The complex Monge-Ampère equation and pluripotential theory

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Abstract. We collect here results on the existence and stability of weak solutions of complex Monge-Ampère equation proved by applying pluripotential theory methods and obtained in past three decades. First we set the stage introducing basic concepts and theorems of pluripotential theory. Then the Dirichlet problem for the complex Monge-Ampère equation is studied. The main goal is to give possibly detailed description of the nonnegative Borel measures which on the right hand side of the equation give rise to plurisubharmonic solutions satisfying additional requirements such as continuity, boundedness or some weaker ones. In the last part the methods of pluripotential theory are implemented to prove the existence and stability of weak solutions of the complex Monge-Ampère equation on compact Kähler manifolds. This is a generalization of the Calabi-Yau theorem.

Contents

Introduction

In this paper we survey the existence theorems for the complex Monge-Ampère equation which are proved by pluripotential theory methods. In mid-seventies Bedford and Taylor [**BT1**] found plurisubharmonic solutions of the Dirichlet problem for the complex Monge-Ampère equation with continuous data in a strictly pseudoconvex domain. In their subsequent fundamental paper [**BT2**] they developed pluripotential theory in which the Monge-Ampère operator plays a crucial role in establishing many important properties of plurisubharmonic functions. Since the Monge-Ampère equation is fully nonlinear many problems in pluripotential theory are more difficult than their counterparts in classical potential theory where we have nice Poisson's equation to play with. Those difficulties can be often overcome if we apply methods exploiting the basic fact that for a plurisubharmonic function u the form $dd^c u$ (understood in the sense of distributions) is nonnegative. In recent years the Dirichlet problem for the complex Monge-Ampère equation

$$(dd^c u)^n = d\mu, \quad u = \varphi \quad \text{on the boundary,}$$

has been solved for a wide variety of measures. We can now give fairly sharp conditions under which a measure yields a continuous solution as well as characterize those measures which lead to solutions in some larger classes of plurisubharmonic functions.

The complex Monge-Ampère equation is also investigated in connection with the geometry of Kähler manifolds. Here the solution of the equation yields a Kähler metric with prescribed Ricci curvature. In seventies Yau [**Y**] solved the Monge-Ampère equation on compact Kähler manifolds, for smooth, non degenerate data, confirming a famous conjecture of Calabi. In the proof he employs the methods of elliptic PDE: the continuity method coupled with a priori estimates for the derivatives of the solution. In a similar fashion the equation can be studied in strictly pseudoconvex domains as it was first done by Caffarelli, Kohn, Nirenberg and Spruck [**CKNS**]. Then apart from existence we obtain regularity of solutions under suitable assumptions. More about this approach can be found in [**A2**] or [**TI**]. Using the methods described in the present paper one can generalize Yau's theorem by admitting non smooth, degenerate data.

We shall present those results with the necessary background. The paper is organized as follows. We first review, following Lelong [**L**], the basic properties of positive currents. Then the currents associated to plurisubharmonic functions are introduced. The results, for the most part coming from the paper by Bedford and Taylor [**BT2**], include: Chern-Levine-Nirenberg inequalities, convergence theorems, the comparison principle, Josefson's theorem and the theorem on negligible sets. Some relations between the relative and global extremal functions are studied in the next chapter. In particular the Alexander-Taylor inequalities are important for the sequel. The remaining part deals exclusively with solving the Monge-Ampère equation. We start with the result of Bedford and Taylor [**BT1**] who solved the Dirichlet problem with continuous data, and then present its generalizations due to the author (Chapter 4) and Cegrell (Chapter 5). In the last chapter we generalize Yau's theorem showing, in particular, the existence of the solutions for the right hand side belonging to $L^p, p > 1$. The unified approach allows us to simplify many original proofs. Main references are given at the end of each section.

We refer to books by Hörmander [**H2**] and by Klimek [**KL**] for background material on plurisubharmonic functions. There is a good deal of high quality literature on pluripotential theory, besides Klimek's book there is Cegrell's monograph [**C1**] and excellent surveys by Bedford [**B**] and Kiselman [**KI3**]. There are also unpublished lecture notes by Demailly [**D1**] and Błocki [**BL**].

I lectured on the subject at the Jagiellonian University in the period 1999-2001, and also at the summer school in pluripotential theory at TUBITAK, Istanbul, 1999 (first part) and at NCTS, Hsinchu, Taiwan in October 1999 (Chapters 4,6). I would like to thank all the institutions for giving the opportunity for lecturing and many interesting discussions. In particular I thank A. Aytuna, Z. Błocki, P. Guan, C. S. Lin, A. Rashkovski, V. Zahariuta, A. Zeriahi, and students of Jagiellonian University who attended the courses, for their critical comments.

CHAPTER 1

Positive Currents and Plurisubharmonic Functions

Positive forms

We begin with the study of the basic properties of positive forms. Let us denote by $C^\infty_{(p,p)}(\Omega)$ the set of all smooth differential forms of bidegree (p,p) defined in an open set $\Omega \subset \mathbb{C}^n$. Using conventional notation, any form ω from $C^\infty_{(p,p)}(\Omega)$ is given by

$$\omega = i^p \sum_{|J|=p,|K|=p}{}' \omega_{JK} dz_J \wedge d\bar{z}_K,$$

where ω_{JK} are C^∞ functions in Ω, $dz_J = dz_{j_1} \wedge dz_{j_2} \wedge ... \wedge dz_{j_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge ... \wedge d\bar{z}_{j_p}$, and $\sum'$ indicates that we sum up over multi indices $J = (j_1, ..., j_p), K = (k_1, ..., k_p)$ such that $j_1 < j_2 < ... < j_p;\ k_1 < k_2 < ... < k_p$. We call ω *Hermitian* if $\omega = \overline{\omega}$.

When $\omega \in C^\infty_{(p,p)}(\Omega)$ has a representation

$$\omega = i^p \omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2 \wedge ... \wedge \omega_p \wedge \bar{\omega}_p$$

where $\omega_j \in C^\infty_{(1,0)}(\Omega)$, it is said to be a *simple positive form.*

Proposition 1.1. *The space of (p,p) forms with constant coefficients is spanned by simple positive forms.*

Proof. It is enough to represent $dz_j \wedge d\bar{z}_k$ as a linear combination of simple positive forms, and in fact

$$dz_j \wedge d\bar{z}_k = \frac{1}{4} \sum_{s=1}^{4} i^s (dz_j + i^s dz_k) \wedge \overline{(dz_j + i^s dz_k)}.$$

Proposition 1.2. *The pull-back $f^*\omega$ of a simple positive form ω via a holomorphic mapping f is again simple positive.*

Proof. Let $f : \Omega \to \Omega'$ be a holomorphic mapping and let $\alpha = \sum a_j dz_j$ be $(1,0)$ form on Ω'. Then

$$f^*\alpha = \sum a_j df_j = \sum_k (\sum_j a_j \frac{\partial f_j}{\partial w_k}) dw_k$$

and

$$f^*\bar{\alpha} = \sum \bar{a}_j d\bar{f}_j = \sum_k (\sum_j \bar{a}_j \overline{(\frac{\partial f_j}{\partial w_k})}) d\bar{w}_k.$$

Hence
$$f^*(\alpha\wedge\bar\alpha) = f^*\alpha\wedge\overline{(f^*\alpha)},$$
from which the proposition easily follows.

We shall often use the canonical $(1,1)$ form on $\mathbb{C}^n$:
$$\beta = \frac{i}{2}\partial\bar\partial|z|^2 = \frac{i}{2}\sum_1^n dz_j\wedge d\bar z_j.$$
Then $V_n = \frac{1}{n!}\beta^n$ is the volume form in $\mathbb{C}^n$.

Definition. A (p,p) form ω is said to be *positive* if
$$\omega\wedge\alpha = f\beta^n \quad \text{with } f\geq 0,$$
for any simple positive form α of bidegree $(n-p,n-p)$.

Remark. It is enough to verify the above defining condition for simple positive forms with constant coefficients.

Proposition 1.3. *1) A pull-back of a positive form via a biholomorphic mapping is positive.*

2) A (p,p) form is positive if and only if its restriction to any complex analytic submanifold of dimension p (equivalently: any analytic subspace of dimension p) is equal to the volume form of the submanifold multiplied by a nonnegative function.

Proof. 1) Let $f:\Omega\to\Omega'$ be a biholomorphic mapping and let ω be a positive form in Ω'. For a simple positive form $\alpha\in C^\infty_{(p,p)}(\Omega)$ its pull-back $(f^{-1})^*\alpha$ is also a simple positive form. Thus for some nonnegative function g
$$f^*\omega\wedge\alpha = f^*(\omega\wedge(f^{-1})^*\alpha) = f^*(g\beta^n) = g|det\ f'|^2\beta^n.$$
This proves our first claim.

2) Having 1) we may reduce the verification of the defining conditions to the case of simple positive form
$$\alpha_0 = i^{n-p}dz_{p+1}\wedge d\bar z_{p+1}\wedge\ldots\wedge dz_n\wedge d\bar z_n$$
and the subspace $A_0=\{z: z_{p+1}=\ldots=z_n=0\}$. But if the restriction to A_0 of a (p,p) form ω is equal to
$$i^p g dz_1\wedge d\bar z_1\wedge\ldots\wedge dz_p\wedge d\bar z_p = 2^p g V_p$$
then
$$\omega\wedge\alpha_0 = 2^n g V_n.$$

Proposition 1.4. *1) A $(1,1)$ form $\alpha=\frac{i}{2}\sum\alpha_{jk}dz_j\wedge d\bar z_k$ is positive iff (α_{jk}) is a positive (semidefinite) Hermitian matrix.*

2) If, moreover, ω is a positive (p,p) form then so is $\alpha\wedge\omega$.

PROOF. 1) Let us first observe that if α is positive then it is Hermitian. Indeed, for any $(n-1, n-1)$ simple positive form γ we have

$$\alpha \wedge \gamma = \overline{\alpha \wedge \gamma} = \overline{\alpha} \wedge \gamma$$

By Proposition 1.1 the same is true for any $(n-1, n-1)$ form. Therefore $\alpha = \bar{\alpha}$.

If we consider a parameterization of a complex line

$$L : \lambda \to (\lambda w_1, \lambda w_2, ..., \lambda w_n)$$

then

$$L^* \alpha = \frac{i}{2} \sum \alpha_{jk} w_j \bar{w}_k d\lambda \wedge d\bar{\lambda}.$$

Using the preceding proposition we get the desired equivalence as w varies.

2) One can apply a unitary change of coordinates to diagonalize the matrix (α_{jk}) at a given point z_0 so that

$$\alpha(z_0) = i \sum \alpha_{jj} dz_j \wedge d\bar{z}_j, \quad \alpha_{jj} \geq 0.$$

Then for any simple positive form γ

$$\alpha \wedge \omega \wedge \gamma = \sum \alpha_{jj} \omega \wedge (i dz_j \wedge d\bar{z}_j \wedge \gamma)$$

and since the forms in brackets are simple positive the right hand side is nonnegative as a sum of nonnegative terms.

Currents

Since plurisubharmonic functions are not smooth in general we need to study also forms with distribution coefficients which are called currents. Most interesting for us will be positive currents. Let $\mathcal{D}_{(p,q)}(\Omega)$ denote the space of test forms in Ω of bidegree (p, q) equipped with Schwartz' topology.

DEFINITION. Any continuous linear functional on the space $\mathcal{D}_{(p,q)}(\Omega)$ is called a *current* of bidegree $(n-p, n-q)$ (equivalently: of bidimension (p, q)) in Ω. The collection of such currents will be denoted by $\mathcal{D}'_{(p,q)}(\Omega)$.

When for $T \in \mathcal{D}'_{(p,p)}(\Omega)$ we have

$$(T, \omega) \geq 0$$

for any simple positive test form ω we say that T is a *positive current.*

For an increasingly ordered multi index J we denote by J' the unique increasing multi index such that $J \cup J' = \{1, 2, ..., n\}$ and $|J| + |J'| = n$. Let us denote by α_{JK} the form complementary to $dz_J \wedge d\bar{z}_K$, that is

$$\alpha_{JK} = \lambda dz_{J'} \wedge d\bar{z}_{K'},$$

where λ is chosen so that $dz_J \wedge d\bar{z}_K \wedge \alpha_{JK} = V_n$.

Let us observe that one can identify a current $T \in \mathcal{D}'_{(p,q)}(\Omega)$ with a differential form which has distribution coefficients

$$T = \sum_{|J|=n-p, |K|=n-q}{}' T_{JK} dz_J \wedge d\bar{z}_K.$$

The coefficients T_{JK} are defined by

$$(T_{JK}, \phi) = (T, \phi\alpha_{JK}).$$

As it was in the case of differential forms the positivity of the current is not affected by a biholomorphic change of coordinates. Let $f : \Omega \to \Omega'$ be a biholomorphic mapping and let T' be a positive current in Ω'. Then the pull-back $T = f^*T'$ of T' via f defined by

$$(T, \omega) = (T', (f^{-1})^*\omega)$$

is again positive. Given $T \in \mathcal{D}'_{(p,p)}(\Omega)$ we set

$$(f_*T, \omega') = (T, f^*\omega')$$

and call f_*T the *direct image* of T. Then for positive T its direct image f_*T is positive as well. The above statements follow directly from the fact that related pull-backs of simple positive forms are simple positive.

One may also define a wedge product of a current T and a smooth form ω setting

$$(T \wedge \omega, \phi) := (T, \omega \wedge \phi)$$

for any test form ϕ. If T is positive and ω is a positive $(1,1)$ form then $T \wedge \omega$ is again positive. In particular, for a positive (p,p) current T and a $(n-p, n-p)$ simple positive form ω the current $T \wedge \omega$ is a nonnegative Radon measure.

We differentiate currents according to the formula

$$(DT, \phi) = -(T, D\phi)$$

for a first order differential operator D. We shall often use the operator $d^c := i(\overline{\partial} - \partial)$.

PROPOSITION 1.5. *The action of a positive current can be continuously extended to the space of compactly supported forms with continuous coefficients.*

PROOF. We are to show that if

$$T = \sum_{|J|=p, |K|=p}{}' T_{JK} dz_J \wedge d\bar{z}_K$$

then all T_{JK} are Radon measures. Let us represent α_{JK} introduced above in a basis (ω_j) consisting of simple positive forms with constant coefficients (see Proposition 1.1)

$$\alpha_{JK} = \sum_s c_{sJK}\omega_s.$$

Then for any test function g we have

$$(T_{JK}, g) = (T, g\alpha_{JK}) = \sum c_{sJK}(T, g\omega_s) = \sum c_{sJK}(T \wedge \omega_s, g).$$

Thus T_{JK} is a linear combination of nonnegative Radon measures.

For a current T with measure coefficients one can define a norm

$$||T||_E = \sum_{|J|=p, |K|=q}{}' |T_{JK}|_E,$$

where $|T_{JK}|_E$ is the total variation of T_{JK} on a compact set E.

For two (p,p) currents S,T the inequality

$$S \leq T$$

means that $T - S$ is a positive current.

PROPOSITION 1.6. *There exists a constant C depending only on the dimension of the space such that*

$$||T||_E \leq C \int_E T \wedge \beta^{n-p}$$

for positive $T \in \mathcal{D}'_{(n-p,n-p)}(\Omega)$.

PROOF. In the preceding proof we got the representation

$$T_{JK} = \sum c_{sJK} T \wedge \omega_s,$$

where ω_s are simple positive forms with constant coefficients and c_{sJK} depend only on n. Since $\omega_s' s$ are wedge products of $(1,1)$ forms we have reduced the proof to an obvious estimate: given $(1,1)$ form ω with constant coefficients one can find C_1 such that

$$\omega \leq C_1 \beta.$$

It is often convenient to work with smooth forms and then prove statements about currents by using an approximation of a given current by smooth forms. To do this one can apply the standard regularization by means of the convolution with a smoothing kernel to each coefficient T_{JK} of the current T.

Given a nonnegative, rotation invariant function $\rho \in C_0^\infty(B)$ (B stands for the unit ball in $\mathbb{C}^n$), where $\int \rho dV = 1$, define a regularizing sequence $(T_j)_{I,J} = T_{I,J} * \rho_j$. with $\rho_j(z) := j^{2n}\rho(jz)$. Then $T_j \to T$ in the sense of currents which, by definition, means that for any test form ω the sequence (T_j, ω) converges to (T, ω).

Unless otherwise stated the term *convergence* applied to a sequence of currents shall have the above meaning.

CURRENTS ASSOCIATED TO PLURISUBHARMONIC FUNCTIONS

By $PSH(\Omega)$ we denote the set of plurisubharmonic (psh in short) functions in Ω. If $u \in PSH(\Omega)$ then $dd^c u$ is a closed positive (1,1) current. Conversely, if T is a positive closed current of bidegree (1,1) defined in a neighbourhood of a closed ball then there exists a psh function inside the ball such that $dd^c u = T$ (see e.g. [**LG**].

We can define wedge products of currents $dd^c u$ provided that the associated psh functions are locally bounded. Indeed, the following statement is true

PROPOSITION 1.7. *For $u \in PSH \cap L^\infty_{loc}(\Omega)$ and a closed positive current T on Ω the current uT is well defined and so is*

$$dd^c u \wedge T := dd^c(uT).$$

Moreover, the latter current is also closed and positive.

PROOF. The statement is local, so one can use the standard regularization of u by a decreasing sequence of smooth functions u_j which are uniformly bounded. Since we know that distribution coefficients of T are complex measures it follows from Lebesgue's dominated convergence theorem that u_jT converges weakly to uT. Hence $dd^c(u_jT) \to dd^c(uT)$. Functions u_j being smooth we have $dd^c(u_jT) = dd^cu_j \wedge T$ and thus $dd^cu \wedge T$ is equal to the limit of positive closed currents $dd^cu_j \wedge T$ which proves the proposition.

This way, using induction, one may define closed positive currents

$$dd^cu_1 \wedge dd^cu_2 \wedge ... \wedge dd^cu_N,$$

for $u_j \in PSH \cap L^\infty_{loc}(\Omega)$. It is also possible to define

$$du \wedge d^cu \wedge T$$

if u is locally bounded psh function and T a closed positive current. For this we can assume that $u \geq 0$ (therefore u^2 is psh) and use the identity

$$du \wedge d^cu \wedge T = (1/2)dd^cu^2 \wedge T - udd^cu \wedge T$$

in which the right hand side is well defined by the above proposition. If moreover T is of bidegree $(n-1, n-1)$ and v is another locally bounded psh function then

$$du \wedge d^cv \wedge T = dv \wedge d^cu \wedge T$$

are well defined and by definition equal to

$$(1/2)[d(u+v) \wedge d^c(u+v) \wedge T - du \wedge d^cu \wedge T - dv \wedge d^cv \wedge T]$$

This follows from Proposition 1.8 below.

The Monge-Ampère operator $\mathcal{M}$ acts on a C^2 smooth psh function u according to the following formula

$$\mathcal{M}(u) := 4^n n! det(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})\, dV_n = (dd^cu)^n,$$

where the power on the right is taken with respect to the wedge product.

A TOOLKIT FOR THE WORK WITH CURRENTS

Here we gather facts which will be frequently used in the sequel.

STOKES' THEOREM. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^1 boundary and let T be a current of degree $2n-1$ defined in a neighbourhood of $\bar{\Omega}$ and such that T is C^1 smooth in a neighbourhood of $\partial\Omega$. Then*

$$\int_{\partial\Omega} T = \int_\Omega dT.$$

PROOF. Let us apply the standard regularization T_j of T. Fix a test function χ in Ω which is equal to 1 in a neighbourhood of the set where T is not smooth. Set

$$S_j = T(1-\chi) + \chi T_j.$$

Thus $S_j = T$ in a neighbourhood of $\partial\Omega$ and one can apply classical Stokes' theorem to S_j getting

$$\int_{\partial\Omega} T = \int_{\partial\Omega} S_j = \int_{\Omega} dS_j \to \int_{\Omega} dT.$$

PROPOSITION 1.8. *If T is a closed positive current in Ω of bidegree $(n-1,n-1)$ and u,v are locally bounded psh functions then*

$$du \wedge d^c v \wedge T = dv \wedge d^c u \wedge T$$

PROOF. For smooth functions u and v the identity follows from the fact that the parts of bidegree $(1,1)$ of $du \wedge d^c v$ and $dv \wedge d^c u$ are both equal to $i\partial u \wedge \bar\partial v + i\partial v \wedge \bar\partial u$. The general case follows if we apply the standard regularization.

SCHWARZ' INEQUALITY. *If T is a positive current in Ω of bidegree $(n-1,n-1)$ and u,v are linear combinations of locally bounded psh functions then*

$$\int_{\Omega} du \wedge d^c v \wedge T \leq \left(\int_{\Omega} du \wedge d^c u \wedge T\right)^{1/2}\left(\int_{\Omega} dv \wedge d^c v \wedge T\right)^{1/2}$$

PROOF. It is enough to observe that the form

$$(u,u) = \int_{\Omega} du \wedge d^c u \wedge T$$

is positive definite since $du \wedge d^c u = 2i\partial u \wedge \bar\partial u$ is simple positive.

We shall often use in the proofs the following way of reducing the proof to that of a simpler case.

LOCALIZATION PRINCIPLE. *If we are to prove the weak convergence or local estimate for a family of locally uniformly bounded plurisubharmonic functions it is no loss of generality if we assume that the functions are defined in a ball and are all equal on some neighbourhood of the boundary.*

PROOF. Given a compact set K we cover it by balls $B(a_j,r)$. Fix one of them and consider the restrictions u_s of functions from our family to the ball $B = B(a_j,tr),\ t>1$, which is contained in the domain we start with. Since u_s are uniformly bounded we can assume $u_s < 0$ and find an exhaustion plurisubharmonic function h for B which is smaller than any u_s on $B(a_j,r)$. (Note that if h is an exhaustion function so is Mh for a positive constant M). To verify the desired estimates we now can work with $h_s = \max(u_s,h)$ which are equal to u_s on $B(a_j,r)$ and equal to h on some neighbourhood of the boundary of B.

Chern-Levine-Nirenberg (CLN) inequalities. *If $K \subset\subset U \subset\subset \Omega$ then for a constant $C = C(K,U,\Omega)$ the following inequality holds*

$$||dd^c u_0 \wedge dd^c u_1 \wedge ... \wedge dd^c u_k \wedge T||_K \le C||u_0||_U||u_1||_U...||u_k||_U||T||_U,$$

for any closed positive T and any set of $u_j \in PSH \cap L^\infty(\Omega)$. Moreover

$$||dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k||_K \le C(K,\Omega)||u_1||_{L^1(\Omega)}||u_2||_\Omega...||u_k||_\Omega,$$

and

$$||u_0 \wedge dd^c u_1 \wedge ... \wedge dd^c u_k||_K \le C(K,\Omega)||u_0||_{L^1(\Omega)}||u_1||_\Omega...||u_k||_\Omega.$$

Proof. Take a nonnegative test function ϕ in U which is equal to 1 on K and does not exceed 1 elsewhere. Applying Proposition 1.8 and (twice) the Stokes' theorem we get for a $(n-j-1, n-j-1)$ current T:

$$\begin{aligned}||dd^c u_0 \wedge T||_K &\le C_1 \int_U \phi dd^c u_0 \wedge T \wedge \beta^j = C_1 \int_U u_0 dd^c \phi \wedge T \wedge \beta^j \\ &\le C||u_0||_U||T||_U,\end{aligned}$$

where C depends on C_1 and the second order derivatives of ϕ. Iteration of this argument gives the first part of the statement. To obtain the second inequality we apply the localization principle and assume that $-1 \le u_j \le 0,\ j = 1,2,...,k$. Let us fix compact sets $K = K_0 \subset K_1 \subset ... \subset K_k \subset \Omega$ and smooth psh functions in $\Omega : h, h_1, ..., h_k$ such that $h - h_j \ge 1$ on K_{j-1} and $h = h_j$ on $\Omega \setminus K_j$. Then using Stokes' theorem and Proposition 1.8 one gets

$$\begin{aligned}&\int_K dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k \wedge \beta^{n-k} \\ &\le \int_{K_1} (h - h_1) dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k \wedge \beta^{n-k} \\ &= \int_{K_1} (-u_1) dd^c (h_1 - h) \wedge dd^c u_2 \wedge dd^c u_3 \wedge ... \wedge dd^c u_k \wedge \beta^{n-k} \\ &\le \int_{K_1} (-u_1) dd^c h_1 \wedge dd^c u_2 \wedge dd^c u_3 \wedge ... \wedge dd^c u_k \wedge \beta^{n-k} \\ &\le \int_{K_2} (h - h_2) dd^c h_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k \wedge \beta^{n-k} \\ &\quad \text{repeating the argument} \\ &\le \int (-u_k) dd^c h_1 \wedge dd^c h_2 \wedge ... \wedge dd^c h_k \wedge \beta^{n-k} \le C \int_\Omega (-u_k)\beta^n.\end{aligned}$$

In view of Proposition 1.6 this estimate gives the second assertion (if we interchange u_1 and u_k). To get the third one use the localization principle and then the integration by parts and iteration as above give:

$$\int_K u_0 dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k \wedge \beta^{n-k} \le \int_\Omega u_0 (dd^c h)^k \wedge \beta^{n-k}.$$

The relative capacity and the convergence of currents

In pluripotential theory, as it is the case in classical potential theory, capacities play an important role. In particular they help to decide when the convergence of psh functions is "good" enough.

Definition.

$$cap(E,\Omega) = \sup\{\int_E (dd^c u)^n : u \in PSH(\Omega), -1 \le u < 0\}$$

is called relative capacity of the Borel set E (with respect to Ω).

We shall also consider set functions associated to closed positive $(n-k,n-k)$ currents T:

$$cap_T(E,\Omega) = \sup\{\int_E (dd^c u)^k \wedge T : u \in PSH(\Omega), -1 \le u < 0\}.$$

By CLN inequalities those quantities are finite. Moreover, $cap(E,\Omega) \ge C \int_E V_n$ with the constant C depending on the dimension of the space and diameter of Ω. Other easy properties are listed in the following proposition.

Proposition 1.9. *For Borel subsets E_j of bounded domain Ω we have*

1) $cap(E_1,\Omega) \le cap(E_2,\Omega)$ *if* $E_1 \subset E_2$,
2) $cap(E,\Omega) \ge \lim_{j\to\infty} cap(E_j,\Omega)$ *if the sequence is increasing to* E,
3) $cap(E,\Omega) \le \sum cap(E_j,\Omega)$ *for* $E = \cup E_j$.

In the next proposition we estimate the relative capacity of a sublevel set of a negative psh function.

Proposition 1.10. *Let $K \subset\subset U \subset\subset \Omega$. Then there exists a constant C depending on those sets such that for any $u \in PSH(\Omega)$, $u < 0$*

$$cap(K \cap \{u < -j\},\Omega) \le \frac{C}{j}||u||_{L^1(U)}.$$

The same inequality holds for cap_T with C depending also on T.

Proof. Fix $v \in PSH(\Omega)$ with $-1 \le v < 0$. Then by CLN inequalities

$$\int_{K\cap\{u<-j\}} (dd^c v)^n \le (1/j)\int_K |u|(dd^c v)^n \le \frac{C}{j}||u||_{L^1(U)}$$

which in view of the definition of the relative capacity proves the statement. The same argument works for cap_T.

Definition. A sequence u_j of functions defined in Ω is said to converge with respect to capacity to u if for any $t > 0$ and $K \subset\subset \Omega$

$$\lim_{j\to\infty} cap(K \cap \{|u-u_j| > t\},\Omega) = 0.$$

In the same way one defines convergence with respect to cap_T.

The Monge-Ampère operator is continuous with respect to sequences converging in this fashion.

Theorem 1.11 (Convergence theorem). *Let $\{u_k^j\}_{j=1}^{\infty}$ be a locally uniformly bounded sequence of psh functions in Ω for $k = 1, 2, ..., n$; and let $u_k^j \to u_k \in PSH \cap L^{\infty}_{loc}(\Omega)$ with respect to cap_β as $j \to \infty$ for $k = 1, 2, ..., n$. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_n^j \to dd^c u_1 \wedge ... \wedge dd^c u_n$$

in the weak topology of currents. If the sequences are convergent with respect to $cap_{T\wedge\beta}$ then

$$dd^c u_1^j \wedge ... \wedge dd^c u_N^j \wedge T \to dd^c u_1 \wedge ... \wedge dd^c u_N \wedge T$$

for positive $T \in \mathcal{D}'_{(n-N,n-N)}(\Omega)$.

Proof. We shall prove the first statement, the argument for the other one is analogous. Without loss of generality we assume that all psh functions involved take values between -1 and 0. Using the identity

$$\begin{aligned} &dd^c v_1 \wedge ... \wedge dd^c v_N - dd^c u_1 \wedge ... \wedge dd^c u_N \\ =&\sum_j dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_{j-1} \wedge dd^c (v_j - u_j) \wedge dd^c v_{j+1} \wedge ... \wedge dd^c v_N \end{aligned}$$

we reduce the proof to showing that if $u_j \to u$ with respect to cap_β and closed positive currents T_j have the representation $dd^c v_1^j \wedge ... \wedge dd^c v_{n-1}^j$, with $v_s^j \in PSH(\Omega)$, $-1 \leq v_s^j < 0$ then

$$dd^c(u_j - u) \wedge T_j \to 0.$$

Let us fix a test function ϕ in Ω with $supp\, \phi = K \subset\subset \Omega$. For $t > 0$ set $E_j(t) = K \cap \{|u_j - u| > t\}$. Note that for T_j represented as above we have

$$\int_E T_j \wedge \beta \leq \int_E (dd^c \sum v_s^j(z))^{n-1} \wedge \beta \leq (n-1)^{n-1} cap_\beta(E, \Omega)$$

This inequality coupled with Stokes' theorem give the estimate

$$\begin{aligned} &\int_K \phi dd^c(u_j - u) \wedge T_j \leq \int_{E_j(t)} (u_j - u) dd^c \phi \wedge T_j + t||dd^c \phi \wedge T_j||_K \\ \leq&||dd^c\phi||(n-1)^{n-1}(cap_\beta(E_j(t), \Omega) + t cap_\beta(K, \Omega)). \end{aligned}$$

One can make the right hand side arbitrarily small by fixing t small enough and then choosing j such that $cap_\beta(E_j(t), \Omega)$ is very close to 0 as well.

The conclusion of the theorem holds if convergence with respect to cap_β is replaced by the convergence with respect to cap since by definition $cap_\beta \leq n^n cap$. In particular, as the following proposition shows, for decreasing sequences of psh functions we get the convergence of corresponding currents.

Proposition 1.12. *A sequence $u_j \in PSH \cap L^{\infty}(\Omega)$ with $u_j \downarrow u$ in Ω converges to $u \in PSH \cap L^{\infty}(\Omega)$ with respect to capacity. So, for decreasing sequences the conclusion of Theorem 1.11 holds true.*

Proof. Our localization principle applies in this setting, so we can assume that Ω is a ball and u_j form a constant sequence in some fixed neighbourhood $\Omega \setminus E$ of $\partial\Omega$. One can also assume that $-1 < u_j < 0$ on E. We fix $v \in PSH(\Omega)$, $0 < v < 1$ and estimate

$$I_0(v) = I_0 = \int_E (u_j - u)(dd^c v)^n.$$

Note that the supremum over all v as above exceeds $t cap(\{u_j - u > t\} \cap E, \Omega)$. By Stokes' theorem and Schwarz' inequality

$$\begin{aligned} I_k :=& \int_E (u_j - u)(dd^c v)^{n-k} \wedge (dd^c u)^k \\ =& - \int_E d(u_j - u) \wedge d^c v \wedge (dd^c v)^{n-k-1} \wedge (dd^c u)^k \\ \leq& (\int_E d(u_j - u) \wedge d^c (u_j - u) \wedge (dd^c v)^{n-k-1} \wedge (dd^c u)^k)^{1/2} \\ \times& (\int_E dv \wedge d^c v \wedge (dd^c v)^{n-k-1} \wedge (dd^c u)^k)^{1/2} \end{aligned}$$

As for the last term let us observe that

$$dd^c (v+1)^2 \geq dv \wedge d^c v$$

(see the definition of the latter current). Hence as in the preceding proof we can estimate as follows

$$\int_E dv \wedge d^c v \wedge (dd^c v)^{n-k-1} \wedge (dd^c u)^k \leq C^2 = n^n cap(E, \Omega).$$

Furthermore

$$\begin{aligned} & \int_E d(u_j - u) \wedge d^c (u_j - u) \wedge (dd^c v)^{n-k-1} \wedge (dd^c u)^k \\ =& - \int_E (u_j - u) \wedge dd^c (u_j - u) \wedge (dd^c v)^{n-k-1} \wedge (dd^c u)^k \\ \leq& \int_E (u_j - u) \wedge (dd^c v)^{n-k-1} \wedge (dd^c u)^{k+1}. \end{aligned}$$

Thus we have proved

$$I_k \leq C I_{k+1}^{1/2}.$$

and therefore

$$I_0 \leq C' (\int_E (u_j - u)(dd^c u)^n)^{1/2^n} := \epsilon_j.$$

The sequence ϵ_j tends to 0 as j tends to ∞. Since

$$cap(\{u_j - u > t\} \cap E, \Omega) \leq \frac{\epsilon_j}{t}$$

our assertion is thus proved.

Corollary. *For $u_j \in PSH \cap L^\infty_{loc}(\Omega)$ the mapping*

$$(u_1, u_2, ..., u_k) \to dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k$$

is symmetric.

Proof. It is true for smooth functions, and any psh function is a limit of a decreasing sequence of smooth psh functions. Thus the symmetry follows from the convergence theorem.

Theorem 1.13. *For a psh function u defined in Ω and a positive number ϵ one can find an open set $U \Subset \Omega$ with $cap(U, \Omega) < \epsilon$ and such that u restricted to $\Omega \setminus U$ is continuous.*

Proof. Fix a compact set $K \subset \Omega$. By Proposition 1.10 one can find $M > 0$ such that the relative capacity of the set $U_1 = K \cap \{u < -M\}$ is less than $\epsilon/2$. Let us consider the standard regularizing sequence u_j decreasing to $\max(u, -M)$. As we know the sequence converges with respect to capacity. Thus for any integer $k > 1$ there exists $j(k)$ such that

$$cap(U_k, \Omega) < \epsilon 2^{-k},$$

where $U_k := K \cap \{u_{j(k)} > u + k^{-1}\}$. The sequence $u_{j(k)}$ is uniformly convergent to u on $K \setminus \cup U_k$ so u is continuous there. To get the statement it is now enough to take an exhaustive sequence of compact sets $K_j \uparrow \Omega$ and apply the first part of the proof to find $U_j \subset K_j$ with $cap(U_j, \Omega) < \epsilon 2^{-j}$ and the property that u restricted to $K_j \setminus U_j$ be continuous. Then u is continuous on the complement of U which is the union of U_j's. The subadditivity of *cap* and the estimates for the capacity of U_j give

$$cap(U, \Omega) < \epsilon.$$

The proof is completed.

Corollary 1.14. *Let $\mathcal{U}$ be a uniformly bounded family of psh functions in Ω. Suppose T_j, T are wedge products of currents $dd^c u$ with $u \in \mathcal{U}$, and assume $T_j \to T$. Then for any $u \in PSH(\Omega)$:*

$$uT_j \to uT.$$

Proof. For fixed $\epsilon > 0$ we can find a continuous function v such that the relative capacity of the set $\{u \neq v\}$ is less than ϵ. Then for any compact set K we have (see Proposition 1.10)

$$\max(||(u-v)T||_K, ||(u-v)T_j||_K) \leq const.(K)\epsilon.$$

For T_j having measure coefficients we get $vT_j \to vT$. To finish the proof it is now enough to combine those two facts with the triangle inequality.

Theorem 1.15 (Convergence theorem for increasing sequences). *Let $\{u_k^j\}_{j=1}^\infty$ be a locally uniformly bounded sequence of psh functions in Ω for $k = 1, 2, ..., N$; and let $u_k^j \uparrow u_k \in PSH \cap L^\infty_{loc}(\Omega)$ almost everywhere as $j \to \infty$ for $k = 1, 2, ..., N$. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_N^j \to dd^c u_1 \wedge ... \wedge dd^c u_N.$$

Proof. We shall use induction over N. Suppose that for $N < n$

$$T_j = dd^c u_1^j \wedge ... \wedge dd^c u_N^j \to dd^c u_1 \wedge ... \wedge dd^c u_N = T.$$

It is enough to show that for psh functions $v_j \uparrow v$ we have

$$v_j T_j \to vT,$$

since then by the Stokes theorem

$$dd^c v_j \wedge dd^c u_1^j \wedge ... \wedge dd^c u_N^j \to dd^c v \wedge dd^c u_1 \wedge ... \wedge dd^c u_N.$$

Applying the localization principle we assume that $\Omega = B$ and all involved psh functions are equal to $h \in PSH(\Omega)$ in a neighbourhood of ∂B. By Corollary 1.14 one obtains

$$\overline{\lim}\, v_j T_j \le \overline{\lim}\, v T_j = vT.$$

In view of this inequality we are done as soon as we prove that

$$\underline{\lim} \int_B v_j T_j \wedge \alpha \ge \int_B vT \wedge \alpha,$$

for any simple positive $(n-N, n-N)$ form α. The last inequality is obtained by making use of Corollary 1.14 and Stokes' theorem in the following way (with $T = dd^c u_1 \wedge S_1$)

$$\begin{aligned} &\underline{\lim} \int_B v_j T_j \wedge \alpha \ge \underline{\lim}_{j\to\infty} \int_B v_s T_j \wedge \alpha \\ &= \int_B v_s T \wedge \alpha = \int_B v_s dd^c u_1 \wedge S_1 \wedge \alpha \\ &= \int_B u_1 dd^c v_s \wedge S_1 \wedge \alpha \to \int_B u_1 dd^c v \wedge S_1 \wedge \alpha = \int_B vT \wedge \alpha \end{aligned}$$

where the convergence in the last line (with $s \to \infty$) follows from the induction hypothesis.

Comparison principle

The comparison principle is the most effective tool in pluripotential theory. It fully exploits the positivity of $dd^c u$ for psh u.

Theorem 1.16 (Comparison principle). *Let Ω be an open bounded subset of $\mathbb{C}^n$. For $u, v \in PSH \cap L^\infty(\Omega)$ satisfying $\underline{\lim}_{\zeta\to z}(u-v)(\zeta) \ge 0$ for any $z \in \partial\Omega$ we have*

$$\int_{\{u<v\}} (dd^c v)^n \le \int_{\{u<v\}} (dd^c u)^n.$$

PROOF. The proof is easy for $u, v \in C^\infty(\Omega)$ and $E = \{u < v\} \subset\subset \Omega$ having smooth boundary. In this case setting $v_k = \max(v, u+1/k)$ we obtain by the Stokes theorem

$$(1.1)\quad \int_E (dd^c v_k)^n = \int_{\partial E} d^c v_k \wedge (dd^c v_k)^{n-1} = \int_{\partial E} d^c u \wedge (dd^c u)^{n-1} = \int_E (dd^c u)^n$$

since $v_k = u + 1/k$ on neighbourhood of ∂E.

Furthermore, Proposition 1.12 applied to $v_k \downarrow v$ on (open) E gives for any compact $K \subset E$ and $\phi \in C_0^\infty(E), 0 \le \phi \le 1$ with $\phi = 1$ on K

$$\int_K (dd^c v)^n \le \int \phi (dd^c v)^n = \lim \int \phi (dd^c v_k)^n \le \underline{\lim} \int_E (dd^c v_k)^n.$$

Hence

$$\int_E (dd^c v)^n \le \underline{\lim} \int_E (dd^c v_k)^n.$$

This combined with (1.1) implies the statement.

For the general case suppose $||u||, ||v|| < 1$, fix $\epsilon > 0, \delta > 0$, and find an open set U such that $cap(U) < \epsilon, u = u_0, v = v_0$ on $\Omega \setminus U$ for some continuous u_0 and v_0. Let $v_k \downarrow v$ and $u_k \downarrow u$ be the standard regularization such that for $E_0(\delta) := \{u_0 < v_0 - \delta\}$ and $E_k(\delta) := \{u_k < v_k - \delta\}$ we have $E_0(2\delta) \setminus U \subset\subset \cap E_k(\delta) \setminus U$ and $\cup E_k(\delta) \setminus U \subset\subset E_0(0)$ (use uniform convergence). By Sard's theorem we can assume (changing δ if needed) that the boundary of $E_k(\delta)$ is smooth. Since for any $\delta \ge 0$ we have

$$E(\delta) \setminus U = E_0(\delta) \setminus U, \quad E(\delta) := \{u < v - \delta\},$$

we may apply the first part of the proof in the following way

$$\begin{aligned} &\int_{E(2\delta)\setminus U} (dd^c v)^n = \int_{E_0(2\delta)\setminus U} (dd^c v)^n \\ \le &\underline{\lim} \int_{E_k(\delta)\cup U} (dd^c v_k)^n \le \underline{\lim} \int_{E_k(\delta)} (dd^c v_k)^n + \epsilon \\ \le &\underline{\lim} \int_{E_k(\delta)} (dd^c u_k)^n + \epsilon \le \int_{E_0(0)} (dd^c u)^n + 2\epsilon \le \int_{E(0)\cup U} (dd^c u)^n + 2\epsilon. \end{aligned}$$

The statement follows if we let ϵ, δ to zero.

COROLLARY 1.17. *Under the assumptions of Theorem 1.16 the inequality* $(dd^c u)^n \le (dd^c v)^n$ *implies* $v \le u$.

If $(dd^c u)^n = (dd^c v)^n$ *and* $\lim_{\zeta \to z}(u - v)(\zeta) = 0$ *for* $z \in \partial\Omega$ *then* $u = v$.

PROOF. Suppose to the contrary that for $\epsilon > 0$ the set $E = \{u < v - \epsilon\}$ is nonempty and fix a negative strictly psh function ρ which is bigger than $-\epsilon$ in Ω. Then, using Theorem 1.16 we reach the contradiction with our assumptions since

$$\int_{\{u<v+\rho\}} (dd^c v)^n < \int_{\{u<v+\rho\}} (dd^c (v+\rho))^n \le \int_{\{u<v+\rho\}} (dd^c u)^n.$$

The second part follows directly from the first one.

Next we estimate the Monge-Ampère measure of the maximum of two psh functions.

THEOREM 1.18. *Let Ω be an open subset of $\mathbb{C}^n$. For $u, v \in PSH \cap L^\infty_{loc}(\Omega)$. Then*

$$(dd^c \max(u,v))^n \geq \chi_{\{u \geq v\}}(dd^c u)^n + \chi_{\{u<v\}}(dd^c v)^n,$$

where χ_E denotes the characteristic function of the set E.

PROOF. It is enough to show the estimate on any compact $K \subset \{u \geq v\}$. Suppose $||u|| < 1$, fix $\epsilon > 0$ and find open set U such that $cap(U) < \epsilon, u = u_0, v = v_0$ on $\Omega \setminus U$ for some continuous u_0 and v_0. For a sequence u_j decreasing to u and $V_t := \{v_0 < u_0 + t\}, t > 0$ we have $v < u_j + t$ on $V_t \setminus U$. Therefore, by the convergence theorem

$$\int_K (dd^c u)^n \leq \underline{\lim}_{j\to\infty} \int_{V_t \cup U} (dd^c u_j)^n \leq \underline{\lim}_{j\to\infty} \int_{V_t \setminus U} (dd^c u_j)^n + 2\epsilon$$

$$= \underline{\lim}_{j\to\infty} \int_{V_t \setminus U} (dd^c \max(u_j + t, v))^n + 2\epsilon \leq \int_{\bar{V}_t \setminus U} (dd^c \max(u + t, v))^n + 2\epsilon.$$

(Note that $\underline{\lim}\mu_j(K) \leq \mu(K)$ for compact K and μ_j weakly convergent to μ.) Since $\bar{V}_t \setminus U$ decreases to $\{u \geq v\} \setminus U$ as t goes to 0 our estimate follows after another application of the convergence theorem.

THE RELATIVE EXTREMAL FUNCTION

A domain is called *hyperconvex* if there exists nonzero $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$.

DEFINITION. *For a subset E of a domain $\Omega \subset \mathbb{C}^n$ we define the relative extremal function by the formula*

$$u_{E,\Omega} = u_E = \sup\{u \in PSH(\Omega) : u < 0, \text{ and } u \leq -1 \text{ on } E\}.$$

By the Choquet lemma (see e.g. [**D3**]) u_E is the limit of an increasing sequence of psh functions. Thus $u^*_E \in PSH(\Omega)$.

PROPOSITION 1.19.

ι) *If* $E_1 \subset E_2$ *then* $u_{E_2} \leq u_{E_1}$.

$\iota\iota$) *If* $E \subset \Omega_1 \subset \Omega_2$ *then* $u_{E,\Omega_2} \leq u_{E,\Omega_1}$.

$\iota\iota\iota$) *If* $K_j \downarrow K$, *with* K_j *compact in* Ω *then* $(\lim u^*_{K_j})^* = u^*_K$.

PROOF. The first two statements are obvious and so is the inequality " $\leq$ " in the last one. For the reverse inequality consider $u \in PSH(\Omega), u \leq 0$ with $u \leq -1$ on K. For $\epsilon > 0$ the open set $U_\epsilon = \{u < -1 + \epsilon\}$ contains K. Hence, for j large enough $K_j \subset U_\epsilon$ and therefore $u - \epsilon \leq u^*_{K_j}$. Taking supremum over all such functions u we get $u_K - \epsilon \leq \lim u^*_{K_j}$. Letting ϵ to 0 we obtain the conclusion.

The next result shows that for compact sets the supremum in the definition of the relative capacity is attained for $u = u^*_E$. The outer capacity cap^* is defined as follows

$$cap^*(E,\Omega) = \inf\{cap(U,\Omega), \; E \subset U, U \quad \text{open}\}.$$

THEOREM 1.20. *For a relatively compact set E in a hyperconvex domain Ω we have*

$$cap^*(E,\Omega) = \int_\Omega (dd^c u_E^*)^n.$$

If $E_j \downarrow E$ is a sequence of compact sets then

$$\lim_{j\to\infty} cap(E_j,\Omega) = cap(E,\Omega) = cap^*(E,\Omega).$$

PROOF. Applying the Choquet lemma one can find an increasing sequence $u_j \geq -1$ with $(\lim u_j)^* = u_E^*$. Using the solution to the Dirichlet problem for the Monge-Ampère equation (Theorem 3.6 below) we find v_j such that $u_j \leq v_j \leq u_E^*$ and $(dd^c v_j)^n = 0$ on a fixed ball $B(z,r) \subset \Omega \setminus \bar{E}$. Theorem 1.15, applied to the sequence v_j, implies that $(dd^c u_E^*)^n = 0$ on $B(z,r)$ and so on the whole set $\Omega \setminus \bar{E}$. Since $u_E^* = -1$ in $int\, E$ we conclude that $(dd^c u_E^*)^n$ is supported by ∂E.

Now, suppose $E = \bar{E}$ and fix an exhaustion psh function h for Ω with $h < -1$ on E. Then one can choose the sequence u_j above so that $h \leq u_j$. Take arbitrary $v \in PSH(\Omega), -1 \leq v < 0$ and for small $\epsilon > 0$ set

$$h_j = \max(u_j, (1-2\epsilon)v - \epsilon).$$

Observe that $h_j = (1-2\epsilon)v - \epsilon$ on E and $h_j = u_j$ in $\Omega \setminus \bar{\Omega}'$ where $\bar{\Omega}' \subset\subset \Omega$. Moreover $-1+\epsilon \leq h_j \leq 0$ and for ϵ small enough $E \subset \Omega'$. Those properties and the fact that Ω' can be chosen with smooth boundary allow to apply the Stokes theorem to obtain

$$\int_E (1-2\epsilon)^n (dd^c v)^n = \int_E (dd^c h_j)^n \leq \int_{\Omega'} (dd^c h_j)^n = \int_{\Omega'} (dd^c u_j)^n.$$

From Theorem 1.15 we thus infer

$$\int_E (1-2\epsilon)^n (dd^c v)^n \leq \overline{\lim} \int_{\Omega'} (dd^c u_j)^n \leq \int_{\bar{\Omega}'} (dd^c u_E^*)^n = \int_E (dd^c u_E^*)^n,$$

where the last equality follows from the first part of this proof. Hence

$$cap(E,\Omega) = \int_E (dd^c u_E^*)^n = \int_\Omega (dd^c u_E^*)^n. \tag{1.2}$$

By Theorem 1.15 and Proposition 1.19 we thus get

$$\lim cap(E_j,\Omega) = cap(E,\Omega) = cap^*(E,\Omega),$$

with the second equality justified by taking E_j with $E \subset intE_j$.

To get the first part of the statement for arbitrary E let us first note that for relatively compact, open V

$$cap(V,\Omega) = \int_\Omega (dd^c u_V^*)^n,$$

which follows from Theorem 1.11 applied to $u_{K_j}^*$ with K_j being an exhaustion sequence of compact sets for V. If now $E \subset V$ then by Theorem 1.16

$$\int_\Omega (dd^c u_E^*)^n \leq \int_\Omega (dd^c u_V^*)^n = cap(V,\Omega).$$

Hence

$$\int_\Omega (dd^c u_E^*)^n \leq cap^*(E,\Omega).$$

For the reverse inequality let us consider u_j and h chosen above. For $t_j \downarrow 1$ set $V_j = \{t_j u_j < -1\}$. Then V_j is decreasing, $E \subset V_j$ and $t_j u_j \leq u^*_{V_j}$. Hence $u^*_{V_j} \uparrow u^*_E$ almost everywhere and by Theorem 1.15

$$\int_\Omega (dd^c u^*_E)^n = \lim \int_\Omega (dd^c u^*_{V_j})^n.$$

Small sets

Definition. A set E in $\mathbb{C}^n$ is said to be pluripolar if for any $z \in E$ there exists a neighbourhood V of z and $v \in PSH(V)$ such that $E \cap V \subset \{v = -\infty\}$.

If $E \subset \{v = -\infty\}$ for $v \in PSH(\mathbb{C}^n)$ we call E *globally pluripolar.* However, this notion turns out to be redundant since Josefson's theorem proved below says that any pluripolar subset of $\mathbb{C}^n$ is globally pluripolar.

Definition. A subset E of an open set $\Omega \subset \mathbb{C}^n$ is called negligible if $E \subset \{u < u^*\}$, where $u = \sup u_s,\ u_s \in PSH(\Omega)$.

Here the family u_s can be chosen to be countable by Choquet's lemma. It is easy to see that if $E \subset \{v = -\infty\}$ for $v \in PSH(\Omega)$ then E is negligible since $E \subset \{u < u^*\}$ for $u = \sup_{j \in \mathbb{N}} v/j$. We shall see that the converse is also true and negligible sets are pluripolar.

Proposition 1.21. *In a hyperconvex domain Ω the following conditions are equivalent:*

1) $E \subset \{v = -\infty\}$ for $v \in PSH(\Omega), v < 0$.
2) $u^*_{E,\Omega} = 0$.
3) $cap^*(E, \Omega) = 0$.

Proof. (1) $\to$ 2)) If 1) holds then for any $\epsilon > 0$ we have $\epsilon v \leq u_E$. Thus $u_E = 0$ outside the set $\{v = -\infty\}$ which has empty interior. Therefore $u^*_E = 0$.

(2) $\to$ 1)) We can choose u_j as in the proof of Theorem 1.20 with additional property $\int_\Omega |u_j| dV_n < 2^{-j}$ (by the Lebesgue monotone convergence theorem). Then $v = \sum u_j$ is psh in Ω and equal $-\infty$ on E.

The last two statements are equivalent by Theorem 1.20 and Corollary 1.17.

One of the major results in pluripotential theory, widely used in polynomial approximation, complex dynamics and elsewhere, says the the negligible sets and the pluripolar sets are the same.

Theorem 1.22 (Bedford-Taylor). *Negligible sets are pluripolar.*

Proof. By the last proposition it is enough to show that a negligible set E satisfies $cap^*(E, \Omega) = 0$. Let u_j be the sequence from the definition of the negligible set and $u = \sup u_j$. Fix $\epsilon > 0$ and set $\Omega_\epsilon = \{z \in \Omega : dist(z, \partial\Omega) > \epsilon\}$. Using quasicontinuity of u_j choose an open set U such that $cap(U, \Omega) < \epsilon$ and all u_j are continuous on the complement of U in Ω. For rational numbers $s < t$ set

$$K_{st} = \{z \in \bar{\Omega}_\epsilon \setminus U : u \leq s < t \leq u^*\}.$$

Then $(\bar{\Omega}_\epsilon \cap E) \setminus U$ is represented as a (countable) union of such compact sets. Thus it is enough to show that $cap(K, \Omega) = 0$ for $K = K_{st}$. Reasoning by contradiction suppose that this equality is false and there exists $h \in PSH \cap C(\Omega)$ with $-1 \le h < 0$ and $\int_K (dd^c h)^n > 0$. Applying the localization principle one can assume that h is exhaustive and for any j $u_j = h$ outside a compact subset of Ω. Set $v_j = u_j + h$ and $v = u^* + h$. Then by Stokes' theorem and Proposition 1.8 we have

$$\int (-h)[(dd^c v_j)^n - (dd^c v)^n] = \int (v - v_j) dd^c h \wedge \sum_{k=0}^{n-1} (dd^c v_j)^k \wedge (dd^c v)^{n-k-1}$$

$$\ge \int_K (v - v_j)(dd^c h)^n \ge (t-s) \int_K (dd^c h)^n > const. > 0$$

contrary to Theorem 1.15 and Corollary 1.14.

Theorem 1.23 (Josefson). *For any pluripolar subset E of $\mathbb{C}^n$ there exists $h \in PSH(\mathbb{C}^n)$ with $E \in \{h = -\infty\}$.*

Proof. By definition $E = \cup E_j$ where $E_j \subset B(a_j, r_j)$ and for some $v_j \in PSH(B(a_j, r_j))$ we have $v_j = -\infty$ on E_j. Let us fix a sequence of positive integers $j(k)$ in which every integer is repeated infinitely many times and such that $B(a_{j(k)}, r_{j(k)}) \subset B(0, \exp(2^k)) = B_k$. By Propositions 1.19 and 1.21 $u^*_{E_{j(k)}, B_{k+1}} = 0$. Thus one can find $u_k \in PSH(B_{k+1})$ with $-1 \le u_k < 0; u_k = -1$ on $E_{j(k)}$ and $\int_{B_k} |u_k| dV_n < 2^{-k}$. Set

$$h_k(z) = \begin{cases} u_k(z) & \text{on } B_k \\ \max(u_k(z), 2^{-k} \log|z| - 2) & \text{on } B_{k+1} \setminus B_k \\ 2^{-k} \log|z| - 2 & \text{on } \mathbb{C}^n \setminus B_{k+1} \end{cases}$$

Then $h = \sum h_j \in PSH(\mathbb{C}^n)$ since on B_k the terms $h_j, j > k$ are negative and the series is convergent by the choice of u_k. Moreover, infinitely many terms of the series are equal to -1 on $E_{j(k)}$. Hence $E \in \{h = -\infty\}$.

Remark. Note that $h(z) < \log(1 + |z|)$.

Notes. The notion of the positive current was introduced by Lelong who proved most of the results of the first paragraph [**L**]. The main results on Monge-Ampère operator are due to Bedford and Taylor [**BT1**][**BT2**]. The exceptions are: Chern-Levine-Nirenberg inequalities [**CLN**], Josefson's theorem [**J**], Theorem 1.11 which is due to Xing [**X**] and Theorem 1.18 which is due to Demailly [**D1**]. The proof of Theorem 1.23, essentially following [**BT2**], was simplified by Demailly [**D1**]. The global defining function with logarithmic growth in this theorem was first found by El Mir [**EM**] and Siciak [**S2**](independently). The present construction is due to Błocki [**BL**]. Some proofs has been shortened (Theorems 1.11, 1.15, 1.22).

CHAPTER 2

Siciak's Extremal Function and a Related Capacity

In this chapter we shall deal with entire plurisubharmonic functions of logarithmic growth. The Siciak extremal function and a capacity T introduced below were originally defined by means of polynomials. Zahariuta showed that one can equivalently use the entire psh functions for this purpose. In the study of the Monge-Ampère equation an important role is played by inequalities between globally defined capacity T and the relative capacity.

Let us first define the Lelong class and its subset:

$$\mathcal{L} := \{u \in PSH(\mathbb{C}^n) : u(z) - \log(1 + |z|) < c_u\},$$

$$\mathcal{L}_+ := \{u \in PSH(\mathbb{C}^n) : |u(z) - \log(1 + |z|)| < c_u\}.$$

The Siciak extremal function associated to a bounded set E is given by the formula

$$L_E(z) = \sup\{u(z) : u \in \mathcal{L},\ u \leq 0 \text{ on } E\}.$$

The upper semicontinuous regularization L_E^* is a psh function.

THEOREM 2.1. *If E is pluripolar then $L_E^* = +\infty$, otherwise for bounded set E the function L_E^* belongs to $\mathcal{L}_+$.*

PROOF. According to the Remark following Theorem 1.23 for a pluripolar set E there exists $u \in \mathcal{L}$ equal $-\infty$ on the set. Then for any constant c we have $u + c \leq L_E$ which proves the first part if we let c to infinity.

Consider $u \in \mathcal{L}_+$. Then the function $u(z) - \log|z|$ restricted to an extended complex line (Riemann sphere) through zero is subharmonic away from a given disk centered at 0. Hence, by the maximum principle

$$\sup_{\mathbb{C}^n \setminus B(0,r)} (u(z) - \log|z|) = \sup_{\partial B(0,r)} (u(z) - \log|z|)$$

It follows from this that if $u \in \mathcal{L}_+$ and

$$f(t) = \max_{|w|=t} u(w)$$

then

$$f(s) - f(r) \leq \log s/r,\ r < s. \tag{2.1}$$

Suppose now that E is non pluripolar. Since, by Theorem 1.22 the set $L_E < L_E^*$ is pluripolar one can find a point, say 0 where L_E^* is finite. By upper semicontinuity L_E^* is upper bounded by some c in a ball $\bar{B}(0,r),\ r > 0$. Then (2.1) applied to $u = L_E^*$ shows that $L_E^* \leq \log(1 + |z|) + c$. On the other hand if $E \subset B(0,R)$ then obviously $L_E^*(z) \geq \log|z|/R$.

Proposition 2.2. *There exists a uniform constant C_n such that*

$$\sup_B L_E^* < \int L_E^* \, dS - C_n, \quad E \subset B = B(0,1),$$

where dS is the normalized Lebesgue measure on the sphere $S = \partial B$.

Proof. We assume, not violating the generality of the argument, that $f(1) = 0$, where f is the function introduced in the preceding proof and taken here for $u = L_E^*$. For $s < 1$ we apply (2.1) and the Harnack inequality to obtain

$$0 = f(1) \leq f(s) - \log s \leq c(s) \int L_E^* \, dS - \log s.$$

The estimate remains valid for any $u \in \mathcal{L}$.

Other basic properties of L_E^* are listed below.

Proposition 2.3.

ι) *If* $E_1 \subset E_2$ *then* $L_{E_2} \leq L_{E_1}$.

$\iota\iota$) $L_E^* = \lim_{j\to\infty} L_{E_j}^*$ *if the sequence E_j is increasing to E.*

$\iota\iota\iota$) *If* $K_j \downarrow K$, *with* K_j *compact then* $(\lim L_{K_j}^*)^* = L_K^*$.

The Monge-Ampère measures associated to the extremal functions u_E^* and L_E^* are equicontinuous.

Proposition 2.4. *Let E be a nonpluripolar compact set with $\widehat{E} \subset \Omega$ where Ω is hyperconvex and $\widehat{E}$ denotes the polynomial hull of E. Then*

$$(\sup_{\partial\Omega} L_E)^{-1} L_E \leq u_E + 1 \leq (\inf_{\partial\Omega} L_E)^{-1} L_E,$$

and

$$(\sup_{\partial\Omega} L_E)^{-n} (dd^c L_E^*)^n \leq (dd^c u_E^*)^n \leq (\inf_{\partial\Omega} L_E)^{-n} (dd^c L_E^*)^n.$$

Proof. The first part is easy. The measure $(dd^c L_E^*)^n$ vanishes outside E by the same proof as for $(dd^c u_E^*)^n$. Since the set $L_E < L_E^*$ is pluripolar both measures are supported on $\{L_E = 0\}$. The inequalities now follow from the first part of the proposition and Theorem 1.18. Indeed, observe that if $u = \max(u, v)$ in Ω and $u = v$ on E then by Theorem 1.18 $(dd^c u)^n \geq (dd^c v)^n$ on E.

Comparing the Monge-Ampère measures of the relative extremal functions with that of the global one we obtain the following corollary.

Corollary 2.5. *For hyperconvex $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2$ there exists $c_1 > 0$ such that for any compact $K \subset \Omega$*

$$c_1 cap(K, \Omega_1) \leq cap(K, \Omega_2) \leq cap(K, \Omega_1).$$

Proposition 2.6. *For $u \in \mathcal{L}_+$ we have $\int_{\mathbb{C}^n} (dd^c u)^n = (2\pi)^n$.*

PROOF. Take any two functions u, v from $\mathcal{L}_+$. Fix a compact set K and a number $t > 1$. Adding a constant to v one gets that $tv < u$ on K and the inequality holds on a bounded set in $\mathbb{C}^n$. The comparison principle implies

$$\int_K (dd^c u)^n \leq t^n \int_{\mathbb{C}^n} (dd^c v)^n.$$

Letting t to 1 and exchanging the roles of u and v:

$$\int_{\mathbb{C}^n} (dd^c u)^n = \int_{\mathbb{C}^n} (dd^c v)^n.$$

To complete the proof it now enough to compute $\int (dd^c \log \frac{1}{2} \log(1 + |z|^2))^n$, which can be verified by an elementary calculation.

By means of the Siciak extremal function we define the capacity

$$T_R(K) := exp(-\sup\{L_K^*(z) : |z| \leq R\}),$$

for some fixed $R > 0$. We shall write T for T_1. This capacity is comparable with the relative capacity in the following manner.

THEOREM 2.7. *If $B_R := B(0, R)$ and $K \subset B(0, r), r < R$ is compact, then*

$$exp(-A(r)(cap(K, B_R))^{-1}) \leq T_R(K) \leq exp(-2\pi(cap(K, B_R))^{-1/n}).$$

PROOF. By Proposition 1.21 and Theorem 2.1 both capacities are equal to 0 when K is pluripolar. Suppose now it is non pluripolar. For $C = \sup\{L_K^*(z) : z \in B_R\}$ we have $-1 \leq C^{-1}L_K^* - 1 < 0$ in B_R and by Proposition 2.4 and Proposition 2.6

$$C^{-n}(2\pi)^n = C^{-n} \int (dd^c L_K^*)^n \leq cap(K, B_R),$$

which proves the right hand side inequality. For the proof of the other one take $u = u_{K,\Omega}^*$ where Ω is the ball $B(0, eR)$. The function f from the proof of Theorem 2.1, with $u = L_K^*$, is bounded by $C + 1$ on $\partial\Omega$. Hence for any $v \in \mathcal{L}_+$ with $v < 0$ on K we get that $v_1 = (C + 1)^{-1}(v - C - 1)$ is less than 0 on Ω and less than -1 on K. Thus $v_1 \leq u$ and taking supremum over v

$$(C + 1)^{-1}(L_K^* - C - 1) \leq u.$$

At a point $z_0 \in \bar{B}(0, R)$ where L_E^* equals C we have $u(z_0) \geq -(C + 1)^{-1}$. Since u is subharmonic $-u(z_0) \geq C(R)||u||_{L^1(\Omega)}$. The last two inequalities combined with CLN inequalities lead to

$$cap(K, \Omega) = \int_\Omega (dd^c u)^n \leq C_0||u||_{L^1(\Omega)}||u||^{n-1} \leq C_1 C^{-1}.$$

The proof is completed by use of Corollary 2.5.

LEMMA 2.8. *For any $\alpha < 2$ there exists $c(\alpha, n)$ such that for all $u \in \alpha\mathcal{L}$ the following inequality holds with $B = B(0, 1)$*

$$\int_B \exp(\sup_B u - u)\, dV < c(\alpha, n).$$

Proof. One can assume that $\sup_B u = u(a) = 0$ for some $a \in \bar{B}$. Set for $k = 2, 3, \ldots$ $E_k = \{z \in B(0,2) : \log(k-1) < -u \le \log k\}$, $F_k = \cup_k^\infty E_j$. The function $v(z) = \frac{1}{\alpha}(u(z) + \log(k-1))$ belongs to $\mathcal{L}$ and $v \le 0$ on F_k, $v(a) = \frac{1}{\alpha}\log(k-1)$. Since v is dominated by the extremal function of F_k we conclude that for any complex line l with $a \in l$:

$$T(l \cap F_k) \le (k-1)^{-\frac{1}{\alpha}},$$

where T is considered as a capacity in the plane. By Proposition 2.2 the logarithmic capacity and T are equivalent for $n = 1$. Therefore one may infer from Theorem III 10 in [**TS**] that for some independent constant c_0

$$V_1(l \cap F_k) \le c_0 T^2(l \cap F_k) \le c_0 (k-1)^{-\frac{2}{\alpha}}.$$

Hence, via Fubini's theorem

$$V(F_k) \le c_1 (k-1)^{-\frac{2}{\alpha}}.$$

Using this inequality we shall estimate (with $B' = B(0,2)$)

$$\int_{B'} \exp(-u)\, dV = \sum_{k=2}^\infty \int_{E_k} \exp(-u)\, dV \le \sum_{k=2}^\infty kV(E_k)$$

$$= 2V(F_2) + \sum_{k=3}^\infty V(F_k) \le 2c_1 \sum_{k=1}^\infty k^{-\frac{2}{\alpha}} = c(\alpha, n).$$

Thus the lemma follows.

Notes. The extremal function was introduced by Siciak [**S1**], and the definition given here is due to Zahariuta [**Z**]. Theorem 2.1 was proved by Siciak [**S2**], Proposition 2.2 by Alexander [**A**], Proposition 2.4 by Levenberg [**LV**] (it is true for Borel sets, see [**BKL**]), Proposition 2.6 by Taylor [**T**]. The proof of Theorem 2.7 is due to Alexander and Taylor [**AT**], and its presentation simplified by Demailly [**D1**]. Lemma 2.8 is shown in Zeriahi's paper [**ZE**] by means of Skoda's integrability theorem [**SK**].

CHAPTER 3

The Dirichlet Problem for the Monge-Ampère Equation with Continuous Data

Throughout this chapter we shall work in a strictly pseudoconvex domain Ω. The goal is to find the solution to the following Dirichlet problem

$$(*)\qquad \begin{aligned} &u \in PSH \cap C(\overline{\Omega}) \\ &(dd^c u)^n = f\,dV \\ &\lim_{z' \to z} u(z') = \varphi(z) \quad z \in \partial\Omega,\ \varphi \in C(\partial\Omega), \end{aligned}$$

for any nonnegative f which is continuous in the closure of Ω. Such a solution is always unique by Corollary 1.17.

Let $\mathcal{C}$ denote the cone of $n \times n$ nonnegative Hermitian matrices and define on $\mathcal{C}$ a homogeneous superadditive functional

$$\mathcal{F}(A) = det^{1/n} A, \quad A \in \mathcal{C}.$$

We also consider the space $\mathcal{M}$ of $\mathcal{C}$- valued measures on Ω and set

$$\mathcal{F}\mu(E) = \inf \sum_j \mathcal{F}(\mu(E_j)),$$

where the infimum is taken over all partitions $\{E_j\}$ of E into a finite number of disjoint Borel sets. This construction is due to Goffmann and Serin [**GS**] who also proved the following properties (except the last one - shown in [**BT**]) of $\mathcal{F}\mu$.

Lemma 3.1.
a) $\mathcal{F}\mu$ *is a scalar measure.*
b) $\mathcal{F}(t\mu) = t\mathcal{F}\mu$ *for* $t > 0$ *and* $\mathcal{F}(\mu + \nu) \geq \mathcal{F}\mu + \mathcal{F}\nu$.
c) $\mathcal{F}(\mu + \nu) = \mathcal{F}\mu + \mathcal{F}\nu$ *if* μ *and* ν *are mutually singular.*
d) $|\mathcal{F}\mu - \mathcal{F}\nu| \leq |\mu - \nu|$ *where* $|\cdot|$ *denotes the total variation of the measure.*
e) If ν *is a nonnegative measure,* h - $\mathcal{C}$ *valued function and* $\mu(E) = \int_E h\,d\nu$ *then* $\mathcal{F}\mu(E) = \int_E \mathcal{F}(h)\,d\nu$.
f) If a sequence μ_j *of* $\mathcal{C}$*-valued measures tends weakly to* μ *then* $\mathcal{F}\mu \geq \lim \mathcal{F}\mu_j$.
g) If ρ *is a test function then* $\mathcal{F}(\mu * \rho) \geq \mathcal{F}\mu * \rho$.

Proof of $g)$. By Jensen's inequality

$$\begin{aligned} \mathcal{F}(\mu * \rho)(E_j) &= \mathcal{F}(\int \rho(z)\mu(E_j - \{z\})\,dV(z) \\ &\geq \int \rho(z)\mathcal{F}(\mu(E_j - \{z\}))\,dV(z). \end{aligned}$$

Summing over j

$$\sum_j \mathcal{F}(\mu*\rho)(E_j) \geq \int \rho(z)\sum_j \mathcal{F}(\mu(E_j-\{z\}))\,dV(z))$$
$$\geq \int \rho(z)\mathcal{F}\mu(E-\{z\})\,dV(z) = \mathcal{F}\mu*\rho(E).$$

For plurisubharmonic u define

$$\Phi(u) = 4(n!)^{1/n}\mathcal{F}(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}).$$

Note that for smooth u we have $(dd^c u)^n = \Phi^n(u)\,dV$. The operator Φ has the following properties.

Proposition 3.2.
a) $\Phi(tu) = t\Phi(u)$ *for* $t > 0$ *and* $\Phi(u+v) \geq \Phi(u)+\Phi(v)$.
b) If ρ *is a test function then* $\Phi(u*\rho) \geq \Phi(u)*\rho$.
c) If a sequence of plurisubharmonic functions u_j *tends weakly to* u *and* $\Phi(u_j)$ *is weakly convergent then* $\Phi(u) \geq \lim \Phi(u_j)$.
d) For the standard regularization $\lim \Phi(u_\epsilon) = \Phi(u)$.
e) $\Phi(\max(u,v)) \geq \min(\Phi(u),\Phi(v))$.

Proof. The first three assertions follow from Lemma 3.1. As for d) observe that by b) and c) we have

$$\Phi(u) \geq \lim \Phi(u_\epsilon) \geq \lim \Phi(u)*\rho_\epsilon = \Phi(u).$$

The last part of the statement is true for smooth u and v (see Theorem 1.18). In general, consider the standard regularization $u_j \downarrow u$ and $v_j \downarrow v$. Passing to a subsequence one may assume that $\Phi(u_j), \Phi(v_j), \Phi(\max(u_j,v_j))$ and $\min(\Phi(u_j),\Phi(v_j))$ are all weakly convergent. Then applying b) and c) we get

$$\Phi(\max(u,v)) \geq \lim_j \Phi(\max(u_j,v_j)) \geq \lim_j \min(\Phi(u_j),\Phi(v_j))$$
$$\geq \lim_j \min(\Phi(u)*\rho_j, \Phi(v)*\rho_j) \geq \lim_j \min(\Phi(u),\Phi(v))*\rho_j = \min(\Phi(u),\Phi(v)).$$

We now return to the Dirichlet problem $(*)$. Let us define the family of subsolutions:

$$\mathcal{S} = \{v \in PSH(\Omega)\cap C(\overline{\Omega}) : \Phi(v) \geq f^{\frac{1}{n}}\,dV,\ v_{|\partial\Omega} \leq \varphi\}$$

and its upper envelope

$$u = \sup_{\mathcal{S}} v.$$

This function will turn out to be the solution of the Dirichlet problem. Note that $\mathcal{S}$ is nonempty since if ρ is C^2 smooth, strictly plurisubharmonic, exhaustion function for Ω then for sufficiently big constants $A, B > 0$ $A\rho - B \in \mathcal{S}$. Furthermore for $u, v \in \mathcal{S}$ we have $\max(u,v) \in \mathcal{S}$ (see Theorem 1.18).

Proposition 3.3. *The upper envelope is continuous and belongs to* $\mathcal{S}$. *If moreover* $f^{\frac{1}{n}}$ *and* φ *are Lipschitz then so is* u.

PROOF. Suppose first that the boundary data φ is smooth and extend it to a smooth function in the closure of Ω. With ρ as above and A large enough we thus get $v_0 = A\rho + \varphi \in \mathcal{S}$ and $\Phi(v_0) \geq \max f^{\frac{1}{n}}\, dV$. Then for h harmonic in Ω and equal to φ on the boundary

$$v_0 \leq u \leq h,$$

which shows that u is continuous on $\partial\Omega$.

Fix $\epsilon > 0$ and a compact set $K \subset \Omega$. Take $z_0 \in K$. Find $v \in \mathcal{S}$ with $v(z_0) > u(z_0) - \epsilon$ and $v_0 \leq v$. To show the continuity of u on K we shall prove that for small $|a|$ the function $v(a + \cdot)$ modified close to the boundary also belongs to $\mathcal{S}$. One can find $\delta > 0$ such that for any $w \in \partial\Omega$

$$|h(z) - \varphi(w)| < \epsilon \text{ and } |v_0(z) - \varphi(w)| < \epsilon \text{ if } |z - w| < \delta.$$

(Note that if φ and v_0 are Lipschitz with constant M then $\delta = \epsilon/M$ is fine.) Hence

$$|v(z) - \varphi(w)| < \epsilon$$

for such z. Therefore, if $|a| < \delta$ and $z + a \in \partial\Omega$ then

$$v(z + a) - \epsilon < \varphi(z + a) < v(z) + \epsilon.$$

It now follows that

$$v_1(z) = \begin{cases} v(z) & \text{if } z + a \notin \overline{\Omega} \\ \max(v(z), v(z + a) - 2\epsilon) & \text{if } z + a \in \overline{\Omega} \end{cases}$$

is well defined and $v_1 = \varphi$ on $\partial\Omega$. Let ω denote the modulus of continuity of $f^{\frac{1}{n}}$. Since

$$\Phi(v(\cdot + a)) \geq f^{\frac{1}{n}}(\cdot + a)$$

we get, using Proposition 3.2 e) that

$$\Phi(v_1) \geq \min(f^{\frac{1}{n}}, f^{\frac{1}{n}}(a + \cdot)).$$

Therefore for $v_2 = v_1 + \omega(|a|)v_0$

$$\Phi(v_2) \geq \Phi(v_1) + \omega(|a|)\Phi(v_0) \geq f^{\frac{1}{n}}\, dV.$$

Thus $v_2 - \omega(|a|)||v_0|| \in \mathcal{S}$ and

$$\begin{aligned} u(z_0 - a) &\geq v_2(z_0 - a) - \omega(|a|)||v_0|| > v(z_0) - 2\epsilon - \omega(|a|)||v_0|| \\ &> v(z_0) - 3\epsilon > u(z_0) - 4\epsilon \end{aligned} \tag{3.1}$$

for $|a|$ small enough. Therefore u is continuous. Moreover, if $f^{\frac{1}{n}}$ is Lipschitz then $\omega(|a|)$ is proportional to $|a|$ and therefore δ can be chosen proportional to ϵ.

By the Choquet lemma there exist $u_j \in \mathcal{S}$ increasing (uniformly) to u. One can assume that $\Phi(u_j)$ is convergent and use Proposition 3.2 c) to conclude that $u \in \mathcal{S}$.

If φ is not smooth then we approximate it by a decreasing sequence of smooth φ_j and observe that the corresponding envelopes u_j are uniformly convergent. The limit function belongs to $\mathcal{S}$ by Proposition 3.2 c). It is the largest minorant of the sequence u_j and therefore it is the envelope we are looking for.

PROPOSITION 3.4. *The upper envelope has bounded second order derivatives under extra assumptions: Ω is equal to the unit ball B, $f^{\frac{1}{n}} \in C^{1,1}(\bar{B})$ and φ is $C^{1,1}$.*

PROOF. For the proof we shall estimate the expression

$$u(z+h)+u(z-h)-2u(z).$$

Since this expression is not defined in the whole B we shall first replace the translations by vectors h and $-h$ with automorphisms T_a and T_{-a}, where for given z we have $h = a - < z,a > z$. The mappings are defined as follows

$$T_a(z) = \frac{(P_a(z)-a)+\sqrt{1-|a|^2}(z-P_a(z))}{1-<z,a>}, \quad P_a(z) = \frac{<z,a>a}{|a|^2}$$

where $< \cdot,\cdot >$ denotes the Hermitian product in $\mathbb{C}^n$. Then, by computation

$$\begin{aligned} T_a(z) &= z-h+\psi(a,z)|a|^2, \\ \det(T_a'(z)) &= 1+2<z,a>+O(|a|^2), \end{aligned} \tag{3.2}$$

with some bounded smooth ψ and T_a' denoting the Jacobian of T_a. Hence for any $g \in C^{0,1}(\overline{B})$,

$$|g\circ T_a(z)-g(z-h)| \le c_1||g||_{C^{0,1}(B)}|a|^2, \tag{3.3}$$

Since, by (3.2)

$$(\det(T_a'(z)))^{2/n} = 1+\frac{4}{n}<z,a>+O(|a|^2)$$

and, by the assumptions and (3.3) ,

$$f^{\frac{1}{n}}\circ T_a(z) = f^{\frac{1}{n}}(z)+\psi_1(a,z)+O(|a|^2)$$

where (from the Taylor expansion) $\psi_1(-a,z) = -\psi_1(a,z)$, we may conclude that for a constant c_2 the following inequality holds

$$(\det T_a')^{\frac{2}{n}}(f^{\frac{1}{n}}\circ T_a)+(\det T_{-a}')^{\frac{2}{n}}(f^{\frac{1}{n}}\circ T_{-a}) \ge 2f^{\frac{1}{n}}-c_2|a|^2, \tag{3.4}$$

Furthermore, from the assumptions and (3.3) applied to $g=\varphi$ we have upon enlarging c_2

$$\varphi\circ T_a+\varphi\circ T_{-a} \le 2\varphi+c_2|a|^2. \tag{3.5}$$

Let us consider

$$v_a(z) = (u\circ T_a+u\circ T_{-a})(z).$$

By the chain rule $\Phi(u\circ T_a) = (\det T_a')^{\frac{2}{n}}(f^{\frac{1}{n}}\circ T_a)$. From this fact, Proposition 3.2 a) and (3.4) we get

$$\Phi(v_a) \ge (2f^{\frac{1}{n}}-c_3|a|^2)\,dV.$$

Hence (see (3.5)) there exists a constant c_4 such that

$$v(z) = \frac{1}{2}v_a(z)-c_2|a|^2(1+c_4(1-|z|^2)) \in \mathcal{S},$$

and therefore $v \le u$. Thus

$$2u(z) \ge (u\circ T_a+u\circ T_{-a})(z)-c_5|a|^2.$$

Applying (3.3) with $g=u$ (which is Lipschitz by Proposition 3.3) we obtain

$$2u(z) \ge u(z+h)+u(z-h)-c_6|a|^2.$$

One can regularize this inequality (on slightly smaller ball) to get the estimate

$$2u_\epsilon(z) \ge u_\epsilon(z+h)+u_\epsilon(z-h)-c_7|a|^2.$$

Now, fix ϵ and let a to 0 to conclude that the Hessians D^2u_ϵ are locally uniformly bounded from above (by $c(K)$ on a compact set K). Since u_ϵ is psh we also have $D^2u_\epsilon \cdot h^2 + D^2u_\epsilon \cdot (ih)^2 \geq 0$ and further

$$D^2u_\epsilon \cdot h^2 \geq -D^2u_\epsilon \cdot (ih)^2 \geq -c(K).$$

It follows that second order derivatives of u are locally bounded.

THEOREM 3.5. *Suppose* $0 \leq f^{\frac{1}{n}} \in C^{1,1}(\bar{B})$ *and* $\varphi \in C^{1,1}(\partial B)$. *Then the envelope* u *belongs to* $C^{1,1}(B)$ *and solves the Dirichlet problem* $(*)$ *in the unit ball.*

PROOF. The smoothness of u has been shown in Proposition 3.4. We also know that $\Phi(u) \geq f^{\frac{1}{n}}\, dV$. For $u \in C^{1,1}$ the density of $\Phi(u)$ is equal to

$$4(n!)^{1/n}(\det(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}))^{1/n}$$

at any point where the second order derivatives exist (that is almost everywhere). Arguing by contradiction, suppose that we have the strict inequality at a point z_0 where second order derivatives are defined.

The Taylor expansion of u at z_0 has the form

$$u(z_0 + h) = u(z_0) + \Re P(h) + H(h) + o(|h|^2),$$

where P is a complex polynomial (so $\Re P$ is pluriharmonic) and

$$H(h) = \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} h_j \bar{h}_k.$$

Since H is strictly positive definite we have for $t < 1$ close enough to 1, and some positive r and δ that

$$u(z_0) + (\Re P + tH)(h) < u(z_0 + h) - \delta, \quad |h| = r,$$

and the function

$$v(z) = \begin{cases} u(z) & \text{if } z \notin B(z_0, r) \\ \max(u(z), u(z_0) + (\Re P + tH)(z - z_0) + \delta) & \text{if } z \in B(z_0, r) \end{cases}$$

belongs to $\mathcal{S}$. Then we reach contradiction $u(z_0) \leq v(z_0) - \delta$. The proof is complete.

THEOREM 3.6. *The upper envelope* u *solves the Dirichlet problem* $(*)$ *in any strictly pseudoconvex domain.*

PROOF. In the case $\Omega = B$ we approximate f and φ uniformly by smooth functions f_j and φ_j respectively. Applying Theorem 3.5 we obtain solutions u_j of $(*)$ corresponding to the set of data f_j, φ_j. It easily follows from the comparison principle that $u_j \to u$ uniformly in $\bar{B}$ and so $(dd^c u_j)^n \to (dd^c u)^n$ by the convergence theorem. Thus u solves $(*)$. For general Ω it remains to prove that $(dd^c u)^n = f\, dV$ (see Proposition 3.3). Fix a ball $B_0 \subset \Omega$ and denote by u_1 the solution of the Dirichlet problem $(dd^c u)^n = f\, dV$ in B_0, $u_1 = u$ on ∂B_0. Then v equal to u_1 in B_0 and equal to u elsewhere in Ω belongs to $\mathcal{S}$. Hence $v \leq u$. Since, due to the comparison principle, $u_1 \geq u$ in B_0 we conclude that u_1 and u are equal in B_0 which shows that $(dd^c u)^n = f\, dV$ in Ω since the above is true for any ball in Ω.

Notes. The results here are due to Bedford and Taylor [**BT1**]. The presentation derives also from Demailly [**D1**] (who dealt with the homogeneous case).

CHAPTER 4

The Dirichlet Problem Continued

We shall generalize Theorem 3.6 weakening restrictions on the right hand side of the equation. We call a continuous increasing function $h : \mathbb{R}_+ \to (1, \infty)$ *admissible* if it satisfies

$$\int_1^\infty (xh^{1/n}(x))^{-1}\, dx < \infty,$$

and if for some $a > 1$ $b > 1$ and $x_0 > 0$ we have

$$h(ax) \le bh(x) \quad \text{for} \quad x > x_0.$$

Let us define the family of nonnegative Borel measures in Ω associated to an admissible function h and a positive constant A:

$$\begin{aligned} \mathcal{F}(A, h) = \{\mu : \mu(K) \le F(cap(K, \Omega)) \quad &\text{for } F(x) = \frac{Ax}{h(x^{-1/n})} \\ &\text{and any compact } K \subset \Omega\}. \end{aligned}$$

For a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\frac{\psi(x)}{x}$ increases to ∞ as $x \to \infty$ we define

$$L^\psi(c_0) = \{f \in L^1(\Omega) : f \ge 0, \int_\Omega \psi(f)\, dV \le c_0\}.$$

and

$$\begin{aligned} &\mathcal{P}(A, h, \psi, c_0; \varphi) \\ &= \{u \in PSH(\Omega) \cap C(\overline{\Omega}) : (dd^c u)^n \in \mathcal{F}(A, h) \cap L^\psi(c_0),\ u = \varphi \text{ on } \partial\Omega\} \end{aligned}$$

Set

$$\psi_h(t) = |t|(log(1 + |t|))^n h(log(1 + |t|)),$$

for some admissible h. First we shall prove that

$$L^{\psi_h}(c_0) \subset \mathcal{F}(A, h)$$

for some positive A. Then, a priori estimates for $||\cdot||_\infty$ norm of solutions of the Dirichlet problem for the measures from $\mathcal{F}(A, h)$ will be shown which imply that for $f \in L^{\psi_h}(c_0)$ the equation $(*)$ has a solution.

Lemma 4.1. *Suppose $u \in PSH(\Omega) \cap C(\overline{\Omega}), u = 0$ on $\partial\Omega, \int (dd^c u)^n \le 1$. Then for any $\alpha < 2$ the Lebesgue measure $V(\Omega_s)$ of the set $\Omega_s := \{u < s\}$ is bounded from above by $c \exp(-2\pi\alpha|s|)$, where c does not depend on u.*

Proof. Assume $\overline{\Omega}$ to be contained in a ball $B = B(0, R)$. We denote by V_k the Lebesgue measure in $\mathbb{C}^k$. Let us write the coordinates of a point $z \in \mathbb{C}^n$ in the form $z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$, and denote by B_1 (resp. B') the balls $\{z \in \mathbb{C} : |z| < R\}$ (resp. $\{z \in \mathbb{C}^{n-1} : |z| < R\}$). Consider the slices of the set Ω_s

$$\Omega_s(z') := \{z_1 \in \mathbb{C} : (z_1, z') \in \Omega_s\}.$$

For fixed s, the Siciak extremal function of of Ω_s in $\mathbb{C}^n$ will be denoted by L. We shall use the capacity T from the previous section.

For $n = 1$ the set function T_R dominates the logarithmic capacity multiplied by a constant depending on R. Hence by Theorem III 10 from [**TS**] we get

$$V_1(\Omega_s(z')) \leq C_1 T_R^\alpha(\Omega_s(z')),$$

where C_1 is an independent constant. Thus, making use of the Fubini theorem and Lemma 2.8 we can estimate as follows

(4.1)

$$V(\Omega_s) = \int V_1(\Omega_s(z'))\, dV_{n-1}(z') \leq C_1 \int T_R^\alpha(\Omega_s(z'))\, dV_{n-1}(z')$$

$$\leq C_1 \int exp(-\alpha \sup_{|z_1|<R} L(z_1, z'))\, dV_{n-1}(z') \leq C_2 exp(-\alpha \sup_{B_1 \times B'} L(z)) \leq C_2 T_R^\alpha(\Omega_s).$$

From Theorem 2.7 it follows that

$$T_R(\Omega_s) \leq exp[-2\pi(cap(\Omega_s, B))^{-1/n}] \leq exp[-2\pi(cap(\Omega_s, \Omega))^{-1/n}].$$

So, continuing the estimate (4.1) we finally arrive at

$$V(\Omega_s) \leq C_2 exp[-2\pi\alpha(cap(\Omega_s, \Omega))^{-1/n}].$$

To complete the proof it remains to show that

$$cap(\Omega_s, \Omega) \leq |s|^{-n}. \tag{4.2}$$

Fix $t > 1$ and a regular compact set $K \subset \Omega_s$. Then by the comparison principle we have

$$cap(K, \Omega) = \int_K (dd^c u_K)^n = \int_{\{-ts^{-1}u < u_K\}} (dd^c u_K)^n$$

$$\leq t^n |s|^{-n} \int_\Omega (dd^c u)^n \leq t^n |s|^{-n}.$$

Thus (4.2) holds and the lemma follows.

Lemma 4.2. *For any admissible h satisfying $h(x) \leq const.(1 + x)^k$ for some $k < \infty$, and for any $c_0 > 0$ there exists $A > 0$ such that*

$$L^{\psi_h}(c_0) \subset \mathcal{F}(A, h).$$

PROOF. We are going to verify that for some $A > 0$, any $f \in L^{\psi_h}(c_0)$ and any compact regular set $K \subset \Omega$ the following inequality holds

$$\int_K f\,dV \le A cap(K,\Omega)[h((cap(K,\Omega))^{-1/n})]^{-1}. \tag{4.3}$$

First, let us note that (4.3) follows from

$$\int_\Omega |v|^n h(|v|) f\,dV \le A, \tag{4.4}$$

where $v \in PSH(\Omega)$ is of the form $v = cap^{-1/n}(K,\Omega)u_K$, with u_K the relative extremal function of K with respect to Ω. Indeed, from (4.4) we have

$$\begin{aligned} A &\ge \int_\Omega |v|^n h(|v|) f\,dV \ge \int_K |v|^n h(|v|) f\,dV \\ &\ge cap^{-1}(K,\Omega) h((cap(K,\Omega))^{-1/n}) \int_K f\,dV, \end{aligned}$$

which proves (4.3). To prove (4.4) we shall use Young's inequality applied to $G(r) = g(log(1+r)) = (log(1+r))^n h(log(1+r))$ and its inverse. Then

$$\begin{aligned} g(|v(z)|)f(z) &\le \int_0^{f(z)} g(log(1+r))\,dr + \int_0^{g(|v(z)|)} [exp(g^{-1}(t)) - 1]\,dt \\ &\le f(z) g(log(1+f(z))) + \int_0^{|v(z)|} e^s g'(s)\,ds \\ &\le \psi_h(f(z)) + g(|v(z)|)e^{|v(z)|}. \end{aligned}$$

Since the integral $\int_\Omega \psi_h(f)\,dV$ is bounded by c_0, we obtain by integrating the above inequality over Ω

$$\int_\Omega |v(z)|^n h(|v(z)|) f(z)\,dV \le c_0 + \int_\Omega g(|v(z)|)e^{|v(z)|}\,dV.$$

It remains to find a uniform bound (independent of v) for the last term. To do this we make use of Lemma 4.1 and the extra assumption on h

$$\begin{aligned} &\int_\Omega g(|v(z)|)e^{|v(z)|}\,dV = \sum_{s=0}^{\infty} \int_{\{-s-1<v<-s\}} g(|v(z)|)e^{|v(z)|}\,dV \\ &\le \sum_{s=0}^{\infty} (s+1)^n h(s+1) e^{s+1} V(\{v < -s\}) \le c \sum_{s=0}^{\infty} (s+1)^n h(s+1) e^{1+s(1-2\pi)} \\ &\le c[h(1) + \sum_{s=1}^{\infty} (s+1)^{n+k} e^{1+s(1-2\pi)}] \le const. < \infty. \end{aligned}$$

The proof is completed.

LEMMA 4.3. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Assume $v \in PSH \cap C(\Omega)$, and $u \in PSH \cap L^\infty(\Omega)$. Suppose that for some positive number A and an admissible function h the following inequality holds*

$$\int_K (dd^c u)^n \leq A cap(K,\Omega) h^{-1}((cap(K,\Omega))^{-1/n}),$$

for any compact set K. If the sets $U(s) := \{u - s < v\}$ are nonempty and relatively compact in Ω for $s \in [S, S+D]$ Then

$$D \leq \kappa(cap(U(S+D),\Omega)), \tag{4.5}$$

where

$$\kappa(s) = c(n)A^{1/n}[\int_{s^{-1/n}}^{\infty} y^{-1}h^{-1/n}(y)\,dy + h^{-1/n}(s^{-1/n})],$$

and the constant $c(n)$ depends only on n.

PROOF. Let us introduce the following notation

$$a(s) := cap(U(s),\Omega), \quad b(s) = \int_{U(s)} (dd^c u)^n.$$

Then

$$t^n a(s) \leq b(s+t) \quad \text{for } 0 < t < S + D - s. \tag{4.6}$$

Indeed, consider a compact regular set $K \subset U(s)$, the psh function $w := \frac{1}{t}(u-s-t)$ and the set $V := \{w < u_K + \frac{1}{t}v\}$, where u_K denotes the relative extremal function of K with respect to Ω. Let us first verify the inclusions $K \subset V \subset U(s+t)$.

Take $z \in K \subset U(s)$. Then $u(z) - s < v(z)$ and so $w(z) = \frac{1}{t}(u(z) - s - t) \leq u_K(z) + \frac{1}{t}v(z)$ which means that $z \in V$. To see the latter inclusion, note that if $z \in V$ then $\frac{1}{t}(u(z) - s - t) \leq u_K(z) + \frac{1}{t}v(z) \leq \frac{1}{t}v(z)$ since u_K is negative. Once we have the inclusions we can apply the comparison principle and Theorem 1.20 to the effect

$$\begin{aligned} cap(K,\Omega) &\leq \int_K [dd^c(u_K + \frac{1}{t}v)]^n \leq \int_V [dd^c(u_K + \frac{1}{t}v)]^n \leq \int_V (dd^c w)^n \\ &\leq t^{-n}\int_V (dd^c u)^n \leq t^{-n}\int_{U(s+t)} (dd^c u)^n = t^{-n} b(s+t). \end{aligned}$$

This way (4.6) follows.

Next we define an increasing sequence $s_0, s_1, ..., s_N$, setting $s_0 := S$ and

$$s_j := \sup\{s : a(s) \leq \lim_{t \to s_{j-1}+} da(t)\}$$

for $j = 1, 2, ..., N$, where d is a fixed number such that $1 < d < 2$. Then this sequence is increasing and

$$a(s_j) \geq da(s_{j-2}). \tag{4.7}$$

(Note that if $a(s_{j-1}) < da(s_{j-2})$ then by definition od s_{j-1} for any $s > s_{j-1}$ we have $a(s) \geq da(s_{j-2})$. In particular it is true for s_j.)

The integer N is chosen to be the greatest one satisfying $s_N \leq S + D$. Then

$$a(S+D) \leq \lim_{t \to s_N+} da(t)$$

(otherwise we would have $s_{N+1} \leq S+D$.) From the last inequality, the assumptions and (4.6) it follows that for any $t \in (s_N, S+D)$ we have

$$\begin{aligned}(S+D-t)^n a(t) &\leq b(S+D) \leq Aa(S+D)h^{-1}([a(S+D)]^{-1/n}) \\ &\leq Ada(t)h^{-1}([a(S+D)]^{-1/n}).\end{aligned}$$

Hence

$$S + D - s_N \leq (Ad)^{1/n} h^{-1/n}([a(S+D)]^{-1/n}). \tag{4.8}$$

Now we shall estimate $s_N - S$. Consider two numbers $S < s' < s < S+D$ such that $a(s) \leq da(s')$ and set $t := s - s'$. Then by the assumptions and (4.6) we have

$$\begin{aligned}a(s') &\leq t^{-n} b(s) \leq At^{-n} a(s) h^{-1}([a(s)]^{-1/n}) \\ &\leq Adt^{-n} a(s') h^{-1}([a(s)]^{-1/n}).\end{aligned}$$

Hence

$$t \leq (Ad)^{1/n} h_1(a(s)),$$

where $h_1(x) := [h(x^{-1/n})]^{-1/n}$. Letting $s \to s_{j+1}-$ and $s' \to s_j+$ we thus get

$$t_j := s_{j+1} - s_j \leq (Ad)^{1/n} h_1(a(s_{j+1})).$$

Using this inequality, (4.7) and the fact that the function $h_2(x) := h_1(d^x) = h^{-1/n}(d^{-x/n})$ is increasing one can estimate as follows

$$\begin{aligned}\sum_{j=0}^{N-1} t_j &\leq (Ad)^{1/n} \sum_{j=0}^{N-1} h_2(\log_d a(s_{j+1})) \\ &\leq (Ad)^{1/n} [\sum_{j=1}^{N-2} \int_{\log_d a(s_j)}^{\log_d a(s_{j+2})} h_2(x)\, dx + 2h_2(\log_d a(s_N))] \\ &\leq 2(Ad)^{1/n} [\int_{\log_d a(S)}^{\log_d a(S+D)} h_2(x)\, dx + h_2(\log_d a(S+D))].\end{aligned}$$

The change of variable $y = d^{-x/n}$ leads to the following transformation of the above integral

$$\begin{aligned}\int_{\log_d a(S)}^{\log_d a(S+D)} h_2(x)\, dx &= \int_{\log_d a(S)}^{\log_d a(S+D)} [h(d^{-x/n})]^{-1/n}\, dx \\ &= \frac{n}{\ln d} \int_{[a(S+D)]^{-1/n}}^{[a(S)]^{-1/n}} [(h(y))^{1/n} y]^{-1}\, dy.\end{aligned}$$

Hence finally

$$s_N - S \leq (Ad)^{1/n} (\frac{2n}{\ln d} \int_{[a(S+D)]^{-1/n}}^{[a(S)]^{-1/n}} [yh^{1/n}(y)]^{-1}\, dy + 2[h(a(S+D)^{-1/n})]^{-1/n}),$$

which combined with (4.8) gives the desired estimate.

THEOREM 4.4. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Assume that $u_j \in PSH(\Omega) \cap C(\bar{\Omega})$ is a sequence converging weakly to $u \in PSH(\Omega)$ and for any j*

$$\underline{\lim}_{z \to \partial\Omega} (u_j - u)(z) \geq 0.$$

Suppose further that

$$(dd^c u_j)^n = f_j \, dV$$

with $f_j \in \mathcal{F}(A, h) \cap L^\psi(c_0)$, where $\frac{\psi(x)}{x}$ is increasing to ∞ as x goes to ∞. Then $u_j \to u$ uniformly in Ω.

PROOF. Denote by $a_j(\delta) = cap(E_j(2\delta), \Omega)$ the relative capacity of the set $E_j(2\delta) = \{u_j + 2\delta \leq u\}$. The set $E_j(2\delta)$ is compact in view of our assumptions. Let us denote by v_j the relative extremal function of the set $E_j(2\delta)$. By Theorem 1.20

$$\int_{E_j(2\delta)} (dd^c v_j)^n = a_j(\delta).$$

Observe that for $V = \{u_j \leq \delta v_j + u - \delta\}$ the following inclusions hold

$$E_j(2\delta) \subset V \subset E_j(\delta).$$

Applying the comparison principle we thus get

$$\begin{aligned} a_j(\delta)\delta^n &\leq \int_{E_j(2\delta)} [dd^c(\delta v_j + u)]^n \leq \int_V (dd^c u_j)^n \\ &\leq \int_{E_j(\delta)} f_j \, dV. \end{aligned} \tag{4.9}$$

Hence for any $M > 0$ and $u_+ := \max(u, 0)$ we have

$$\begin{aligned} a_j(\delta)\delta^{n+1} &\leq \int_\Omega (u - u_j)_+ f_j \, dV \\ &= \int_{\{f_j > M\}} (u - u_j)_+ f_j \, dV + \int_{\{f_j \leq M\}} (u - u_j)_+ f_j \, dV \\ &\leq \max_\Omega (u - u_j)_+ \int_{\{f_j > M\}} f_j \, dV + M \int_\Omega (u - u_j)_+ \, dV \\ &\leq \max_\Omega (u - u_j)_+ \frac{M}{\psi(M)} \int_\Omega \psi(f_j) \, dV + M \int_\Omega (u - u_j)_+ \, dV. \end{aligned} \tag{4.10}$$

For the last inequality we use the assumption that $\frac{\psi(x)}{x}$ is increasing. Fix $\epsilon > 0$. By the previous lemma, applied for $v = 0$, there exists $c_1 > 0$ such that $w \geq -c_1$ for any $w \in \mathcal{F}(A, h)$ with $w \geq u$ on $\partial\Omega$. So, in view of our assumptions the quantities

$$\max_\Omega (u - u_j)_+ \int_\Omega \psi(f_j) \, dV$$

are uniformly bounded. Using the assumptions on ψ we can make $\frac{M}{\psi(M)}$ arbitrarily small by taking M big enough. We choose M so that the first term on the right hand side (4.10) is less than $\epsilon/2$ for any j. Since $u_j \to u$ in $L^1(\Omega)$ (see e.g. [**H1**]) the other term is less than $\epsilon/2$ for $j > j_0$. Therefore

$$a_j(\delta) \leq \epsilon\delta^{-n-1} \quad \text{for} \quad j > j_0.$$

Suppose for a while that $E_j(3\delta)$ were nonempty. Then, applying Lemma 4.3 we would get

$$\delta \leq \kappa(a_j(\delta)) \leq \kappa(\epsilon\delta^{-n-1}), \quad j > j_0.$$

Since, by the assumption on h we have $\lim_{s\to 0}\kappa(s) = 0$ the last inequality yields a contradiction if we take ϵ small enough. Thus $E_j(3\delta)$ is empty for $j > j_0$ which together with Hartogs' lemma implies the uniform convergence of the sequence u_j.

THEOREM 4.5. *For ψ as in the previous theorem the set $\mathcal{P}(A, h, \psi, c_0; \varphi)$ is equicontinuous.*

PROOF. Arguing by contradiction suppose that for some $\epsilon > 0, u_j \in \mathcal{F}(A, h) \cap L^\psi(c_0)$ and two sequences z_j, w_j of points in Ω we had $||z_j - w_j|| < j^{-1}$ and $u_j(z_j) - u_j(w_j) > \epsilon$. Since $\mathcal{P}(A, h, \psi, c_0; \varphi)$ is uniformly bounded (due to Lemma 4.3), one can pick subsequences z_{j_k}, w_{j_k} converging to $z \in \overline{\Omega}$ and u_{j_k} converging in L^1 norm to $u \in PSH(\Omega) \cap C(\overline{\Omega})$. By Theorem 4.4 u_{j_k} converges uniformly. Thus for k large enough we have the following inequalities

$$\begin{aligned} |u(z_{j_k}) - u(z)| &< \epsilon/4, \\ |u(w_{j_k}) - u(z)| &< \epsilon/4, \\ |u_{j_k}(z_{j_k}) - u(z_{j_k})| &< \epsilon/4, \\ |u_{j_k}(w_{j_k}) - u(w_{j_k})| &< \epsilon/4. \end{aligned}$$

Combined they yield

$$|u_{j_k}(z_{j_k}) - u_{j_k}(w_{j_k})| < \epsilon,$$

which contradicts the choice of the sequences. The proof is completed.

THEOREM 4.6. *For any $f \in L^{\psi_h}(c_0)$ the Dirichlet problem (∗) has a solution.*

PROOF. Set $h_1(x) = \min(h(x), x + 1)$. Then, by Lemma 4.2, for some $A > 0$ we have

$$L^{\psi_h}(c_0) \subset L^{\psi_{h_1}}(c_0) \subset \mathcal{F}(A, h_1).$$

Take a sequence of continuous functions $f_j \in L^{\psi_h}(c_0)$ tending to f in L^1. The sequence u_j of solutions of $(*)$ of with f_j in place of f (obtained in Theorem 3.6) is uniformly bounded by Lemma 4.3. From the previous theorem we can therefore conclude that, after passing to a subsequence, u_j is uniformly convergent to a continuous plurisubharmonic function u. From the convergence theorem it follows that

$$(dd^c u)^n = f\, dV$$

which completes the proof.

Example. Take $\psi(t) = |t|(log(1+|t|))^n(1+log(1+log(1+|t|)))^m, m > n$. Then, by Lemma 4.2, the Dirichlet problem $(*)$ is solvable for any $f \in L^{\psi}(c_0)$. On the other hand, if $\chi(t) = |t|(log(1+|t|))^m, m < n$ then by the result of Persson [**P**], the Monge-Ampère equation admits unbounded solutions with pointwise singularities for some radially symmetric densities from L^{χ}. Indeed, one may verify that the function $f(z) = |z|^{-2n} \log^{-k} 2|z|^{-1}$ belongs to $L^{\chi}(B)$ for $k > m+1$ and the corresponding solution is equal $-\infty$ at 0 for $k < n+1$.

Example. For any $p > 1$ we have $L^p(\Omega) \subset L^{\psi}(\Omega)$, where ψ is the function from the previous example. Thus for $f \in L^p, p > 1$ the equation $(*)$ is solvable.

How subsolutions lead to solutions

Let us consider the Dirichlet problem in a strictly pseudoconvex domain where we admit bounded plurisubharmonic solutions.

$$(**) \qquad \begin{aligned} & u \in PSH \cap L^{\infty}(\Omega), \\ & (dd^c u)^n = d\mu, \\ & \lim_{\zeta \to z} u(\zeta) = \varphi(z) \text{ for } z \in \partial\Omega. \end{aligned}$$

Theorem 4.7. *If there exists a subsolution for the Dirichlet problem* $(**)$ *then the problem is solvable.*

Proof. Let us first state some additional assumptions and observe that by doing this we do not affect the generality of the proofs. It is enough to consider only measures μ which have compact support. Then, given non-compactly supported measure μ one can find solutions corresponding to $\chi_j\mu$, where χ_j is a non-decreasing sequence of cut-off functions $\chi_j \uparrow 1$ on Ω. The solutions will be bounded from below by the given subsolution (due to the comparison principle) and they will decrease to the solution for μ by the convergence theorem.

Next, the subsolution v given by the hypothesis can be modified so that the function is defined in a neighbourhood of $\overline{\Omega}$, and $\lim_{\zeta \to z} v(\zeta) = 0$ for any $z \in \partial\Omega$. Furthermore, using the balayage procedure, one can make the support of $d\nu := (dd^c v)^n$ compact in Ω. For such v one can define the regularizing sequence $w_j \downarrow v$ in the closure of Ω. Let $(dd^c w_j)^n = g_j \, dV$. By Theorem 3.6 there exits $v_j \in PSH(\Omega) \cap C(\overline{\Omega})$, $v_j = 0$ on $\partial\Omega$ and such that $(dd^c v_j)^n = g_j \, dV$. Since $|v_j - w_j|$ attains its maximum on $\partial\Omega$ and w_j tends to 0 uniformly on the boundary we conclude that $v_j \to v$ uniformly on each compact set where the restriction of v is continuous. Thus it is convergent with respect to capacity.

By the Radon-Nikodym theorem $d\mu = h \, d\nu, \ 0 \le h \le 1$. Applying Theorem 4.6 we solve the following Dirichlet problem

$$\begin{cases} u_j \in PSH(\Omega) \cap C(\overline{\Omega}) \\ (dd^c u_j)^n = h g_j \, dV \\ u_j(z) = \varphi(z) \text{ for } z \in \partial\Omega. \end{cases}$$

As we shall see the function $u = (\limsup u_j)^*$ solves the equation $(**)$. Passing to a subsequence we assume that u_j converge in $L^1(\Omega)$.

LEMMA 4.8. *The function u defined above solves the Dirichlet problem* $(**)$ *provided that for any $a > 0$ and any compact $K \subset \Omega$ we have*

$$\lim_{j\to\infty} \int_{E_j(a)\cap K} (dd^c u_j)^n = 0, \quad \textit{where } E_j(a) := \{u - u_j \geq a\}. \tag{4.11}$$

PROOF. Indeed, if (4.11) holds then for any s one can find $j(s)$ such that

$$\int_{E_j(1/s)\cap K} (dd^c u_j)^n < 1/s, \quad j \geq j(s).$$

Set $\rho_s := \max(u_{j(s)}, u - 1/s)$. Then $(dd^c\rho_s)^n = (dd^c u_{j(s)})^n$ on $(int\, K) \setminus E_{j(s)}(1/s)$, and so the above inequality implies that any accumulation point of $\{(dd^c\rho_s)^n\}$ is $\geq d\mu$ on $int\, K$. On the other hand, by the definition of ρ_s and a version of the Hartogs lemma given in Theorem 4.1.9 from [**H1**] $\rho_s \to u$ uniformly on any compact E such that $u_{|E}$ is continuous. So it follows from Theorem 1.13 that ρ_s converge to u with respect to capacity. Therefore applying Theorem 1.11 we obtain $(dd^c\rho_s)^n \to (dd^c u)^n$, and further

$$(dd^c u)^n \geq d\mu. \tag{4.12}$$

To get the reverse inequality note that $\rho_s = u_{j(s)}$ on a neighbourhood of $\partial\Omega$ since all the u_j's (and therefore u as well) are bounded from above by the solution of the homogeneous Monge-Ampère equation with the same boundary data, and this solution is continuous in the closure of Ω. Hence, due to the Stokes theorem, $\int_\Omega (dd^c\rho_s)^n = \int_\Omega (dd^c u_{j(s)})^n$. By the construction, the integrals on the right tend to $\int_\Omega d\mu$, so the measures in (4.12) must be equal. Thus the lemma follows.

LEMMA 4.9. *Suppose that u_j do not fulfil the hypothesis of the previous lemma, and so, after passing to a subsequence (which does not change u since u_j converge in L^1) we have*

$$\int_{E_j(a_0)} (dd^c u_j)^n > A_0, \quad A_0 > 0, a_0 > 0.$$

Then there exist $a_m > 0, A_m > 0, k_1 > 0$ such that

$$\int_{E_j(a_m)} (dd^c v_j)^{n-m} \wedge (dd^c v_k)^m > A_m, \ j > k > k_1. \tag{4.13}$$

PROOF. We shall proceed by induction over m. For $m = 0$ the statement holds by the hypothesis. We assume that (4.13) is true for some fixed $m < n$ and now we shall prove it for $m + 1$.

Let us observe that by the CLN inequalities there exists $C > 0$ such that

$$\int_\Omega T \leq C, \tag{4.14}$$

for currents T which are wedge products of $dd^c v_j$ or $dd^c u_j$. Indeed, we can extend all the functions involved to a slightly larger domain as it was done in Chapter 3 and apply CLN inequalities. We also assume that those functions have L^∞ norm bounded by one.

Denote by $T = T(j,k,m)$ the current $(dd^c v_j)^{n-m-1} \wedge (dd^c v_k)^m$. For fixed $\epsilon \in (0, \frac{a_m A_m}{4C+4})$ we choose an open set U such that

$$cap(U,\Omega) < \epsilon/2^{n+1},$$

and both u and v are continuous on $\Omega \setminus U$. Note that, assuming for simplicity, that $-1 < v_j, u_j < 0$, we have

$$\int_U (dd^c(v_j + v_k))^n \le 2^n cap(U,\Omega) < \epsilon/2. \tag{4.15}$$

Moreover one can replace v_j by u_j in this inequality. Then for $k > k_0$ and we have

$$\begin{aligned} v_k &\le v + \epsilon, \\ u_k &\le u + \epsilon, \end{aligned} \tag{4.16}$$

on $\Omega \setminus U$. Indeed, the inequalities are valid in a neighbourhood of $\partial\Omega$ because all u_j (resp. v_j) are bounded from above by the maximal function in Ω with boundary data φ (resp. 0). On the remaining part of $\Omega \setminus U$ one obtains (4.16) by the Hartogs lemma. Set

$$\begin{aligned} J'(j,k) &:= \int_\Omega (u - u_j) dd^c v_j \wedge T, \\ J(j,k) &:= \int_\Omega (u - u_j) dd^c v_k \wedge T, \quad j > k > k_0. \end{aligned}$$

Integrating by parts we get

$$J'(j,k) - J(j,k) = \int_\Omega (v_j - v_k) dd^c(u - u_j) \wedge T = \int_{\Omega \setminus U} ... + \int_U ... \, .$$

Since $v_j \to v$ uniformly away from U one can find $k_1 > k_0$ such that $||v_j - v_k|| < \epsilon/2C$ on $\Omega \setminus U$ for $j > k > k_1$. Thus, using (4.14) and (4.15), we conclude that each integral on the right hand side does not exceed $\epsilon/2$ for such j, k. So

$$J'(j,k) - J(j,k) \le \epsilon \le \frac{a_m A_m}{4}, j > k > k_1. \tag{4.17}$$

Using the induction hypothesis, (4.14), (4.15), and (4.16) we have

$$\begin{aligned} J'(j,k) &\ge a_m \int_{E_j(a_m)} dd^c v_j \wedge T - \epsilon \int_{\Omega \setminus U} dd^c v_j \wedge T - \int_U dd^c v_j \wedge T \\ &\ge a_m \int_{E_j(a_m)} dd^c v_j \wedge T - \epsilon(C+1) \ge a_m A_m - \epsilon(C+1) \ge \frac{3 a_m A_m}{4}, \end{aligned}$$

for $j > k > k_2 > k_1$. Combined with (4.17) this gives

$$J(j,k) \ge \frac{a_m A_m}{2}, \quad j > k > k_2. \tag{4.18}$$

Fixing $d > 0$ one can estimate $J(j,k)$ from above as follows

$$\begin{aligned} J(j,k) &\le \int_{\{u_j<u-d\}} dd^c v_k \wedge T + d\int_\Omega dd^c v_k \wedge T \\ &\le \int_{\{u_j<u-d\}} dd^c v_k \wedge T + dC. \end{aligned}$$

Setting $a_{m+1} := d = \frac{a_m A_m}{4C}$ in the last formula and combining it with (4.18) we finally arrive at

$$\int_{E_j(a_{m+1})} dd^c v_k \wedge T \ge \frac{a_m A_m}{4} := A_{m+1}, \quad j > k > k_2,$$

which concludes the proof of the inductive step. Thus the lemma follows.

Now we shall prove Theorem 4.7 reasoning by contradiction. So, suppose the hypothesis of Lemma 4.9 is valid. Then using its statement for $m = n$ we can fix $k > k_1$ such that

$$\int_{E_j(a_n)} (dd^c v_k)^n > A_n \quad \text{if } j > k.$$

Since, by the construction, $(dd^c v_k)^n \le M_k dV$ for some $M_k > 0$ one infers from the last inequality that

$$V(E_j(a_n)) \ge M_k^{-1} \int_{E_j(a_n)} (dd^c v_k)^n > \frac{A_n}{M_k}, \quad j > k,$$

which contradicts the fact that $u_j \to u$ in L^1_{loc}. Thus the theorem follows.

Notes. The main results of this section come from [**KO1**][**KO2**][**KO3**][**KO5**]. The proof of Theorem 4.7 is considerably simplified. The improved estimate of Lemma 4.1 has been shown (in a different way) by Kiselman [**KI2**] and Zeriahi [**ZE**].

CHAPTER 5

The Monge-Ampère Equation for Unbounded Functions

Applying Proposition 1.7 we have defined

$$dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k$$

for any collection of locally bounded psh functions. For unbounded psh functions the matter becomes complicated as the following example from [**Kl1**] shows

Example. The function

$$u(z) = (-\log |z_1|)^{1/2}(|z'|^2 - 1)$$

for $z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$ is psh in a neighbourhood of the origin but

$$\int_{B(0,r)\setminus L} (dd^c u)^n = \infty$$

for $L = \{z : z_1 = 0\}$ and $r > 0$.

However, the Monge-Ampère operator can be defined on some classes of psh functions in such a way that $(dd^c u)^n$ is locally finite and that it is continuous with respect to monotone sequences of psh functions. Throughout this section Ω will denote a fixed hyperconvex domain in $\mathbb{C}^n$. Recall that a domain is called hyperconvex if there exists nonzero $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$. The set of such functions satisfying $\int_\Omega (dd^c u)^n < \infty$ we denote by $\mathcal{E}$.

Fact. $\mathcal{E}$ is a convex cone.

Definition. *We say that a plurisubharmonic function u belongs to $\mathcal{F}_p$ if there exists $u_j \in \mathcal{E}$ with $u_j \downarrow u, \sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. If the sequence u_j fulfils the above conditions but the last one then u belongs to $\mathcal{E}_p$.*

Fact. $\mathcal{E} \subset \mathcal{F}_p \subset \mathcal{E}_p$, $\mathcal{F}_q \subset \mathcal{F}_p$ for $q > p$.

The following estimate is crucial for the sequel.

Theorem 5.1. *For $u, v \in \mathcal{E}$ and $p \geq 1$*

$$\int_\Omega (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j}$$
$$\leq C(j,p)(\int_\Omega (-u)^p (dd^c u)^n)^{(p+j)/(n+p)} (\int_\Omega (-v)^p (dd^c v)^n)^{(n-j)/(n+p)}$$

with $C(j,p) = 1$ if $p = 1$ and $C(j,p) = p(p+j)(n-j)/(p-1)$ otherwise.

Proof (sketch). Suppose

$$\int_\Omega (dd^c u)^j \wedge (dd^c v)^{n-j} < \infty,$$

and denote

$$x_j = \log \int_\Omega (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j},$$
$$y_j = \log \int_\Omega (-v)^p (dd^c u)^{n-j} \wedge (dd^c v)^j.$$

Applying integration by parts (which is justified by using the additional assumption, see [**CP**]) and Hölder's inequality one gets

$$\begin{aligned} e^{x_j} &= -\int dv \wedge d^c(-u)^p \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &= \int v dd^c(-u)^p \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &= p(p-1) \int v(-u)^{p-2} du \wedge d^c u \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &+ p \int (-v)(-u)^{p-1} (dd^c u)^{j+1} \wedge (dd^c v)^{n-j-1} \\ &\leq p \int (-v)(-u)^{p-1} (dd^c u)^{j+1} \wedge (dd^c v)^{n-j-1} \\ &\leq (p \int (-v)^p (dd^c u)^{j+1} \wedge (dd^c v)^{n-j-1})^{1/p} \\ &\times (p \int (-u)^p (dd^c u)^{j+1} \wedge (dd^c v)^{n-j-1})^{(p-1)/p}. \end{aligned}$$

Take logarithms of both sides to obtain the system of inequalities

$$x_j \leq \frac{p-1}{p} x_{j+1} + \frac{1}{p} y_{n-j-1} + \log p$$
$$y_j \leq \frac{p-1}{p} y_{j+1} + \frac{1}{p} x_{n-j-1} + \log p.$$

The system in matrix notation is given by

$$A(x_0, y_0, ..., x_n, y_n)^T \leq \log p(1, 1, ..., 1)^T, \tag{5.1}$$

where $A = (a_{jk}), j = 1, ..., 2n; k = 1, ..., 2n+2$ has coefficients $a_{jj} = 1, a_{j,j+2} = (1-p)/p, a_{j,2n-j+1} = -1/p$ for $j = 1, ..., 2n$. Removing from A the last two columns we obtain $2n \times 2n$ matrix denoted by C. After showing that C has an inverse with nonnegative coefficients we shall multiply the system by C^{-1} reducing it to the row-echelon form. To invert C consider the system of equations

$$C(x_0, y_0, ..., x_{n-1}, y_{n-1})^T = (b_0, c_0, ..., b_{n-1}, c_{n-1})^T$$

and compute x_j, y_j which turn out to be linear combinations of $b_j's$ and $c_j's$ with nonnegative coefficients. The same calculation shows that $C^{-1}A$ is equal to the $2n \times 2n$ identity matrix complemented by two last columns given by $\frac{1}{p+n}(A_0, ..., A_{n-1})^T$

where $A_j = (a^j_{kl})$ is 2×2 matrix with $a^j_{11} = a^j_{22} = p + j$ and $a^j_{12} = a^j_{21} = n - j$ Multiplying (5.1) by C^{-1} one obtains (by calculation)

$$x_j - \frac{p+j}{p+n} x_n - \frac{n-j}{p+n} y_n \leq \frac{(p+j)(n-j)}{p-1} \log p$$
$$y_j - \frac{n-j}{p+n} x_n - \frac{p+j}{p+n} y_n \leq \frac{(p+j)(n-j)}{p-1} \log p.$$

This gives the assertion for $p > 1$ and passing to the limit for $p = 1$.

To get rid of the extra assumption from the beginning of the proof one applies the above reasoning to the standard regularizations u_j, v_j of u and v on slightly smaller domains Ω_j. It turns out that the integrals from the statement are the limits of analogous integrals for u_j, v_j, Ω_j. We refer to [**CP**] for details.

The following two facts can easily be deduced from Theorem 5.1.

FACT. *$\mathcal{E}_p$ and $\mathcal{F}_p$ are convex cones.*

FACT. *$\mathcal{E}_p$ and $\mathcal{F}_p$ are closed with respect to the operation of taking maximum of a finite number of functions.*

THEOREM 5.2. *Suppose $u \in PSH(\Omega)$ is the limit of a decreasing sequence $u_j \in \mathcal{E}$ such that $a = \sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$. Then $(dd^c u_j)^n$ is weakly convergent to a measure $d\mu$ which is independent of the choice of u_j satisfying the condition above. Thus one can define $(dd^c u)^n = d\mu$.*

PROOF. Take a nonnegative test function χ with $||\chi|| = 1$. We shall use the notation

$$v_{[k} := \max(v, -k).$$

Since $u_j = u_{j[k}$ on $\{u_j > -k\}$ we get

$$\begin{aligned}
&|\int \chi[(dd^c u_j)^n - (dd^c u_{j[k]})^n]| \leq \int_{\{u_j \leq -k\}} \chi[(dd^c u_j)^n + (dd^c u_{j[k]})^n] \\
\leq & k^{-p} \int_{\{u_j \leq -k\}} k^p [(dd^c u_j)^n + (dd^c u_{j[k]})^n] \\
\leq & k^{-p} \int (-u_j)^p (dd^c u_j)^n + (-u_{j[k]})^p (dd^c u_{j[k]})^n \\
\leq & 2ak^{-p}.
\end{aligned}$$

Hence, by the convergence theorem, if $d\mu$ is the weak limit of a subsequence of $(dd^c u_j)^n$ then

$$|\int \chi(d\mu - (dd^c u_{[k]})^n| \leq 2ak^{-p},$$

which gives the statement.

THEOREM 5.3. *For $u_j \in \mathcal{E}_p$, $u_j \uparrow u$ we have $u \in \mathcal{E}_p$ and*

$$\lim_{j \to \infty} (dd^c u_j)^n = (dd^c u)^n.$$

PROOF. Use the estimate from the previous proof and Theorem 1.15.

THEOREM 5.4 (COMPARISON PRINCIPLE). *If $p \geq 1$ and $u, v \in \mathcal{F}_p$ then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

PROOF. Using the fact that $u, v \in \mathcal{F}_p$ one can find U_0 with $cap(U_0, \Omega) < \epsilon$ and

$$\int_{U_0} (dd^c u_j)^n + (dd^c v_j)^n < \epsilon$$

for any j, where u_j, v_j are continuous and $u_j \downarrow u, v_j \downarrow v$. Then after incorporating U_0 into U we may repeat the proof of Theorem 1.16.

COROLLARY. *If $p \geq 1$, $u, v \in \mathcal{F}_p$ and $(dd^c u)^n \leq (dd^c v)^n$ then $v \leq u$ in Ω.*

Following Cegrell [**C2**] one can now characterize the measures for which the Dirichlet problem has a solution in $\mathcal{F}_p$.

THEOREM 5.5. *Let μ be a positive measure with finite total mass in Ω. Then there exists a unique $u \in \mathcal{F}_p$ solving*

$$(dd^c u)^n = d\mu$$

if and only if for some positive A the following inequality holds

$$\int (-u)^p \, d\mu \leq A\left(\int (-u)^p (dd^c u)^n\right)^{\frac{p}{n+p}}, \tag{5.2}$$

for any $u \in \mathcal{E}$.

PROOF. We begin with a version of Lemma 4.8.

LEMMA 5.6. *If $u_j \in \mathcal{E}$ is a sequence converging a.e. to $u \in PSH(\Omega)$ with*

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$$

and

$$\lim_{j\to\infty} \int |u - u_j| (dd^c u_j)^n = 0,$$

then

$$\lim_{j\to\infty} (dd^c u_j)^n = (dd^c u)^n.$$

PROOF. The proof of Lemma 4.8 applies except that we do not know in advance that ρ_s is uniformly bounded. So, to ensure that $\rho_s \to u$ with respect to capacity we need to use the assumption

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$$

to conclude that

$$\lim_{k\to\infty} \sup_s \int_{\{\rho_s < -k\}} (dd^c \rho_s)^n = 0.$$

To this end we need the following proposition.

PROPOSITION. *If $u, v \in \mathcal{E}_1$ and $v \le u$ then*

$$\int (-u)(dd^c u)^n \le \int (-v)(dd^c v)^n.$$

PROOF. Use repeatedly the inequality which follows from integration by parts formula

$$\int (-u)(dd^c u)^{n-k} \wedge (dd^c v)^k \le \int (-v)(dd^c u)^{n-k} \wedge (dd^c v)^k$$
$$= \int (-u)(dd^c u)^{n-k-1} \wedge (dd^c v)^{k+1}.$$

Applying this inequality for $u = \rho_s$ and $v = u_{j(s)}$ we get the desired estimate

$$k \int_{\{\rho_s < -k\}} (dd^c \rho_s)^n \le \int (-\rho_s)(dd^c \rho_s)^n \le \int (-u_{j(s)})(dd^c u_{j(s)})^n.$$

LEMMA 5.7. *Let μ be a nonnegative compactly supported measure that satisfies (5.2) for $p > n/(n-1)$ and let $u_j \in \mathcal{E}$ be a sequence with $\sup_j \int_\Omega (dd^c u_j)^n = a < \infty$ that converges a.e. to $u \in PSH(\Omega)$. Then $\lim_j \int u_j \, d\mu = \int u \, d\mu$.*

PROOF. A simple measure theoretic argument shows that $\overline{\lim}_j \int u_j \, d\mu \le \int u \, d\mu$, thus we need to prove $\underline{\lim}_j \int u_j \, d\mu \ge \int u \, d\mu$. Passing to a subsequence one can assume that $\underline{\lim}_j \int u_j \, d\mu = \lim_j \int u_j \, d\mu$. Set $E(j,k) = \{u_j < -k\} \cap supp\, \mu$ and denote by u_{jk} the relative extremal function of this set. By the assumptions

$$\int_{E(j,k)} d\mu \le A\Big(\int_{E(j,k)} (dd^c u_{jk})^n\Big)^{\frac{p}{n+p}}.$$

By the comparison principle (Theorem 5.4)

$$k^{-n} \int_{E(j,k)} (dd^c u_{jk})^n \le 2^n \int_{\{2u_j < k u_{jk}\}} (dd^c u_j)^n \le 2^n a.$$

Combining the above two inequalities we get

$$\int_{E(j,k)} d\mu \le A(2^n a)^{\frac{p}{n+p}} k^{\frac{-np}{n+p}}.$$

Since we assumed $p > n/(n-1)$ we have $q := \frac{np}{n+p} > 1$. Thus applying the previous estimate one obtains

$$\int_{E(j,2^k)} (-u_j) \, d\mu = \sum_{s=k}^{\infty} \int_{\{-2^{s+1} < u_j \le -2^s\}} (-u_j) \, d\mu$$
$$\le A(2^n a)^{\frac{p}{n+p}} \sum_{s=k}^{\infty} \frac{2^{s+1}}{2^{qs}} =: c_k.$$

Hence

$$\lim_{k \to \infty} \sup_j \int_{E(j,2^k)} (-u_j) \, d\mu = 0$$

and

$$\int_\Omega (-u_j) \, d\mu \le 2^k \int_\Omega d\mu + c_k.$$

Thus $\sup_j \int (-u_j)\, d\mu < \infty$. Having those estimates it is enough to show that

$$\lim_{j\to\infty} \int u_{j[k}\, d\mu = \int u_{[k}\, d\mu$$

or just assume that u_j are uniformly bounded. Then the sequence is also bounded in $L^2(d\mu)$, so, passing to a subsequence one can find $v \in L^2(d\mu)$ with $v_k := k^{-1}\sum_1^k u_j \to v$ in $L^2(d\mu)$. Extracting a subsequence of v_k we also get $v_{k_s} \to v$ $d\mu$ *a.e.* From a.e. convergence of u_j to u we obtain that $v_{k_s} \to u$ a.e. with respect to the Lebesgue measure. Therefore $(\sup_{s>t} v_{k_s})^* \downarrow u$ as $t \to \infty$. Then

$$\begin{aligned}\lim_j \int u_j\, d\mu = \underline{\lim}_j \int u_j\, d\mu = \lim_{s\to\infty} \int v_{k_s}\, d\mu = \int v\, d\mu \\ = \lim_{t\to\infty} \int (\sup_{s>t} v_{k_s})^* d\mu = \int u\, d\mu.\end{aligned}$$

The proof is completed.

Proof of Theorem 5.5. **Case** $p > \frac{n}{n-1}$. As in the proof of Theorem 4.7 one can show that it is enough to prove the statement for μ compactly supported. For such μ we define a regularizing sequence μ_j. Let I_0 denote a unit cube containing Ω and let us consider a sequence $\mathcal{B}_j$ of subdivisions of I_0 into 3^{2sn} congruent open cubes of equal size which are pairwise disjoint but their closures cover I_0. It is no restriction to assume that for each j we have $\mu(\cup_{I\in\mathcal{B}_j} \partial I) = 0$. Set

$$\mu_j := f_j dV, \quad f_j(z) := \frac{\mu(I\cap\Omega)}{V(I\cap\Omega)} \text{ if } z \in I \in \mathcal{B}_j,$$

(for $z \in \partial I$ we put $f_j(z) = 0$).

By Theorem 4.6 one can solve the following Dirichlet problem

$$\begin{cases} u_j \in PSH(\Omega) \cap C(\overline{\Omega}) \\ (dd^c u_j)^n = f_j dV \\ u_j(z) = 0 \text{ for } z \in \partial\Omega. \end{cases}$$

First we are going to show that u_j is bounded in L^1_{loc}. Set $r_j = n3^{-j}$ and $c_j = [V(B(0,r_j)]^{-1}$. Then for $z \in I \in \mathcal{B}_j$ we have $I \subset B(z,r_j)$. By subharmonicity

$$u_j(z) \le c_j \int_{B(z,r_j)} u_j\, dV \le c_j \int_I u_j\, dV.$$

Hence, via Fubini's theorem

$$\begin{aligned}\int_I u_j\, d\mu \le (\sup_I u_j) \int_I f_j\, dV \le c_j (\int_I u_j\, dV)(\int_I f_j\, dV) \\ \le c_j V(I) \int_I u_j\, d\mu_j.\end{aligned}$$

Thus $\int (-u_j)\, d\mu_j \le const. \int (-u_j)\, d\mu$ and the last integral is uniformly bounded by the previous proof. Applying this estimate and Theorem 5.1 with $u = u_j$ and some fixed strictly plurisubharmonic function v we conclude that $||u_j||_{L^1}$ is bounded on any compact subset of Ω. Therefore, passing to a subsequence, we may consider u_j to be convergent to u a.e. in dV.

Our next objective is to prove that $(dd^c u_j)^n \to (dd^c u)^n$. To this end we define the "error term"

$$v_j(z) = c_j \int_{B(0,r_j)} |u(z+w) - u_j(z+w)|\, dV,$$

with r_j, c_j introduced above. Setting $\tilde{u}_j := (\sup_{j \le k} u_k)^*$ we estimate this term as follows:

$$\begin{aligned} v_j(z) &\le c_j \int_{B(0,r_j)} |u(z+w) - \tilde{u}_j(z+w)| + |\tilde{u}_j(z+w) - u_j(z+w)|\, dV(w) \\ &\le c_j \int_{B(0,r_j)} \tilde{u}_j(z+w) - u(z+w)\, dV(w) \\ &+ c_j \int_{B(0,r_j)} \tilde{u}_j(z+w)\, dV(w) - c_j \int_{B(0,r_j)} u_j(z+w)\, dV(w) \\ &\le c_j \int_{B(0,r_j)} \tilde{u}_j(z+w) - u(z+w)\, dV + \sup_{B(0,r_j)} \tilde{u}_j(\cdot + z) - u_j(z). \end{aligned}$$

From the Lebesgue monotone convergence theorem we conclude that $v_j(z) \to 0$ a.e. in dV. Thus, by Lemma 5.7 we have $\lim_j \int v_j\, d\mu = 0$.

Now, observe that by Fubini's theorem

$$\int |u - u_j| (dd^c u_j)^n = \sum_{I \in \mathcal{B}_j} \int_I |u - u_j| f_j\, dV$$

$$\le \sum_{I \in \mathcal{B}_j} \frac{1}{V(I)} \int_I d\mu \int_I |u - u_j|\, dV \le \sum_{I \in \mathcal{B}_j} \frac{1}{c_j V(I)} \int_I v_j\, d\mu = const. \int v_j\, d\mu \to 0.$$

We have thus verified that the assumptions of Lemma 5.6 are fulfilled and therefore

$$(dd^c u)^n = \lim_{j \to \infty} (dd^c u_j)^n = d\mu.$$

We need yet to prove that $u \in \mathcal{F}_p$. If one denotes by χ_k the characteristic function of the set $\{u \ge -k\}$ then by Theorem 1.18 $(dd^c u_{[k]})^n \ge \chi_k d\mu$. Applying Theorem 4.7 we now get a bounded plurisubharmonic function v_k with $\lim_{w \to z} v_k(w) = 0,\ z \in \partial\Omega$ and such that $(dd^c v_k)^n = \chi_k d\mu$. By the assumptions

$$\int (-v_k)^p (dd^c v_k)^n \le \int (-v_k)^p (dd^c u)^n \le A\Big(\int (-v_k)^p (dd^c v_k)^n\Big)^{\frac{p}{n+p}},$$

and so

$$\int (-v_k)^p (dd^c v_k)^n \le A^{\frac{n+p}{n}}. \tag{5.3}$$

Hence $u = \lim v_k \in \mathcal{F}_p$.

Case $p \ge 1$.

Fix $q > n/(n-1)$, a compact K containing $supp\,\mu$ and a constant C satisfying

$$C > C(0,q)\, cap^{\frac{n}{n+q}}(K, \Omega), \tag{5.4}$$

where $C(0,q)$ comes from Theorem 5.1. Let us consider the set of measures

$$M = \{\nu \ge 0, supp\,\nu \subset K, \int (-u)^q\, d\nu \le C\Big(\int (-u)^q (dd^c u)^n\Big)^{\frac{q}{n+q}} \quad \text{for } u \in \mathcal{E}\}$$

and the set N associated to some $\nu_0 \in M$

$$N = \{\nu \geq 0, supp\, \nu \subset K, \int d\nu = 1,$$
$$\int (-u)^q \, d\nu \leq C(1/D_1 + 1/D_2)(\int (-u)^q (dd^c u)^n)^{\frac{q}{n+q}} \quad \text{for } u \in \mathcal{E}\}$$

where $D_1 = \sup\{\int d\nu, \nu \in M\}$ and $D_2 = \int d\nu_0$. Then for $\nu \in M$

$$(1/D_1 D_2)[(D_1 - \int d\nu)\nu_0 + D_2 \nu] \in N.$$

It is easy to check that N is weak* - compact and convex set of probability measures. By a version of Radon-Nikodym theorem from [**R**] there exists $\nu \in N$ and $f \in L^1(d\nu)$ such that $\nu_s = \mu - f\, d\nu$ is nonnegative and orthogonal to N. If $E \subset K$ and $cap(E, \Omega) > 0$ then by Theorem 5.1 and (5.4) $(dd^c u_E^*)^n \in M$. Thus there exists a measure in N which does not vanish on E. Since ν_s is orthogonal to N and, by (5.2), μ puts no mass on pluripolar sets one concludes that $\nu_s = 0$ and $\mu = f\, d\nu$. By the first part of the proof one can find $u_k \in \mathcal{F}_q$ with $(dd^c u_k)^n = f_k\, d\nu$, where $f_k = \min(f, k)$. The sequence u_k is decreasing and the same argument as the one leading to (5.3) shows that $u = \lim u_k$ belongs to $\mathcal{F}_p$.

COROLLARY 5.8. *If μ is a nonnegative compactly supported measure in Ω satisfying*

$$\mu(K) \leq A\, cap^{\frac{p}{n}}(K, \Omega),$$

for some $p > 1, A > 0$ and any compact regular $K \subset \Omega$ then there exists $u \in \mathcal{F}_1$ such that $(dd^c u)^n = d\mu$.

PROOF. As in the proof of Lemma 5.7 we first show that

$$\int_{E(j,k)} d\mu \leq const.k^{\frac{-np}{n+p}},$$

(where $E(j, k) = \{u_j < -k\} \cap supp\, \mu$ and u_j is the sequence constructed in the preceding proof) and then

$$\sup_j \int (-u_j)\, d\mu < \infty.$$

Having this we may continue as in the proof of Theorem 5.5 proving that $(dd^c u_j)^n \to (dd^c u)^n$ and $u = (\limsup u_j)^* \in \mathcal{F}_1$.

THEOREM 5.9. *The Dirichlet problem (*) has a continuous solution for any $d\mu \in \mathcal{F}(A, h)$ with admissible h.*

PROOF. Fix an exhaustion sequence K_j of compact sets in $\Omega = \cup K_j$. Denote by χ_j the characteristic function of K_j. It is clear that $\chi_j d\mu$ satisfies the assumptions of Corollary 5.8 with (for instance) $p = n$. Thus we get $u_j \in \mathcal{F}_1$ satisfying $(dd^c u_j)^n = \chi_j d\mu$. Similarly, denoting by χ_{jk} the characteristic function of $K_j \cap \{u_j \geq -k\}$ we can find $u_{jk} \in \mathcal{F}_1$ solving

$$(dd^c u_{jk})^n = \chi_{jk} d\mu.$$

Now, functions u_{jk} are bounded by the argument from the proof of Theorem 5.5. Since the family of measures $\chi_{jk} d\mu$ fulfils the hypothesis of Lemma 4.3 with the same function h and the same constant A, we have by Lemma 4.3 a uniform bound

$$||u_{jk}|| < B,$$

for all these solutions. Thus in particular the functions $u_j = \lim_k u_{jk}$ are bounded and so is $u = \lim_j u_j$ (the limits exist since the sequences are monotone due to the comparison principle). From the convergence theorem it now follows that

$$(dd^c u)^n = d\mu.$$

The boundary values of u are equal to 0 (see the definition of $\mathcal{F}_1$). To get the general case, consider $v = u + u_\phi$, where u_ϕ is the function which solves $(dd^c u_\phi)^n = 0$ with the given boundary data ϕ. The function v is a subsolution for the Dirichlet problem in the statement of Theorem 4.7. Applying Theorem 4.7 one gets the desired solution.

We need yet to prove the continuity of u. To do this we shall apply Lemma 4.3 once more. Since φ is continuous one can find for any given $\delta > 0$ a compact $K \subset \Omega$ such that $u_j < u + \delta$ on ∂K, where u_j is the standard regularization for u. Then the capacity of the set $\{u_j > u + 2\delta\}$ tends to 0 as j goes to infinity (see Proposition 1.12). Thus for j large enough the right hand side in (4.5) is less than δ when applied for $v = u_j$ which yields a contradiction unless the set $\{u_j > u + 2\delta\}$ is empty.

Notes. This section is based on Cegrell's work [**C2**]. Theorem 5.5 also holds for nonzero continuous boundary data (see [**C2**]) but the proof requires the main result from [**CKNS**] which is beyond the scope of this paper. The important estimate from Theorem 5.1 is due to Cegrell and Persson [**CP**]. Theorem 5.9 comes from [**KO4**].

CHAPTER 6

The Complex Monge-Ampère Equation on a Compact Kähler Manifold

Let us consider a compact n-dimensional Kähler manifold M equipped with the fundamental form

$$\omega = \frac{i}{2}\sum_{k,j} g_{k\bar{j}} dz^k \wedge d\bar{z}^j.$$

By the definition of a Kähler manifold $(g_{k\bar{j}})$ is a positive definite Hermitian matrix and $d\omega = 0$. The volume form associated to the Hermitian metric is given by n-th wedge product of ω multiplied by $1/n!$. For the introduction to Kähler manifolds we refer to [**D3**].

We shall study the Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f\,\omega^n, \tag{6.1}$$

where φ is the unknown function such that $\omega + dd^c\varphi$ is a nonnegative (1,1) form. The given nonnegative function $f \in L^1(M)$ is normalized by the condition

$$\int_M f\,\omega^n = \int_M \omega^n.$$

Since, by the Stokes theorem, the integral over M of the right hand side is equal to $\int_M \omega^n$, this normalization is necessary for the existence of a solution.

Equation (6.1) has the following geometrical meaning when f is smooth and positive. Given the volume form $f\omega^n$ on M we look for a Kähler metric (represented by the fundamental form $\omega + dd^c\varphi$) which yields this volume form. More interestingly, as a short calculation shows (see [**A1**] [**A2**] [**TI**] [**Y**]), equation (6.1) arises when given a closed $(1,1)$ form τ representing the first Chern class of M we want to find a Kähler form ω' such that $\tau = Ricc(\omega')$ (the Ricci form of ω') and ω' lives in the same Chern class as τ. E. Calabi conjectured that this is always possible. He also proved in [**C**] the uniqueness of such ω' which is equivalent to the fact that any two solutions of (6.1) differ by a constant. The Calabi conjecture was confirmed by S.-T. Yau [**Y**] who proved the following theorem.

THEOREM 6.1. *Let $f > 0, f \in C^k(M), k \geq 3$. Then there exists a solution to* (6.1) *belonging to Hölder class $C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$.*

For its proof we refer to [**Y**]. In this section we shall generalize the existence part of this result.

Preliminaries

We work throughout on a compact Kähler manifold M with a fundamental form ω and assume

$$\int_M \omega^n = 1.$$

Denote by $||\cdot||_p$ the norm in $L^p(M)$ for $p \in [1, \infty]$.

For the sake of brevity we shall write

$$\omega_\varphi = \omega + dd^c\varphi$$

and call a continuous function φ *ω-plurisubharmonic (ω-psh* in short) if $\omega_\varphi \geq 0$. The set of such functions will be denoted by $PSH(\omega)$.

If in an open subset of M there exists a potential function v satisfying $\omega = dd^c v$ then for ω-psh φ the function $v + \varphi$ is a true plurisubharmonic function. Thus such properties of plurisubharmonic functions as Hartogs' lemma or the theorem saying that the weak convergence implies convergence in L^p_{loc} (see e.g. [**H2**]) hold also for ω-psh functions. The same goes for the convergence theorems from Chapter 1.

For a Borel set $E \subset M$ one can define a capacity

$$cap_\omega(E) = \sup\{\int_E \omega_\varphi^n : \varphi \in PSH(\omega), 0 \leq \varphi \leq 1\}.$$

Let us consider two open finite coverings $\{V_s\}$, $\{V_s'\}$, $s = 1, 2, ..., N$ of M such that $\bar{V}_s' \subset V_s$ and in each strictly pseudoconvex V_s there exists $v_s \in PSH(V_s)$ with $dd^c v_s = \omega$ and $v_s = 0$ on ∂V_s. Given a compact set $K \subset M$ define $K_s = K \cap \overline{V_s'}$. We are going to show that $cap_\omega(K)$ is comparable with $cap_\omega'(K) = \sum_s cap(K_s, V_s)$, where $cap(K, V)$ denotes the relative capacity from Chapter 1. We know that

$$\begin{aligned} cap(K, V) &:= \sup\{\int_K (dd^c u)^n, u \in PSH(V), u \leq 0, u \leq -1 \quad \text{on } K\} \\ &= \int_K (dd^c u^*_{K,V})^n, \end{aligned}$$

For fixed s put $\varphi_s = u_{K_s,V_s} - v_s$. Then $\varphi_s \geq -1$ on K_s and $\varphi_s = 0$ on ∂V_s. One can find $\psi_s \in PSH(\omega) \cap C^\infty(M)$ such that $\psi_s = 0$ outside V_s and $\psi_s \leq -3\delta < 0$, $\delta < 1/2$, on V_s' with the same δ for all s. (To see this just take any smooth ψ_s' which is equal to 0 outside V_s and negative on V_s' and choose $\epsilon > 0$ so small that $\psi_s = \epsilon\psi_s'$ is ω-psh.) Take χ_s which is equal to $\max(\delta\varphi_s - \delta, \psi_s)$ on V_s and equal to 0 elsewhere on M. Note that this function is ω-psh and equal to $\delta\varphi_s - \delta$ on a neighbourhood of K_s. Therefore

$$\begin{aligned} \int_{K_s} \omega_{\chi_s}^n &= \int_{K_s} [\delta\omega_{\varphi_s} + (1-\delta)\omega]^n \\ &\geq \delta^n \int_{K_s} \omega_{\varphi_s}^n = \delta^n \int_{K_s} (dd^c u^*_{K_s,V_s})^n = \delta^n cap(K_s, V_s). \end{aligned}$$

So

$$cap_\omega(K) \geq cap_\omega(K_s) \geq \delta^n cap(K_s, V_s).$$

By the definitions,

$$cap_\omega(K) \leq (C_1+1)^n \sum_s cap(K_s, V_s),$$

where C_1 is chosen so that $v_s \geq -C_1$ for any s. Indeed, let $\varphi \in PSH(\omega)$ with $-1 < \varphi \leq 0$. Then $(C_1+1)^{-1}(\varphi+v_s)$ is a competitor in the definition of $cap(K_s, V_s)$. So

$$\begin{aligned}(C_1+1)^{-n} \int_K \omega_\varphi^n &\leq (C_1+1)^{-n} \sum_s \int_{K_s} \omega_\varphi^n \\ &\leq \sum_s cap(K_s, V_s)\end{aligned}$$

which proves the preceding inequality. Thus we finally obtain

$$\frac{\delta^n}{N} cap'_\omega(K) \leq cap_\omega(K) \leq (C_1+1)^n cap'_\omega(K). \tag{6.2}$$

A sequence φ_j of functions defined in M is said to converge with respect to capacity to φ if for any $t > 0$

$$\lim_{j\to\infty} cap_\omega(\{|\varphi - \varphi_j| \geq t\}) = 0.$$

The following lemma is shown in [**D3**] and the proof relies on Richberg's approximation theorem [**RI**].

LEMMA 6.2. *If $\varphi \in PSH(\omega)$ and γ is a continuous $(1,1)$ form on M such that $dd^c\varphi \geq \gamma$ (as currents) then given $\delta > 0$ one can find a smooth function ψ satisfying $\varphi < \psi < \varphi + \delta$ and $dd^c\psi \geq \gamma - \delta\omega$ on M.*

Next lemma is well known for smooth forms. We need a more general version (cf. [**D2**]).

LEMMA 6.3. *Suppose $g \in L^1(M)$ and $\varphi, \psi \in PSH(\omega)$ satisfy*

$$\omega_\varphi^n \geq g\omega^n, \quad \omega_\psi^n \geq g\omega^n.$$

Then $\omega_\varphi^k \wedge \omega_\psi^{n-k} \geq g\omega^n$.

PROOF. The statement is local and so it is equivalent to the following one: *For $u, v \in PSH(B) \cap C(\overline{B})$ (B - a ball in $\mathbb{C}^n$) satisfying*

$$(dd^c u)^n \geq g\, dV, \quad (dd^c v)^n \geq g\, dV$$

we have $(dd^c u)^k \wedge (dd^c v)^{n-k} \geq g\, dV$, where $g \in L^1(B)$.

For smooth u, v and $g > 0$ it is a well known matrix inequality which follows from concavity of the mapping $A \to \log \det^{1/n} A$ defined on the set of positive definite Hermitian matrices (see Corollary 7.6.9 in [**HJ**]). If $u, v \in C^{1,1}$ then $dd^c u$ and $dd^c v$ can be evaluated pointwise almost everywhere and so the statement follows in that case too. Next we shall prove it for $g \in L^2(B)$. Let g_j be a sequence of smooth functions, positive on $\overline{B}$ and tending in $L^2(B)$ to g. Fix also two sequences f_j, h_j of smooth functions on ∂B such that $f_j \to u$ and $h_j \to v$ uniformly on ∂B.

Applying Theorem 3.5 one can find $u_j, v_j \in PSH(B) \cap C^{1,1}(\overline{B})$ solving the Dirichlet problems for the Monge-Ampère equation

$$
\begin{aligned}
(dd^c u_j)^n &= g_j dV, \quad u_j = f_j \text{ on } \partial B \\
(dd^c v_j)^n &= g_j dV, \quad v_j = h_j \text{ on } \partial B.
\end{aligned}
$$

By Theorem 4.5 u_j and v_j tend uniformly to u and v respectively. Hence one can apply the convergence theorem and the statement for $C^{1,1}$ functions to the effect

$$
\begin{aligned}
(dd^c u)^k \wedge (dd^c v)^{n-k} &= \lim_{j\to\infty} (dd^c u_j)^k \wedge (dd^c v_j)^{n-k} \\
&\geq \lim_{j\to\infty} g_j \, dV = g \, dV.
\end{aligned}
$$

For the general case we take an increasing sequence $g_j \uparrow g$ with $g_j \in L^2(B)$ and repeat the above argument using Theorem 4.6 to solve the suitable Dirichlet problems. Now the convergence of the approximating sequences is not uniform, but the sequences are decreasing due to the comparison principle. So the convergence theorem still applies in this case.

Comparison principle

Now we shall prove the comparison principle for the Monge-Ampère operator on compact Kähler manifolds.

THEOREM 6.4. *If φ and ψ are ω-psh on M then for $\Omega = \{\varphi < \psi\}$ we have*

$$\int_\Omega \omega_\psi^n \leq \int_\Omega \omega_\varphi^n.$$

PROOF. Suppose first that φ, ψ and the boundary of Ω are smooth. Set $\varphi_t = \max(\varphi + t, \psi),\ t > 0$. Then close to $\partial\Omega$ we have $\varphi_t = \varphi + t$. Define the (closed) current

$$T_t = \sum_{k=1}^{n} \binom{n}{k} (dd^c \varphi_t)^{k-1} \wedge \omega^{n-k}$$

and set $T = \lim_{t\to 0} T_t$. By Stokes' theorem

$$
\begin{aligned}
\int_\Omega \omega_{\varphi_t}^n &= \int_\Omega dd^c \varphi_t \wedge T_t + \omega^n = \int_{\partial\Omega} d^c \varphi_t \wedge T_t + \int_\Omega \omega^n \\
&= \int_{\partial\Omega} d^c \varphi \wedge T + \int_\Omega \omega^n = \int_\Omega \omega_\varphi^n.
\end{aligned}
$$

Since $\varphi_t \downarrow \psi$ in Ω as $t \to 0$ we get applying the convergence theorem that

$$\omega_{\varphi_t}^n \to \omega_\psi^n \quad \text{in } \ \Omega.$$

Hence for a test function χ in Ω with $0 \leq \chi \leq 1$ we get

$$\int_\Omega \chi \omega_\psi^n = \lim_{t\to 0} \int_\Omega \chi \omega_{\varphi_t}^n \leq \underline{\lim}_{t\to 0} \int_\Omega \omega_{\varphi_t}^n.$$

So

$$\int_\Omega \omega_\psi^n \leq \underline{\lim}_{t\to 0} \int_\Omega \omega_{\varphi_t}^n = \int_\Omega \omega_\varphi^n,$$

which completes the proof for smooth functions. Suppose now that φ and ψ are continuous and that they satisfy the extra assumption

$$dd^c\varphi \geq (\delta-1)\omega, \quad dd^c\psi \geq (\delta-1)\omega, \tag{6.3}$$

for some $\delta > 0$. Then, applying Lemma 6.2, one can find two sequences of ω-psh functions φ_j and ψ_j converging uniformly to φ and ψ respectively. Given a compact set $K \subset \Omega$ we find $t > 0$ and a positive integer j_0 such that $K \subset \Omega(t,j) = \{\varphi_j < \psi_j - t\} \subset \Omega$ for $j > j_0$ and the boundary of $\Omega(t,j)$ is smooth (using Sard's theorem). Applying the first part of the proof and the convergence theorem we obtain

$$\int_K \omega_\psi^n \leq \underline{\lim}_{j\to\infty} \int_{\Omega(t,j)} \omega_{\psi_j}^n \leq \underline{\lim}_{j\to\infty} \int_{\Omega(t,j)} \omega_{\varphi_j}^n \leq \int_\Omega \omega_\varphi^n.$$

Exhausting Ω by compact sets we get the desired inequality in this case. It remains to get rid of the extra assumption. Note that for fixed $t \in (0,1)$ and ω-psh functions φ, ψ the functions $t\varphi$ and $t\psi$ satisfy (6.3) for some $\delta > 0$. Fix a compact set $K \subset \Omega = \{\varphi < \psi\}$ and consider $\delta > 0$ and $t \in (0,1)$ such that $K \subset \Omega(\delta, t) = \{\varphi < \psi - \delta/t\}$. By the above and the convergence theorem we have

$$\int_K \omega_\psi^n \leq \underline{\lim}_{t\to 1} \int_{\Omega(\delta,t)} \omega_{t\psi}^n \leq \underline{\lim}_{t\to 1} \int_{\Omega(\delta,t)} \omega_{t\varphi}^n \leq \int_\Omega \omega_\varphi^n.$$

Again to complete the proof it is enough to consider an exhaustion sequence of compact subsets of Ω.

L^∞ ESTIMATES

Consider a family of functions

$$\mathcal{F}(A,h) = \{f \in L^1(M) : f \geq 0,\ \int_M f\omega^n = 1,$$
$$\int_E f\omega^n \leq F(cap_\omega(E)) \ \text{ for any Borel set } E \subset M\},$$

where $F(x) = \frac{Ax}{h(x^{-1/n})}$, with $A > 0$ and admissible $h : \mathbb{R}_+ \to [1,\infty)$ (see Chapter 4). By extension, we also call F admissible. Our goal is to prove the existence of continuous solutions of equation (6.1) for $f \in \mathcal{F}(A,h)$. We first prove an analogue of Lemma 4.3 for compact Kähler manifolds. Only the beginning of the proof requires some modification.

LEMMA 6.5. *Let φ and ψ be ω-psh functions on M with $0 \leq \psi \leq C$. Assume that $\{\varphi - S < \psi\}$ is nonempty. Suppose that for some positive number A and an admissible function h the following inequality holds*

$$\int_K \omega_\varphi^n \leq F(cap_\omega(K)), \quad \textit{with} \quad F(x) = \frac{Ax}{h(x^{-1/n})}, \quad A > 0, \tag{6.4}$$

for any compact set K. Then for $D < 1$ we have

$$D \leq \kappa(a(S+D)),$$

where

$$a(s) := cap_\omega(U(s)), \quad U(s) := \{\varphi - s < \psi\},$$

and

$$\kappa(s) = c(n)A^{1/n}(1+C)[\int_{s^{-1/n}}^{\infty} x^{-1}h^{-1/n}(x)\,dx + h^{-1/n}(s^{-1/n})].$$

Proof. Set for $s \in [S, S+D]$

$$b(s) = \int_{U(s)} \omega_\varphi^n.$$

First we shall prove the inequality

$$t^n a(s) \leq b(s+t+Ct) \quad \text{for } t < 1, 0 < t < \frac{S+D-s}{C+1}. \tag{6.5}$$

Indeed, take $\rho \in PSH(\omega)$ with $-1 \leq \rho \leq 0$ and put $V(s) := \{\varphi - s - t - Ct < t\rho + (1-t)\psi\}$. One easily verifies that $U(s) \subset V(s) \subset U(s+t+Ct)$. We can now apply the comparison principle (Theorem 6.4) to obtain

$$\begin{aligned} t^n \int_{U(s)} \omega_\rho^n &\leq \int_{V(s)} (t\omega_\rho + (1-t)\omega_\psi)^n \\ &\leq \int_{V(s)} \omega_\varphi^n \leq \int_{U(s+t+Ct)} \omega_\varphi^n = b(s+t+Ct). \end{aligned}$$

Taking supremum over ρ we get (6.5). The rest of the proof goes on exactly the same way as in the proof of Lemma 4.3.

Corollary. *The family of ω-psh functions such that $\omega_\varphi^n \in \mathcal{F}(A,h)$ and $\max_M \varphi = 0$ is uniformly bounded.*

Proof. For ω-psh function φ we have $\Delta_\omega \varphi \geq -n$. So, using the representation of φ in terms of the Green function on M (see e.g. [**A2**]) we get

$$0 = \max_M \varphi \leq \int_M \varphi\omega^n + C_0,$$

with C_0 depending only on M. Having this L^1 estimate we can use Proposition 1.10 coupled with (6.2) to obtain

$$cap_\omega(U(\varphi, j)) \leq C_1/j, \quad U(\varphi, j) := \{\varphi < -j\},$$

where C_1 does not depend on φ. Now we apply Lemma 6.5 with $\psi = 0$ and S chosen so that

$$\kappa(C_1/S) < 1$$

and conclude that $U(\varphi, S+1)$ must be empty for any φ from the family we consider.

LEMMA 6.6. *If the sequence φ_j is uniformly bounded and*

$$\omega^n_{\varphi_j} = f_j \omega^n,$$

with $||f_j - f||_1 < 2^{-j-1}$ then $\varphi := (\limsup_{j\to\infty} \varphi_j)^*$ solves the equation (6.1).

PROOF. Let us introduce some auxiliary functions

$$\varphi_{kl} = \max_{k\le j\le l} \varphi_j, \quad \psi_k = (\lim_{l\to\infty} \uparrow \varphi_{kl})^*,$$
$$F_{kl} = \min_{k\le j\le l} f_j, \quad G_k = \lim_{l\to\infty} \downarrow F_{kl}.$$

Since, locally, ω is representable by $dd^c v$, where v is a plurisubharmonic function, one can apply Theorem 1.18 to get

$$(\omega + dd^c \varphi_{kl})^n \ge F_{kl}\omega^n.$$

Hence, by the convergence theorem

$$G_k\omega^n \le \lim_{l\to\infty} (\omega + dd^c\varphi_{kl})^n = (\omega + dd^c\psi_k)^n. \tag{6.6}$$

Note that $\varphi = \lim_{k\to\infty} \downarrow \psi_k$, so one can apply the convergence theorem once more to get

$$(\omega + dd^c\psi_k)^n \to \omega^n_\varphi. \tag{6.7}$$

From the assumptions we have $||f - G_k||_{L^1(M)} \le \frac{1}{2^k}$, so $G_k \to f$ in $L^1(M)$. Therefore applying (6.6) and (6.7) one obtains

$$\omega^n_\varphi \ge f\omega^n.$$

Since the integrals over M of both currents in the above inequality are equal to $\int_M \omega^n$ we finally arrive at

$$\omega^n_\varphi = f\omega^n.$$

Thus the lemma follows.

We are now in a position to generalize Theorem 6.2.

THEOREM 6.7. *If h is admissible and $1 \in \mathcal{F}(A, h)$, then for any $f \in \mathcal{F}(A, h)$ there exists a continuous solution of* (6.1). *Moreover there exists $a(A, h) > 0$ such that any solution of*

$$\omega^n_\varphi = f\,\omega^n, \quad \max_M \varphi = 0,$$

with $f \in \mathcal{F}(A, h)$ satisfies $\varphi \ge -a(A, h)$.

PROOF. Suppose first that f is bounded. Then one can find $f_j \in C^\infty(M), 0 < f_j < N, \int_M f_j\omega^n = 1$ and f_j converging in L^1 to f. Since $1 \in \mathcal{F}(A,h)$ we have $f_j \in \mathcal{F}(NA,h)$. Applying Yau's theorem we find ω-psh solutions of

$$\omega^n_{\varphi_j} = f_j\omega^n, \quad \max_M \varphi_j = 0.$$

By Corollary to Lemma 6.5 φ_j are uniformly bounded which allows us to use Lemma 6.6 and conclude that $\varphi = (\limsup \varphi_j)^*$ solves (6.1). For general f construct $f_j = t_j g_j$, where $g_j = \min(f,j)$ and $t_j > 0$ is chosen so that $\int_M f_j\omega^n = 1$. Since $f \in L^1(M)$ we have $\lim_{j\to\infty} t_j = 1$ and so for j big enough $f_j \in \mathcal{F}(2A,h)$. Therefore the ω-psh solutions of

$$\omega^n_{\varphi_j} = f_j\omega^n, \quad \max_M \varphi_j = 0$$

are uniformly bounded (see Corollary to Lemma 6.5). Again, Lemma 6.6 says that $\varphi = (\limsup \varphi_j)^*$ solves

$$\omega^n_\varphi = f\omega^n.$$

The uniform bound for sup norms of the solutions follows from the corollary to Lemma 6.5. The proof is finished.

Define

$$L^\psi(c_0) = \{f \in L^1(M) : f \geq 0, \int_M f\,\omega^n = 1, \int_\Omega \psi(f)\,\omega^n \leq c_0\}.$$

and recall that in our notation

$$\psi_h(t) = |t|(log(1+|t|))^n h(log(1+|t|)),$$

for some admissible h. In analogy to Lemma 4.2 we have the following inclusion

THEOREM 6.8. *For any admissible h, with $h(x) \leq (1+x)^k,\ k > 0$, and $c_0 > 0$ there exists $A > 0$ such that*

$$L^{\psi_h}(c_0) \subset \mathcal{F}(A,h).$$

PROOF. Fix $f \in L^{\psi_h}(c_0)$ and a compact $K \subset M$. Consider the covering V_s, the sets K_s and numbers δ, N as in Preliminaries of this chapter. We can assume that $\int_{K_s} f\,\omega^n \leq \int_{K_1} f\,\omega^n$. The following chain of inequalities is obtained by using the properties of h, Lemma 4.2 and (6.1).

$$\begin{aligned}\int_K f\,\omega^n &\leq \sum_{s=1}^N \int_{K_s} f\,\omega^n \leq N\int_{K_1} f\,\omega^n \\ &\leq A_0 F(cap(K_1,V_1)) \leq AF(cap_\omega(K)),\end{aligned}$$

where $F(x) = \frac{Ax}{h(x^{-1/n})}$. The proof is complete.

Combining the last two results one obtains the following corollary.

COROLLARY 6.9. *For admissible h and $f \in L^{\psi_h}(c_0)$ we can solve*

$$\omega^n_\varphi = f\,\omega^n, \quad -C \leq \varphi \leq 0,$$

where C depends on h and c_0.

Uniqueness and stability of the solutions

The uniqueness (up to an additive constant) of solutions of the Monge-Ampère equation on compact Kaähler manifolds has been proved by Calabi [**C**] in the case of smooth data. The result holds for data belonging to $\mathcal{F}(A,h)$. It follows from a stability estimate which we are now about to prove. Let us fix $\mathcal{F}(A,h)$ for some admissible h. From Theorem 6.7 it follows that there exists a constant denoted by $a(A,h)$ such that for any $f \in \mathcal{F}(A,h)$ the ω-psh solution φ of

$$\omega_\varphi^n = f\omega^n, \quad \max_M \varphi = 0,$$

satisfies $\varphi \geq -a(A,h)$. We shall denote by $\kappa_{A,h}$ the function κ from Lemma 6.5 with $C = a(3A,h)$. So

$$\kappa_{A,h}(s) = c(n)A^{1/n}(1 + a(3A,h))[\int_{s^{-1/n}}^{\infty} x^{-1}h^{-1/n}(x)\,dx + h^{-1/n}(s^{-1/n})].$$

We shall need an estimate similar to the one given in Theorem 4.4.

Lemma 6.10. *Let φ and ψ be ω-psh functions on M with $0 \leq \varphi \leq C-1$ and let*

$$\omega_\psi^n = g\omega^n.$$

Then

$$cap_\omega(\{\psi + 2s < \varphi\}) \leq C^n s^{-n} \int_{\{\psi+s<\varphi\}} g\omega^n.$$

Proof. Denote

$$E_j(s) = \{\psi + s < \varphi\} \quad \text{and} \quad a_j := cap_\omega(E_j(2s)).$$

Take $\rho \in PSH(\omega)$ with $-1 \leq \rho \leq 0$ and set $V := \{\psi < \frac{s}{C}\rho + (1-\frac{s}{C})\varphi - s\}$. Then

$$E_j(2s) \subset V \subset E_j(s).$$

By the comparison principle for $s < C$ we have

$$\begin{aligned} \frac{s^n}{C^n}\int_{E_j(2s)} \omega_\rho^n &\leq \int_V (\frac{s}{C}\omega_\rho + (1-\frac{s}{C})\omega_\varphi)^n \\ &\leq \int_V \omega_{\varphi_j}^n \leq \int_{E_j(s)} g\omega^n. \end{aligned}$$

Taking supremum over ρ one gets

$$\frac{s^n}{C^n} a_j \leq \int_{E_j(s)} g\omega^n.$$

The proof is finished.

THEOREM 6.11. *Let us consider two functions f and g belonging to $\mathcal{F}(A,h)$ for some admissible h with $1 \in \mathcal{F}(A,h)$ and the corresponding solutions of*

$$\omega_\varphi^n = f\omega^n, \quad \omega_\psi^n = g\omega^n,$$

normalized by

$$\max_M(\varphi - \psi) = \max_M(\psi - \varphi).$$

Set $q = q(n) = (\frac{3}{2})^{1/n}$ and define an increasing function on $\mathbb{R}_+$ by

$$\gamma(t) = \frac{(2a(3A,h))^n}{(a(3A,h)+1)^n} \frac{q-1}{3} \kappa_{A,h}^{-1}(t)$$

(where $\kappa_{A,h}^{-1}(t)$ denotes the inverse of $\kappa_{A,h}(t)$). Then the inequality

$$||f - g||_1 \leq \gamma(t)t^{n+3}$$

implies

$$||\varphi - \psi||_\infty \leq (4a(3A,h)+2)t$$

for $t < t_0$ with $t_0 > 0$ depending on γ.

PROOF. Put $a = a(3A,h)$. One can assume that

$$\int_{\{\psi<\varphi\}} (f+g)\omega^n \leq 1 \tag{6.8}$$

since otherwise we may interchange the roles of φ and ψ. Since $\lim_{t\to 0}\gamma(t) = 0$ one can fix $t_0 < (q-1)/2$ and such that $\gamma(t_0)t_0^{n+3} < 1/3$. From now on we shall work with fixed $t < t_0$. Denote by E_k the set $\{\psi < \varphi - kat\}$ and put

$$C_0 = \int_{E_2} g\omega^n.$$

Then by (6.8) and the assumptions

$$\begin{aligned}\int_{E_0} g\omega^n &= \frac{1}{2}\int_{E_0}[(f+g)+(g-f)]\omega^n \\ &\leq \frac{1}{2}(1+\gamma(t_0)t_0^{n+3}) \leq \frac{2}{3}.\end{aligned} \tag{6.9}$$

Define ω-psh function ρ as the solution of

$$\omega_\rho^n = g_1\omega^n, \quad \max_M \rho = 0,$$

where $g_1 = (3/2)g$ on E_0 and g_1 is equal to a constant $c_0 \geq 0$ elsewhere, with c_0 chosen so that $\int_M g_1\omega^n = 1$. (Observe that (6.9) implies $c_0 \geq 0$.) Since

$$\int_E g_1\omega^n \leq \int_E [(3/2)g+1]\omega^n$$

and $1 \in \mathcal{F}(A, h)$ the solution ρ belongs to $\mathcal{F}(3A, h)$ and so

$$\rho \geq -a.$$

Adding the same constant to φ and ψ (which does not influence neither the hypothesis nor the assertion) one can assume that

$$-a \leq \varphi \leq 0.$$

The last two inequalities entail

$$E_2 \subset E := \{\psi < (1-t)\varphi + t\rho - at\} \subset E_0.$$

Let us denote by G the set $\{f < (1-t^2)g\}$. From Lemma 6.3 we know that for $k \leq n$ the following inequalities hold on $E_0 \setminus G$:

$$\omega_\varphi^k \wedge \omega_\rho^{n-k} \geq q^{n-k}(1-t^2)^{k/n} g\omega^n.$$

Therefore on $E_0 \setminus G$

$$\begin{aligned} \omega^n_{t\rho+(1-t)\varphi} &= \sum_{k=0}^{n} \binom{n}{k} (1-t)^k t^{n-k} \omega_\varphi^k \wedge \omega_\rho^{n-k} \\ &\geq [(1-t)(1-t^2)^{1/n} + qt]^n g\omega^n \geq [(1-t)(1-t^2) + qt]^n g\omega^n \\ &\geq [1 + t(q-1) - t^2] g\omega^n \geq [1 + \frac{t}{2}(q-1)] g\omega^n, \end{aligned} \tag{6.10}$$

where the last inequality follows from $t < t_0 < (q-1)/2$.

By the assumptions we have

$$t^2 \int_G g\omega^n \leq \int_G (g-f)\omega^n \leq \gamma(t) t^{n+3}.$$

Hence

$$\int_G g\omega^n \leq \gamma(t) t^{n+1}. \tag{6.11}$$

The following chain of inequalities is obtained by applying, in turn, formula (6.10), the comparison principle, and formula (6.11):

$$\begin{aligned} [1 + \frac{t}{2}(q-1)] \int_{E\setminus G} g\omega^n &\leq \int_E \omega^n_{t\rho+(1-t)\varphi} \leq \int_E g\omega^n \\ &\leq \int_{E\setminus G} g\omega^n + \gamma(t) t^{n+1}. \end{aligned} \tag{6.12}$$

We infer from (6.12) that

$$\frac{q-1}{2} \int_{E\setminus G} g\omega^n \leq \gamma(t) t^n.$$

Therefore

$$\frac{q-1}{2}(C_0 - \gamma(t)t^{n+1}) \leq \frac{q-1}{2}(\int_{E_2} g\omega^n - \int_G g\omega^n) \leq \gamma(t)t^n,$$

and further

$$C_0 \leq (t + \frac{2}{q-1})\gamma(t)t^n \leq \frac{3}{q-1}\gamma(t)t^n.$$

We also have from Lemma 6.10

$$cap_\omega(E_4) \leq \frac{(a+1)^n}{(2at)^n} \int_{E_2} g\omega^n.$$

So coupling the last two estimates one obtains

$$\begin{aligned} cap_\omega(E_4) &\leq (2ta)^{-n}(a+1)^n C_0 \leq (a+1)^n (2a)^{-n} \frac{3}{q-1}\gamma(t) \\ &\leq (a+1)^n (2a)^{-n} \frac{3}{q-1}\gamma(t). \end{aligned}$$

Suppose $E' = \{\psi < \varphi - (4a+2)t\}$ were nonempty. Then by Lemma 6.5, the above estimate and the definition of γ we would have

$$2t \leq \kappa_F(cap_\omega(E_4)) \leq \kappa_F((a+1)^n (2a)^{-n} \frac{3}{q-1}\gamma(t)) = t,$$

which is a contradiction. Therefore E' is empty which translates into the desired estimate

$$\max(\psi - \varphi) = \max(\varphi - \psi) \leq (4a+2)t.$$

The proof is completed.

It follows from Theorem 6.11 that for $f \in \mathcal{F}(A, h)$ with admissible h the solution of (6.1), normalized by $\max_M \varphi = 0$, is unique.

Corollary. *If φ_1 and φ_2 solve*

$$\omega_{\varphi_1}^n = f\omega^n = \omega_{\varphi_2}^n$$

with $f \in \mathcal{F}(A, h)$ for some admissible h then $\varphi_1 - \varphi_2 = const.$

Example. We shall make the estimate from Theorem 6.11 more explicit for $h(x) = x^n$. Then $\psi_h(t) = |t| \log^{2n}(1 + |t|)$. The function $\kappa_{A,h}$ is easy to compute:

$$\kappa_{A,h}(t) = const.(At)^{1/n}$$

and so $\gamma(t) = Ct^n$ with C depending on A. Therefore, by Theorem 6.11, suitably normalized ω-psh solutions of the equations

$$\omega_\varphi^n = f\omega^n, \quad \omega_\psi^n = g\omega^n$$

for $f, g \in L^{\psi_h}(c_0)$ satisfy

$$||\varphi - \psi||_\infty \leq c||f - g||_1^{1/(2n+3)},$$

with c depending on c_0.

Notes. The results are taken from [**KO3**] and [**KO6**].

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