $1 \leq \beta \leq n$ with $\beta \neq k, l$. It remains to prove the case $n = 2$. Consider

$$\sum_{\gamma=1}^n \xi_{i\bar{j}, \bar{k}l}^{\alpha\gamma, \bar{\beta}\gamma} = 0$$

for $(\alpha, \beta, i, j, k, l) = (1, 1, 1, 2, 1, 1), (1, 2, 1, 2, 2, 2), (2, 2, 1, 2, 2, 2)$. That is,

$$(8.21) \quad \begin{cases} \partial_{\bar{1}2} f^{11} \overline{\partial_{11} f^{11}} - \partial_{\bar{1}1} f^{11} \overline{\partial_{12} f^{11}} + \partial_{\bar{1}2} f^{12} \overline{\partial_{11} f^{12}} - \partial_{\bar{1}1} f^{12} \overline{\partial_{12} f^{12}} = 0 \\ \partial_{\bar{1}2} f^{11} \overline{\partial_{22} f^{21}} - \partial_{\bar{2}2} f^{11} \overline{\partial_{12} f^{21}} + \partial_{\bar{1}2} f^{12} \overline{\partial_{22} f^{22}} - \partial_{\bar{2}2} f^{12} \overline{\partial_{12} f^{22}} = 0 \\ \partial_{\bar{1}2} f^{21} \overline{\partial_{22} f^{21}} - \partial_{\bar{2}2} f^{21} \overline{\partial_{12} f^{21}} + \partial_{\bar{1}2} f^{22} \overline{\partial_{22} f^{22}} - \partial_{\bar{2}2} f^{22} \overline{\partial_{12} f^{22}} = 0. \end{cases}$$

By (8.20),

$$\partial_{\bar{1}1} f^{\alpha\beta} = \partial_{\bar{2}2} f^{\alpha\beta} = 0 \quad \text{for } 1 \leq \alpha, \beta \leq 2.$$

By (8.19)

$$\begin{aligned} \partial_{11} f^{11} &= \partial_{22} f^{22} = \partial_{22} f^{22} = 1, \\ \partial_{11} f^{12} &= \partial_{22} f^{21} = 0. \end{aligned}$$

It follows from (8.21) that

$$\partial_{\bar{1}2} f^{11} = \partial_{\bar{1}2} f^{12} = \partial_{\bar{1}2} f^{22} = 0.$$

Case 2. $p < n$. It follows from (8.10) and (8.13)_i, $1 < i \leq n$, that

$$\partial_{\bar{j}n} f^{in} = \delta_{ij} \quad \text{for } 1 \leq i \leq j \leq n.$$

Since $f^{ni} = f^{ni}$, it follows that the rank of the $n \times (1/2)n(n+1)$ matrix

$$(\partial_{\bar{i}j} f^{n\alpha})_{1 \leq \alpha \leq n, 1 \leq i \leq j \leq n}$$

is n . After we apply the transformation

$$Z \longmapsto {}^t A \bar{Z} A,$$

where A is the matrix obtained from I_n by interchanging the first and last columns, we reduce this case to Case 1 and we therefore conclude, by the arguments of Case 1, that $\partial_{ij} f^{\alpha\beta} = 0$ for all $1 \leq i \leq j \leq n$ and $1 \leq \alpha \leq \beta \leq n$, which is a contradiction. Hence Case 2 cannot occur and $\partial_{ij} f^{\alpha\beta} = 0$ for all $1 \leq i \leq j \leq n$ and $1 \leq \alpha \leq \beta \leq n$. Q.E.D.

9. Adequate negativity of the curvature of D_n^{IV}

PROPOSITION 8. *For $n \geq 3$ the curvature tensor $R_{\alpha\bar{\rho}\gamma\bar{\sigma}} (1 \leq \alpha, \rho, \beta, \sigma \leq n)$ of D_n^{IV} is adequately negative.*

Proof. Let U be an open neighborhood of 0 in $\mathbb{C}^n$. Let $w_i (1 \leq i \leq n)$ be the coordinates of $\mathbb{C}^n$. Denote $\partial/\partial w^i, \partial/\partial \bar{w}^i$ by $\partial_i, \partial_{\bar{i}}$. Let $f: U \rightarrow D_n^{\text{IV}}$ be a

smooth map which sends the origin to the origin and which is locally diffeomorphic at 0. Let $f^\alpha (1 \leq \alpha \leq n)$ be the components of f and let

$$\xi_{ij}^{\alpha\bar{\beta}} = (\partial_i f^\alpha)(0)(\overline{\partial_j f^\beta})(0) - (\partial_j f^\alpha)(0)(\overline{\partial_i f^\beta})(0).$$

Assume

$$(9.1) \quad \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\rho}\bar{\beta}\bar{\sigma}} \xi_{ij}^{\alpha\bar{\rho}} \xi_{ij}^{\gamma\bar{\sigma}} = 0$$

at 0 for all $1 \leq i, j \leq n$. We have to show that

$$(9.2) \quad \text{either } \bar{\partial} f^\alpha = 0 \text{ at } 0 \text{ for } 1 \leq \alpha \leq n \text{ or } \partial f^\alpha = 0 \text{ at } 0 \text{ for } 1 \leq \alpha \leq n.$$

For notational simplicity we will denote $(\partial_i f^\alpha)(0)$, $(\partial_i \bar{f}^\alpha)(0)$, $(\partial_i \bar{f}^\alpha)(0)$, $(\partial_i \bar{f}^\alpha)(0)$ simply by $\partial_i f^\alpha$, $\partial_i \bar{f}^\alpha$, $\partial_i \bar{f}^\alpha$, $\partial_i \bar{f}^\alpha$ respectively. From (9.1) and (5.4) it follows that

$$(9.3) \quad \begin{cases} \sum_{\alpha=1}^n \xi_{ij}^{\alpha\bar{\alpha}} = 0 & \text{for all } i, j \\ \xi_{ij}^{\alpha\bar{\beta}} - \xi_{ij}^{\beta\bar{\alpha}} = 0 & \text{for all } i, j, \alpha, \beta. \end{cases}$$

We will prove (9.2) only from the second equation of (9.3). The second equation of (9.3) is invariant under the automorphism $z \mapsto \bar{z}$ of D_n^{IV} .

For $1 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq n$ let $V_{\alpha_1 \alpha_2 \alpha_3}$ be the vector subspace of the tangent space $T_{U,0}$ of U at 0 consisting all tangent vectors whose images under df are annihilated by $dz_{\alpha_\nu}, d\bar{z}_{\alpha_\nu}, 1 \leq \nu \leq 3$. Since f is locally diffeomorphic at 0, $V_{\alpha_1 \alpha_2 \alpha_3}$ is of real codimension 6 in $T_{U,0}$. By Lemma 1, we can choose a basis $e_1, \dots, e_n$ of $T_{U,0}$ over $\mathbb{C}$ such that

$$V_{\alpha_1 \alpha_2 \alpha_3} \cap (\bigoplus_{\nu=1}^{n-3} \mathbb{C} e_{i_\nu}) = 0$$

for all $1 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq n$ and all $1 \leq i_1 < \dots < i_{n-3} \leq n$. We denote again by $w_i, 1 \leq i \leq n$, the linear coordinate system of U such that $e_i = 2 \operatorname{Re}(\partial/\partial w_i)$ at 0, $1 \leq i \leq n$. Then for any $1 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq n$ and any $1 \leq i_1 < i_2 < i_3 \leq n$, the map $(f^{\alpha_1}, f^{\alpha_2}, f^{\alpha_3})$ whose domain variables are $w_{i_1}, w_{i_2}, w_{i_3}$ (the other variables being fixed at 0) is locally diffeomorphic at 0. For the conclusion of the proposition, clearly it suffices to show that for any $1 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq n$ and any $1 \leq i_1 < i_2 < i_3 \leq n$ either $\partial_{i_\mu} f^{\alpha_\lambda} = 0$ for $1 \leq \mu, \lambda \leq 3$ or $\partial_{i_\mu} \bar{f}^{\alpha_\lambda} = 0$ for $1 \leq \mu, \lambda \leq 3$. Hence in order to derive the conclusion of the proposition from the second equation of (9.3), it suffices to consider the case $n = 3$. We assume now $n = 3$.

We write out below in full the nine equations $\xi_{ij}^{\alpha\bar{\beta}} = \xi_{ij}^{\beta\bar{\alpha}}, \alpha \neq \beta, i \neq j$ from the second equation of (3.9):

$$(9.4) \quad \partial_1 f^1 \bar{\partial}_2 f^2 - \partial_2 f^1 \bar{\partial}_1 f^2 = \partial_1 f^2 \bar{\partial}_2 f^1 - \partial_2 f^2 \bar{\partial}_1 f^1,$$

$$(9.5) \quad \partial_2 f^1 \bar{\partial}_3 f^2 - \partial_3 f^1 \bar{\partial}_2 f^2 = \partial_2 f^2 \bar{\partial}_3 f^1 - \partial_3 f^2 \bar{\partial}_2 f^1,$$

$$(9.6) \quad \partial_3 f^1 \bar{\partial}_1 f^2 - \partial_1 f^1 \bar{\partial}_3 f^2 = \partial_3 f^2 \bar{\partial}_1 f^1 - \partial_1 f^2 \bar{\partial}_3 f^1,$$

$$(9.7) \quad \partial_1 f^3 \bar{\partial}_2 f^2 - \partial_2 f^3 \bar{\partial}_1 f^2 = \partial_1 f^2 \bar{\partial}_2 f^3 - \partial_2 f^2 \bar{\partial}_1 f^3,$$

$$(9.8) \quad \partial_{\bar{z}} f^3 \overline{\partial_3 f^2} - \partial_{\bar{z}} f^3 \overline{\partial_2 f^2} = \partial_{\bar{z}} f^2 \overline{\partial_3 f^3} - \partial_{\bar{z}} f^2 \overline{\partial_2 f^3},$$

$$(9.9) \quad \partial_3 f^3 \overline{\partial_1 f^2} - \partial_1 f^3 \overline{\partial_3 f^2} = \partial_3 f^2 \overline{\partial_1 f^3} - \partial_1 f^2 \overline{\partial_3 f^3},$$

$$(9.10) \quad \partial_1 f^1 \overline{\partial_2 f^3} - \partial_{\bar{z}} f^1 \overline{\partial_1 f^3} = \partial_1 f^3 \overline{\partial_2 f^1} - \partial_{\bar{z}} f^3 \overline{\partial_1 f^1},$$

$$(9.11) \quad \partial_{\bar{z}} f^1 \overline{\partial_3 f^3} - \partial_3 f^1 \overline{\partial_2 f^3} = \partial_{\bar{z}} f^3 \overline{\partial_3 f^1} - \partial_3 f^3 \overline{\partial_2 f^1},$$

$$(9.12) \quad \partial_3 f^1 \overline{\partial_1 f^3} - \partial_1 f^1 \overline{\partial_3 f^3} = \partial_3 f^3 \overline{\partial_1 f^1} - \partial_1 f^3 \overline{\partial_3 f^1}.$$

Let p be the rank of the 3×3 matrix $(\partial_i f^\alpha)_{1 \leq i, \alpha \leq 3}$ and q be the rank of the 3×3 matrix $(\bar{\partial}_i f^\alpha)_{1 \leq i, \alpha \leq 3}$. We claim that one of p and q is at least 2. Suppose the contrary. Then we can choose $1 \leq \lambda, \mu \leq 3$ such that for $1 \leq \alpha \leq 3$,

$$\partial f^\alpha = \lambda_\alpha \partial f^\lambda,$$

$$\bar{\partial} f^\alpha = \mu_\alpha \bar{\partial} f^\mu$$

at 0 for some $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ (which may be zero). It follows that at 0 the 6-form

$$\wedge_{\alpha=1}^3 (df^\alpha \wedge d\bar{f}^\alpha)$$

is a linear combination of exterior products of $\partial f^\lambda, \bar{\partial} f^\mu, \overline{\partial f^\lambda}, \bar{\bar{\partial} f^\mu}$ and hence must be 0, contradicting the assumption that f is locally diffeomorphic at 0. By applying the transformation $z \mapsto \bar{z}$ if necessary, we can assume without loss of generality that $p \geq 2$.

Case 1. $p = 3$. By Lemma 3 a), after a linear coordinate change in w_i , $1 \leq i \leq 3$, we can assume without loss of generality that

$$\partial_i f^\alpha = \delta_{i\alpha} \quad \text{for } 1 \leq i, \alpha \leq 3.$$

From (9.4)–(9.12) it follows that

$$(9.13) \quad \partial_1 f^1 = -\partial_2 f^2,$$

$$\partial_3 f^1 = 0,$$

$$0 = \partial_{\bar{z}} f^2,$$

$$\partial_1 f^3 = 0,$$

$$(9.14) \quad -\partial_3 f^3 = \partial_{\bar{z}} f^2,$$

$$0 = \partial_1 f^2,$$

$$0 = \partial_{\bar{z}} f^3,$$

$$\partial_{\bar{z}} f^1 = 0,$$

$$(9.15) \quad -\partial_1 f^1 = \partial_{\bar{z}} f^3.$$

From (9.13), (9.14), and (9.15) it follows that

$$\partial_1 f^1 = -\partial_{\bar{z}} f^2 = \partial_{\bar{z}} f^3 = -\partial_1 f^1.$$

Hence $\partial_i f^\alpha = 0$ for $1 \leq i, \alpha \leq 3$.

Case 2. $p = 2$. At 0, two of $\partial f^1, \partial f^2, \partial f^3$ are linearly independent. After renumbering f^1, f^2, f^3 if necessary, we can assume without loss of

generality that $\partial f^1, \partial f^2$ are linearly independent at 0. After applying a linear coordinate change to $w_i, 1 \leq i \leq 3$, we have

$$\partial f^1 = dw_1,$$

$$\partial f^2 = dw_2$$

at 0; i.e., $\partial_j f^\alpha = \delta_{j\alpha}$ for $1 \leq j \leq 3, 1 \leq \alpha \leq 2$. Since ∂f^3 is a linear combination of ∂f^1 and ∂f^2 at 0, we have $\partial_3 f^3 = 0$. From (9.5), (9.6), and (9.12) it follows that $\partial_3 f^\alpha = 0$ for $1 \leq \alpha \leq 3$. Combining this with $\partial_3 f^\alpha = 0$ for $1 \leq \alpha \leq 3$, we conclude that $df^\alpha, d\bar{f}^\alpha, 1 \leq \alpha \leq 3$, are linear combinations of $dw_1, dw_2, d\bar{w}_1, d\bar{w}_2$ at 0. Hence

$$\wedge_{\alpha=1}^3 (df^\alpha \wedge d\bar{f}^\alpha)$$

vanishes at 0, contradicting the assumption that f is locally diffeomorphic at 0. Thus Case 2 cannot occur and we have $\partial_i f^\alpha = 0$ for $1 \leq i, \alpha \leq 3$.

10. Strong rigidity and other applications

THEOREM 6. *$k \geq 2$. Suppose $g: N \rightarrow M$ is a continuous map of compact Kähler manifolds. If the curvature tensor of M is negative of order k and if the map $H_l(N, \mathbf{R}) \rightarrow H_l(M, \mathbf{R})$ induced by f is nonzero for some $l \geq 2k$, then g is homotopic to a holomorphic or conjugate holomorphic map from N to M .*

Proof. There exists a harmonic map $f: N \rightarrow M$ which is homotopic to g . Since the map $H_l(N, \mathbf{R}) \rightarrow H_l(M, \mathbf{R})$ induced by f is nonzero, it follows that $\text{rank}_{\mathbf{R}} df \geq 2k$ at some point of N . By Theorem 5, f is either holomorphic or conjugate holomorphic. Q.E.D.

The following theorem follows from Theorem 6 and Lemma 2.

THEOREM 7. *Suppose M is a compact Kähler manifold whose curvature tensor is strongly negative. Then for $k \geq 2$ an element of $H_{2k}(M, \mathbf{Z})$ can be represented by a complex-analytic subvariety of M if it can be represented by the continuous image of a compact Kähler manifold.*

THEOREM 8. *Let $g: N \rightarrow M$ be a continuous map of compact Kähler manifolds, both of complex dimension $n \geq 2$. Suppose the curvature tensor of M is adequately negative. Suppose g is of degree 1 and the map $H_{2n-2}(N, \mathbf{R}) \rightarrow H_{2n-2}(M, \mathbf{R})$ induced by g is injective. Then g is homotopic to a biholomorphic or conjugate biholomorphic map from N to M .*

Proof. By Theorem 6, g is homotopic to a holomorphic or conjugate holomorphic map $f: N \rightarrow M$. Let V be the set of points of N where f is not locally homeomorphic. Since f is of degree 1, $V \neq N$. Suppose V is nonempty.

We want to derive a contradiction. V is a complex-analytic subvariety of pure complex codimension 1 in N , because locally V is defined by $\det(\partial z^\alpha/\partial w^i)$ when f is holomorphic and by $\det(\partial \bar{z}^\alpha/\partial w^i)$ when f is conjugate holomorphic, where (z^α) (respectively (w^i)) is a holomorphic local coordinate system of M (respectively N). $f(V)$ is a complex-analytic subvariety in M . Since f is of degree 1, f maps $N - f^{-1}(f(V))$ bijectively onto $M - f(V)$. The complex codimension of $f(V)$ in M is at least two. For otherwise there exists $v \in V$ such that v is an isolated point of $f^{-1}(f(v))$ and, by using a local coordinate chart (w^i) of N at v and by applying the Riemann removable singularity theorem to $w^i \circ f^{-1}$ or $\overline{w^i} \circ f^{-1}$ on $U - f(V)$ for some open neighborhood of $f(v)$ in M , we conclude that f is locally diffeomorphic at v , contradicting $v \in V$. Hence the element in $H_{2n-2}(N, \mathbf{R})$ defined by V is mapped by f to the zero element in $H_{2n-2}(M, \mathbf{R})$, contradicting the injectivity of the map $H_{2n-2}(N, \mathbf{R}) \rightarrow H_{2n-2}(M, \mathbf{R})$ induced by f . Q.E.D.

Now Theorems 2 and 4 follow from Theorem 8, Lemma 2, and Propositions 5-8.

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