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On Fundamental Groups of Complete Affinely Flat Manifolds

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This note tries to decide which groups can occur as fundamental groups of complete affinely flat manifolds.

THEOREM 1.2. *If a group is torsion-free and contains a polycylic subgroup of finite index, then it is isomorphic to the fundamental group $\pi_1(M)$ for some complete affinely flat manifold M.*

For definitions and proof, see Section 1. The remaining sections center around two unresolved questions: *Are these the only fundamental groups which can occur* (Section 3)? *Can the manifold M always be chosen to be compact* (Section 4)?

1. CONSTRUCTING AFFINE ACTIONS

First some definitions. A group Γ is *virtually polycyclic* if it has a subgroup Γ_0 of finite index which is *polycyclic*, that is, admits a finite composition series $\Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{1\}$ so that each quotient Γ_i/Γ_{i+1} is cyclic. The number of Γ_i/Γ_{i+1} which are free cyclic is an invariant called the *rank* of Γ.

More generally, given any property P (such as being commutative, solvable, or torsion-free), a group is *virtually P* if it admits a subgroup of finite index with property P.

An action $(\gamma, x) \mapsto \gamma x$ of a group Γ on a locally compact space X is *properly discontinuous* if for each compact $K \subset X$ the set of γ with $\gamma K \cap K \neq \varnothing$ is finite. This action is *effective* if the equation $\gamma x = x$ for all x implies that $\gamma = 1$, and *free* if $\gamma x = x$ for any x implies that $\gamma = 1$.

THEOREM 1.1. *Any virtually polycyclic group admits an effective and properly discontinuous action by affine transformations $x \mapsto Ax + b$ of some Euclidean space $\mathbf{R}^n$.*

Proof. Let $\Gamma_0 \subset \Gamma$ be a polycyclic subgroup of finite index. Auslander [2] and Swan [26] have shown that Γ_0 can be embedded into the general linear group $GL(m, \mathbf{Z})$ for some m. *Hence Γ_0 can be embedded as a discrete subgroup of the complex general linear group* $GL(m, \mathbf{C})$.

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Consider the Zariski closure of Γ_0, that is, the smallest complex algebraic variety in GL(m, **C**) which contains Γ_0. This closure is both a smooth algebraic variety, and a solvable complex Lie group. (Compare [7].) Its identity component S is a subgroup of finite index, hence the intersection $\Delta = S \cap \Gamma_0$ is a subgroup of finite index in Γ_0.

By Lie's theorem, S is conjugate to a subgroup of the standard Borel group B consisting of upper triangular matrices in GL(m, **C**). (Compare [17, p. 50] or [15, p. 134].) Hence, after applying an inner automorphism, we may assume that $\Delta \subset B$.

Recall that B splits as a semidirect product UD, where U is the group of uni-triangular matrices and D is the commutative group consisting of diagonal matrices. Note that U, being a complex $m(m-1)/2$-dimensional plane in the vector space of $m \times m$ matrices, has the structure of a complex affine space. Since U is a normal subgroup of B, there is a short exact sequence

$$1 \longrightarrow U \longrightarrow B \xrightarrow{\pi} D \longrightarrow 1.$$

Using this projection homomorphism π, each group element b of B operates on U by the affine transformation

$$u \mapsto bu\pi(b)^{-1}.$$

(Or, equivalently, we could say that B operates on the coset space $B/D \cong U$ by left translation.) Therefore, the discrete subgroup $\Delta \subset B$ also operates by affine transformations of U. This action need not be either effective or properly discontinuous. However, the subgroup $\Delta \cap U$, which operates on U just by left translation, certainly acts effectively and properly discontinuously.

Next consider the quotient group $\Delta/(\Delta \cap U)$ which is finitely generated (since Δ is) and commutative (since it embeds in D). This quotient splits as a direct sum $\Lambda_1 \oplus \cdots \oplus \Lambda_p$ of cyclic groups. Each cyclic group operates effectively and properly discontinuously either by translations or by rotations of the complex numbers **C**. Hence $\Delta/(\Delta \cap U)$ acts effectively and properly discontinuously by affine transformations of the product $\mathbf{C} \times \cdots \times \mathbf{C} = \mathbf{C}^p$.

The diagonal action of Δ on the product affine space $U \times \mathbf{C}^p$ now has all of the required properties. It is clearly an effective action by affine transformations, and it is not difficult to show that it is properly discontinuous.

Finally we must pass from the subgroup Δ to the larger group Γ. Let q be the index of Δ in Γ. Given a representation of Δ by affine transformations of $\mathbf{R}^k$, we can form the induced representation of Γ by affine transformations of the space $\mathrm{Hom}_\Delta(\Gamma, \mathbf{R}^k)$. By definition, this is the kq-dimensional affine space consisting of all functions $f\colon \Gamma \to \mathbf{R}^k$ which are Δ-equivariant:

$$f(\delta\gamma) = \delta f(\gamma) \qquad \text{for} \quad \delta \in \Delta,\ \gamma \in \Gamma.$$

If $\gamma_1, \ldots, \gamma_q$ are coset representatives, so that $\Gamma = \Delta\gamma_1 \cup \cdots \cup \Delta\gamma_q$, note that f

is uniquely determined by the values $f(\gamma_1),\ldots, f(\gamma_q) \in \mathbf{R}^k$, which can be prescribed arbitrarily.

Each group element φ in Γ acts on this affine space $\mathrm{Hom}_\Delta(\Gamma, \mathbf{R}^k)$ by the affine transformation $f \mapsto {}^\varphi f$, where ${}^\varphi f(\gamma) = f(\gamma\varphi)$. If the action of Δ on $\mathbf{R}^k$ is effective and properly discontinuous, then it is easy to check that this induced action of Γ on $\mathrm{Hom}_\Delta(\Gamma, \mathbf{R}^k)$ is also. ▮

It is now easy to prove Theorem 1.2, as stated in the Introduction.

First the necessary definitions. An *affinely flat* structure on an n-dimensional manifold M can be specified by a collection of coordinate homeomorphisms

$$f_\alpha : U_\alpha \to V_\alpha \subset \mathbf{R}^n,$$

where the U_α are open sets covering M and the V_α are open subsets of $\mathbf{R}^n$. Whenever $U_\alpha \cap U_\beta \neq \varnothing$, it is required that the change of coordinate homeomorphism

$$f_\beta f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

should extend to an affine transformation $x \mapsto Ax + b$ from $\mathbf{R}^n$ to itself.

One can define more restrictive kinds of flatness by requiring that the matrix A should belong to some proper subgroup of $\mathrm{GL}(n, \mathbf{R})$. For example, if A is required to be orthogonal, then we obtain the concept of a *Riemannian-flat manifold*, locally isometric to Euclidean space. Similarly, if A must belong to the Lorentz group $O(n-1, 1)$, we obtain the concept of a *Lorentz-flat* manifold (cf. [5]), and if A is required to have determinant ± 1 we obtain the concept of an *area-preserving affinely flat* manifold.

By a *geodesic* in an affinely flat manifold M is meant a mapping $\mathbf{R} \to M$ compatible with the affine structure. The manifold M is said to be *complete* if every geodesic segment $[0, 1] \to M$ can be extended to a full geodesic, or equivalently if the universal covering manifold $\tilde{M}$ is affinely diffeomorphic to $\mathbf{R}^n$.

In order to actually construct a complete affinely flat manifold with given fundamental group Γ, it is usually easiest to start with an action of Γ by affine transformations of $\mathbf{R}^n$. If this action is free and properly discontinuous, then the quotient $M = \Gamma\backslash\mathbf{R}^n$ is the required manifold. For details, the reader is referred to [29, pp. 39, 45].

Proof of 1.2. If Γ is virtually polycyclic, then it admits a properly discontinuous, affine action on some $\mathbf{R}^n$ by Theorem 1.1. Note that the subgroup fixing any point of $\mathbf{R}^n$ must be finite. Now suppose that the group Γ is also *torsion-free*, i.e., suppose that every finite subgroup is trivial. Then the action of Γ on $\mathbf{R}^n$ must be free, and the conclusion follows. ▮

If we sharpen the requirements, and ask that Γ act by Euclidean isometries on $\mathbf{R}^n$, then the situation was thoroughly analyzed by Bieberbach more than 60 years ago. Here is an outline. (Compare [20].)

THEOREM 1.3. *The group Γ can occur as the fundamental group of a complete Riemannian-flat manifold M if and only if it is torsion-free, finitely generated, and contains a commutative subgroup of finite index. The manifold M can always be chosen to be compact.*

In fact, if Γ is finitely generated and virtually commutative, then it contains a free Abelian subgroup, $\Delta \cong \mathbf{Z}^k$, of finite index q. Letting Δ operate on $\mathbf{R}^k$ by translation, it follows that Γ operates on the Euclidean space $\mathrm{Hom}_\Delta(\Gamma, \mathbf{R}^k) \cong \mathbf{R}^{kq}$ by Euclidean isometries. This action is effective and properly discontinuous. If Γ is also torsion-free, then the action is free, and $M = \Gamma\backslash\mathbf{R}^{kq}$ is the required complete Riemannian-flat manifold.

Given any group Γ which acts properly discontinuously by Euclidean isometries of $\mathbf{R}^n$, Bieberbach showed that there always exists a Euclidean subspace $\mathbf{R}^k$ on which Γ acts with compact fundamental domain. If Γ is torsion-free, then it follows that the quotient $\Gamma\backslash\mathbf{R}^k$ is a compact flat manifold, embedded as a deformation retract in the complete manifold $\Gamma\backslash\mathbf{R}^n$. (In the language of Cheeger and Gromoll, this compact totally geodesic submanifold is called the "soul" of $\Gamma\backslash\mathbf{R}^n$.)

Finally, Bieberbach showed that Γ contains, as subgroup of finite index, a free Abelian group of rank k which acts on $\mathbf{R}^k$ by translation. For details, the reader is referred to [6; 29, pp. 99–106]. ∎

2. Lemmas Concerning Lie Groups and Discrete Subgroups

This section proves three lemmas, for use later. The first is an immediate consequence of a theorem of Tits [27].

LEMMA 2.1. *Let Γ be a discrete subgroup of a Lie group G which has finitely many components. Then either Γ is virtually polycyclic or Γ contains a subgroup $\mathbf{Z} * \mathbf{Z}$ which is free on two generators.*

These two possibilities are mutually exclusive.

It is natural to ask which Lie groups have the property that *every* discrete subgroup is virtually polycyclic. Recall that a topological group G is said to be *amenable* if the space of bounded continuous functions $f\colon G \to \mathbf{R}$ admits a *left invariant mean,* that is, a linear real-valued function $f \mapsto m(f)$ which is invariant under left translation and satisfies

$$\inf f \leqslant m(f) \leqslant \sup f.$$

Using Lemma 2.1, the following statement is more or less well known. (Cf. [12; 22].)

LEMMA 2.2. *If G is a Lie group with finitely many components, then the following four conditions are equivalent*:

(a) *G is amenable*,

(b) *every discrete subgroup of G is virtually polycyclic*,

(c) *every semisimple connected subgroup of G is compact*,

(d) *G contains a solvable normal subgroup with compact quotient.*

Proof of Lemma 2.1. If Γ is any group which admits a faithful finite dimensional representation, so that

$$\Gamma \subset \mathrm{GL}(n, \mathbf{R})$$

for some n, then Tits' theorem asserts that either Γ is virtually solvable or Γ contains a free noncyclic subgroup.

The same dichotomy holds for any Γ which can be embedded in a Lie group G with finitely many components. For, according to Ado's theorem [17, p. 202], the Lie algebra of G admits a faithful linear representation. Hence some group G' which is locally isomorphic to G embeds in $\mathrm{GL}(n, \mathbf{R})$ for some n. Thus there exist homomorphisms

$$G \leftarrow \tilde{G}_0 \rightarrow G'$$

with discrete central kernel, where $\tilde{G}_0$ is the universal covering of the identity component of G. For any $\Gamma \subset G$, we can construct an associated subgroup $\Gamma' \subset G'$ by first intersecting with the identity component G_0, then passing to the full inverse image in $\tilde{G}_0$, and finally projecting into G'. It is easy to check that Γ is virtually solvable [or contains a free noncyclic subgroup] if and only if Γ' is virtually solvable [or contains a free noncyclic subgroup].

To conclude the proof, we make use of Mostow's theorem which asserts that any discrete solvable subgroup of a connected Lie group must actually be polycyclic. Hence, for discrete groups, virtually solvable implies virtually polycyclic. ∎

Proof of Lemma 2.2. The proof that (d) $\Rightarrow$ (a) $\Rightarrow$ (b) is based on theorems from Greenleaf [13]. Clearly, every compact group is amenable. Since every commutative group is amenable, and every extension of an amenable group by an amenable group is amenable [13, pp. 5, 30], it follows that (d) $\Rightarrow$ (a).

Since discrete subgroups of amenable groups are amenable, but free noncyclic groups are not amenable [13, pp. 30, 6], it follows using Lemma 2.1 that (a) $\Rightarrow$ (b).

Proof that (b) $\Rightarrow$ (c). Let H be any semisimple subgroup of G. We make use of the Iwasawa decomposition

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

where $\mathfrak{h}$ is the Lie algebra of H. (Compare [14, p. 222].) Here $\mathfrak{k}$ is a subalgebra with the property that the associated Lie group $K \subset H$ is compact, at least when H has finite center. The subalgebra $\mathfrak{a}$ is commutative, and the subalgebra $\mathfrak{n}$ is nilpotent, with the property that $\mathrm{ad}(x)\colon \mathfrak{h} \to \mathfrak{h}$ is nilpotent for each $x \in \mathfrak{n}$. Closely related is the *Cartan decomposition* $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a vector space containing $\mathfrak{a}$ and satisfying the condition $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ (see [14, p. 157]).

We first show that the nilpotent summand $\mathfrak{n}$ must be trivial; for otherwise the Lie algebra $\mathfrak{h}$ would contain an element $x \neq 0$ with $\mathrm{ad}(x)$ nilpotent. By the Jacobson–Morozov theorem, x would be contained in a subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ (see [17, pp. 100, 203, 79, 14]). Let $S \subset H$ be the corresponding Lie group, and let $\bar{S}$ be the closure of S in G. Then evidently the adjoint action of S on $\mathfrak{s}$ extends to a continuous action of $\bar{S}$ on $\mathfrak{s}$. In this way we obtain a continuous homomorphism from $\bar{S}$ onto the group $PSL(2, \mathbf{R})$ of automorphisms of $\mathfrak{s}$. Now choosing a discrete subgroup isomorphic to $\mathbf{Z} * \mathbf{Z}$ in $PSL(2, \mathbf{R})$, and lifting both generators to $\bar{S} \subset G$, we obtain a free noncyclic discrete subgroup of G, contradicting (b).

Hence we may assume that $\mathfrak{n} = 0$. It follows that the two vectors spaces $\mathfrak{a} \subset \mathfrak{p}$ have the same dimension, and hence are equal. Therefore $\mathfrak{a} = \mathfrak{p}$ is a commutative ideal in the semisimple Lie algebra $\mathfrak{h}$, hence $\mathfrak{a} = \mathfrak{p} = 0$. This means that $\mathfrak{h} = \mathfrak{k}$, so that the Lie group H, modulo its center, must be compact. By a theorem of Weyl [14, p. 123], this implies that H itself is compact, proving (c).

Proof that (c) $\Rightarrow$ (d). Let $N \subset G$ be the largest connected solvable normal subgroup. (See, for example, [8, p. 241].) By Levi's theorem, there exists a complementary semisimple subgroup $H \subset G$ (see [17, p. 91]). Then H maps onto the identity component of the quotient G/N, so if H is compact (or is contained in a compact subgroup of G) then it follows that G/N is compact, proving (d), and completing the proof of 2.2. ∎

We need one further lemma, concerning a Lie group which acts affinely on Euclidean space.

Lemma 2.3. *If the Lie group G is compact, or connected and semisimple, then any smooth representation of G by affine transformations of $\mathbf{R}^n$ admits a fixed point.*

Proof. This is an immediate consequence of the complete reducibility theorem for linear representations of such groups (see, for example, [8, p. 246]). We identify the affine space $\mathbf{R}^n$ with the hyperplane $\mathbf{R}^n \times 1$ in $\mathbf{R}^{n+1}$. Clearly any representation of G by affine transformations of $\mathbf{R}^n \times 1$ extends uniquely to a representation by linear transformations of $\mathbf{R}^{n+1}$. Since the linear subspace $\mathbf{R}^n \times 0$ is invariant, there must exist a complementary invariant linear subspace L. The intersection $L \cap (\mathbf{R}^n \times 1)$ is now the required fixed point. ▮

3. Are There Other Properly Discontinuous Affine Groups?

Let M be a complete affinely flat manifold. Then one can conjecture that

(1) *the fundamental group* $\pi_1(M)$ *must be virtually polycyclic.*

If this is true, then it follows in particular that:

(2) *the group* $\pi_1(M)$ *must be finitely generated, and*

(3) *the Euler characteristic* $\chi(M) = \sum (-1)^i \operatorname{rank} H_i(M)$ *must be defined and equal to zero, except for the special case* $M \simeq \mathbf{R}^n$, *with* $\chi(\mathbf{R}^n) = 1$.

The proof that (1) implies (2) and (3) is not difficult. (Compare [29, p. 106].) Conversely, if (1) is false, then it follows from 2.1 that (2) and (3) are false also. For the group $\pi_1(M)$ certainly embeds discretely in the Lie group consisting of all affine transformations of $\tilde{M} \simeq \mathbf{R}^n$. Hence, if $\pi_1(M)$ is not virtually polycyclic, then it must contain a subgroup $\Phi \simeq \mathbf{Z} * \mathbf{Z}$ which is free on two generators. The quotient $M' = \Phi \backslash \mathbf{R}^n$ would then be a complete affinely flat manifold having the homotopy type of a figure 8. Its Euler characteristic $\chi(M')$ would be -1. The maximal Abelian covering manifold of M' would have a fundamental group $[\Phi, \Phi] \simeq \mathbf{Z} * \mathbf{Z} * \cdots$ which is not even finitely generated. ▮

I do not know whether such a manifold exists even in dimension 3. One could try to construct a Lorentz-flat example by starting with a discrete subgroup $\mathbf{Z} * \mathbf{Z} \subset O(2, 1)$, then adding translation components to obtain a group of isometries of Lorentz 3-space; but it seems difficult to decide whether the resulting group action is properly discontinuous.

If we consider affinely flat manifolds which are not complete, then the case $\pi_1(M) \supset \mathbf{Z} * \mathbf{Z}$ definitely can occur. As an example, the compact manifold

$$M = S^1 \times (\text{surface of genus } 2)$$

admits an affinely flat structure having the open cone $x > (y^2 + z^2)^{1/2}$ in $\mathbf{R}^3$ as universal covering space. The proof (which I learned from M. Hirsch) is not difficult.

It would be interesting to know whether the Euler characteristic of such a compact, but not complete, flat manifold must be zero (Cf. [30]). If M is compact *and* complete, then Kostant and Sullivan have shown that $\chi(M) = 0$. On the other hand, using a weaker concept of flatness, Smillie has constructed compact examples with $\chi(M) \neq 0$.

It would also be interesting to know whether a compact manifold with an area-preserving flat-affine structure is necessarily complete.

As a partial justification for conjecture (1) above, we can consider the corresponding situation for a connected Lie group G which acts smoothly by affine transformations of $\mathbf{R}^n$. Such a group is said to act *properly* if for each compact set $K \subset \mathbf{R}^n$ the set of group elements g with $gK \cap K \neq \varnothing$ is compact.

THEOREM 3.1. *If the connected Lie group G acts properly by affine transformations of $\mathbf{R}^n$, then G is amenable, hence every discrete subgroup is virtually polycyclic.*

Proof. Let H be any semisimple subgroup of G. Then H has a fixed point p in $\mathbf{R}^n$ by Lemma 2.3. Since the action is proper, the subgroup consisting of all elements in G which fix p is compact. Thus H is contained in a compact group, and it follows [14, p. 128] that H itself is compact. Using Lemma 2.2, the conclusion follows. ∎

If we require that G acts freely on $\mathbf{R}^n$, then the situation can be described even more precisely.

THEOREM 3.2. *A connected Lie group acts freely by affine transformations of some $\mathbf{R}^n$ if and only if it is simply connected and solvable.*

For, if G admits such an action, then every semisimple subgroup is trivial by Lemma 2.3, hence G is solvable. Furthermore, its maximal compact subgroup is also trivial by Lemma 2.3, so G is simply connected.

Conversely, if G is solvable, then using Ado's theorem we can embed a locally isomorphic group into $GL(m, \mathbf{C})$, and hence into its Borel subgroup. The argument is then analogous to the proof of Theorem 1.1. ∎

4. THE COMPACT CASE

Let Γ be torsion-free and virtually polycyclic of rank k. Then we have seen that Γ is isomorphic to the fundamental group of a *complete* affinely flat manifold. Furthermore, Johnson has shown that Γ is isomorphic to the fundamental group of a *compact* k-dimensional manifold, with universal covering $\tilde{M}$ diffeomorphic to $\mathbf{R}^k$. (See [4, 18]. This is proved by embedding Γ as a discrete cocompact subgroup of a Lie group G having finitely many components, and then letting Γ operate on the coset space $G/K \cong \mathbf{R}^k$, where K is a maximal compact subgroup.)

Now let us ask whether we can combine these two results, constructing a manifold M which is both compact and complete affinely flat with $\pi_1(M) \cong \Gamma$. Many examples are known (Cf. Theorem 1.3, as well as [1, 5, 23]), but the general case seems quite difficult.

As a first step, we can ask a corresponding question for a k-dimensional Lie group.

Does every solvable Lie group G admit a complete affinely flat structure which is invariant under left translation, or equivalently, does the universal covering group $\tilde{G}$ operate simply transitively by affine transformations of $\mathbf{R}^k$?

If G admits such a structure, then evidently, for any discrete subgroup Γ, the coset space $\Gamma\backslash G$ is a complete affinely flat manifold. In many cases, Γ can be chosen so that $\Gamma\backslash G$ is also compact.

All that is known about this question is a list of a few special cases:

If the Lie algebra of G admits a nonsingular derivation, then an affirmative answer has been given by Scheuneman [23]. Such a Lie algebra $\mathfrak{g}$ is necessarily nilpotent. (Compare 10, 11, 16.) As an example, if $\mathfrak{g}$ is graded by positive integers, with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, then a nonsingular derivation can be constructed by setting $Dx = jx$ for $x \in \mathfrak{g}_j$.

Similarly, if G is 3-step nilpotent, then an affirmative answer has been given by Scheuneman [24].

Another interesting class of nilpotent examples has been studied by Auslander [3]. Let A be an associative algebra, finite-dimensional over $\mathbf{R}$, which is nilpotent. Introducing the new associative product operation

$$a, b \mapsto a + b + ab,$$

we make the underlying flat manifold A into a Lie group, denoted by A^*. Note that the given complete affinely flat structure on A^* is invariant under both left and right translation. (In fact, this is the most general example of a simply connected Lie group with such a left and right invariant complete flat structure.)

Here is an explicit case. Suppose that A is commutative, with basis $a, a^2, \ldots, a^k$ where $a^{k+1} = 0$. Then the Lie group A^* is also commutative. Choosing any lattice $\Gamma \simeq \mathbf{Z}^k$ in A^*, we can form the quotient k-torus $\Gamma\backslash A^*$, which is provided with an exotic complete affinely flat structure. (Compare [28].)

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