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Classification of overtwisted contact structures on 3-manifolds

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Introduction

A contact structure on a $(2n + 1)$-dimensional manifold is a codimension 1 tangent distribution which can be defined (at least locally) by a 1-form α with $\alpha \wedge (d\alpha)^n$ nowhere 0. In this paper we will study a special class: *overtwisted* contact structures on 3-manifolds. Examples of overtwisted structures were studied by Lutz [11], Gonzalo and Varela [10], Erlandsson [5], and Bennequin [1]. We give here the complete classification of overtwisted structures. For example, it follows from the classification that all known examples of non-standard (but homotopically standard) contact structures on S^3 are equivalent.

The paper has the following organization. In Sect. 1 we give basic definitions and formulate main results. In Sect. 2 we prove miscellaneous lemmas needed for the main theorem. In Sect. 3 we prove the main theorem, and in Sect. 4 discuss open questions around the subject.

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1. Basic definitions and the statement of the main theorem

1.1. The induced foliation on a surface in a contact 3-manifold

Let M be a 3-dimensional contact manifold with a contact structure ξ and $S \subset M$ be a surface tangent to ξ at a set $\Sigma \subset S$. Intersections of ξ with tangent planes to S define a one-dimensional distribution on $S \setminus \Sigma$. Its integral curves form

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a one-foliation S_ξ on S with singularities at Σ. Leaves of the foliation are by construction legendrian curves of ξ (i.e. curves tangent to ξ).

1.2. The sign of a contact structure

A contact structure on a 3-manifold M defines an orientation because the sign of the form $\alpha \wedge d\alpha$ does not depend on the sign of α. If M is connected and already oriented then we call the contact structure *positive* if the two orientations coincide, and call it *negative* in the opposite case. Furthermore, we will consider only positive contact structures. The term *contact structure* will always denote *positive contact structure.* Negative contact structures could be considered in the same way.

1.3. Contact structures on $\mathbb{R}^3$

We will consider two special contact structures on $\mathbb{R}^3$: the *standard* structure ζ_0 and the *standard overtwisted* structure ζ_1, which are defined, respectively, in cylindric coordinates (ρ, ϕ, z) by equations

$$dz + \rho^2 d\phi = 0 \qquad \text{and} \qquad \cos\rho\, dz + \rho \sin\rho\, d\phi = 0\,.$$

We will also denote by ζ_0 the standard contact structure on S^3 formed by complex tangent lines to the boundary of the unit ball in $\mathbb{C}^2$. At the complement of a point $p \in S^3$ these two standard structures ζ_0 are equivalent. The contact structure on $\mathbb{R}^3$ which is equivalent to ζ_0 can also be defined by the equation $dz - ydx = 0$ in cartesian coordinates (x, y, z). Note that for both structures ζ_0 and ζ_1 the rays $\{z = c_1, \phi = c_2\}$ perpendicular to the z-axis, are legendrian. The planes of the distributions ζ_0 and ζ_1 turn around these rays when they move along them away from the z-axis. The angle of turning is always less than $\pi/2$ for ζ_0 and goes to infinity for ζ_1.

Denote by Δ the disc $\{z = 0, \rho \leqq \pi\} \subset \mathbb{R}^3$. The disc Δ with the germ of the contact structure ζ_1 on it will be called the *standard overtwisted disc.* The boundary $\partial\Delta$ is a legendrian curve for ζ_1 and the structure ζ_1 is tangent to Δ along $\partial\Delta$. Thus Δ is not in general position to ζ_1. But for any $\varepsilon > 0$ the embedded disc $\Delta^\varepsilon = \{z = \varepsilon\rho^2, \rho \leqq \pi\}$ is close to Δ, is tangent to ζ_1 only at the origin, and has the induced foliation (see Fig. 1) with only one singular point of focus type and with one limit cycle at the boundary.

1.4. Overtwisted structures

A contact structure ξ on a connected 3-manifold M is called *overtwisted* if there is a contact embedding of the standard overtwisted disc (Δ, ζ_1) into (M, ξ). As follows from 1.2, an overtwisted structure contains an embedded 2-disc $\mathscr{D}$ such that the induced foliation $\mathscr{D}_\xi$ has the form as in Fig. 1. It is easy to see that the converse is also true. If a contact manifold contains a disc with the foliation as in Fig. 1, then it is overtwisted.

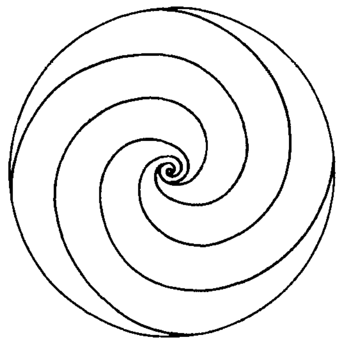

Fig. 1

Any contact structure can be "spoiled" and made overtwisted using so-called *Lutz twisting* (see [10] or [1] for definitions and discussions) which is a surgery of the structure (but not of the manifold) along a closed transversal. It is possible to make Lutz twisting without changing the homotopy class of the contact structure as a plane distribution.

1.5. The spaces Distr(M), Cont(M), *and* Cont$^{ot}(M)$

Let M be an oriented connected 3-manifold. Let us fix a point $p \in M$ and an embedded 2-disc $\Delta \subset M$ centered at the point p. Denote by Distr(M) the space of all tangent 2-plane distributions on M fixed at the point $p \in M$ provided with the C^∞-topology. Denote by Cont(M) the subspace of Distr(M) which consists of (positive) contact structures and by Cont$^{ot}(M)$, the subspace of Cont(M) consisting of all overtwisted structures which have the disc $\Delta \subset M$ as the standard overtwisted disc. If M is closed then any two contact structures belonging to the same component of Cont(M) are isotopic (see [7]). Any overtwisted structure on M is evidently isotopic to a structure from Cont$^{ot}(M)$.

Consider the inclusions

$$i\colon \text{Cont}(M) \hookrightarrow \text{Distr}(M)$$
$$j\colon \text{Cont}^{ot}(M) \hookrightarrow \text{Distr}(M)$$

R. Lutz proved [11] that the mapping $i_*\colon \pi_0(\text{Cont}(M)) \to \pi_0(\text{Distr}(M))$ is surjective. D. Bennequin [1] showed that this mapping is not injective in the case $M = S^3$. Namely, he proved that the standard contact structure ζ_0 on S^3 is not overtwisted. Hence it is not equivalent to (and does not belong to the same component of Cont(S^3)) any overtwisted structure on S^3 which we can get by Lutz twisting of ζ_0.

1.6. The main results

Theorem 1.6.1. *The inclusion* j: Cont$^{ot}(M) \to$ Distr(M) *is a homotopy equivalence.*

In particular, two (positive) overtwisted structures on a closed M are isotopic if they are homotopic as plane distributions. If M is open then 1.6.1 follows immediately from the classical theorem of Gromov [8]. A non-trivial result for non-closed manifolds is the extension theorem (see 3.1.1 below).

Corollary 1.6.2. *There exists an overtwisted structure ξ on S^3, a ball $B \subset S^3$ and a contact embedding $\psi: (B, \xi) \to (S^3, \xi)$ such that ψ cannot be extended to a contact diffeomorphism $S^3 \to S^3$ and cannot be connected with the inclusion $B \subsetneq S^3$ by a contact isotopy.*

Proof of 1.6.2. Any contact isotopy of a submanifold in a contact manifold can be extended to a contact diffeotopy of the whole manifold. Thus it is enough to prove only the first assertion. Let us take S^3 with the standard structure ζ_0 and make the Lutz twisting of ζ_0 along a closed transversal inside a smal ball $B \subset S^3$. The resulting overtwisted structure ζ can be made to belong to the same component of $\mathrm{Distr}(S^3)$ as ζ_0. Now take ζ and make the same perturbation inside another small ball $B' \subset S^3 \setminus B$. The new contact structure ζ' is also overtwisted and by 1.6.1 there is an isotopic to the identity contact diffeomorphism h: $(S^3, \zeta) \to (S^3, \zeta')$. Let $B'' = h(B)$. By the construction there exists a contact diffeomorphism g: $(B', \zeta') \to (B, \zeta)$. Now I claim that the contact diffeomorphism $h \circ g$: $(B', \zeta') \to (B'', \zeta')$ cannot be extended to a contact diffeomorphism $(S^3, \zeta') \to (S^3, \zeta')$. Indeed, Lutz *untwisting* in B'' moves ζ' into ζ while untwisting in B' transforms ζ' into ζ_0. But ζ is equivalent to ζ' and is not equivalent to ζ_0.

2. Miscellaneous lemmas

In this section we define notions and prove propositions needed to prove 1.6.1.

2.1. Contact structures near 2-surfaces

2.1.1. Simple and almost horizontal foliations. A one-dimensional oriented foliation on S^2 with two singular points of focus type is said to be *simple* if all its limit cycles are isolated and placed on parallels between the two focuses (see Fig. 2) and if one of its focuses is stable and the other is unstable. We will call the focuses north and south poles respectively.

A simple foliation $\mathscr{F}$ on S^2 is called *almost horizontal* if there is a transversal ℓ to $\mathscr{F}$ connecting its poles.

If an oriented contact structure ξ, defined in a neighborhood of a sphere S embedded in a 3-manifold, generates a simple or almost horizontal foliation S_ξ on S, then we say that the contact structure itself is simple or almost horizontal near S.

2.1.2. Diffeomorhism of holonomy. For an almost horizontal foliation $\mathscr{F}$ on S^2 the holonomy along the leaves defines a diffeomorphism of the transversal and hence (defined up to a conjugacy) a diffeomorphism $h(\mathscr{F})$: $I \to I$ of the unit interval $I = [0, 1]$. Given a diffeomorphism $h: I \to I$, we denote by $\mathscr{F}(h)$ the almost horizontal foliation with $h(\mathscr{F}(h)) = h$.

If $\mathcal{F}$ is simple but not almost horizontal, then the holonomy is defined near the poles and the limit cycles.

2.1.3. The diagram of a simple foliation. A limit cycle of a simple foliation $\mathcal{F}$ divides the sphere S into two hemispheres: the lower one (containing the south pole) and the upper one. We call the limit cycle *positive* if it is oriented as the boundary of lower hemisphere and negative in the opposite case. Let us order limit cycles of $\mathcal{F}$ in the direction from the south to the north poles: $\ell_1, \dots, \ell_p$. The *diagram* $\mathcal{D}(\mathcal{F})$ of the simple foliation $\mathcal{F}$ consists of p points $q_1, \dots, q_p$ on a line with the following additional structure: to each point q_i, $i = 1, \dots, p$, is assigned the sign of the limit cycle ℓ_i; each interval $q_i q_{i+1}$, $i = 1, \dots, p$, is oriented from q_i to q_{i+1} (resp. from q_{i+1} to q_i) if the cycle ℓ_i is unstable (resp. stable) for the foliation $\mathcal{F}$ restricted to the band between ℓ_i and ℓ_{i+1} (see Fig. 3).

The following proposition is a corollary of the Poincaré-Bendixson theorem.

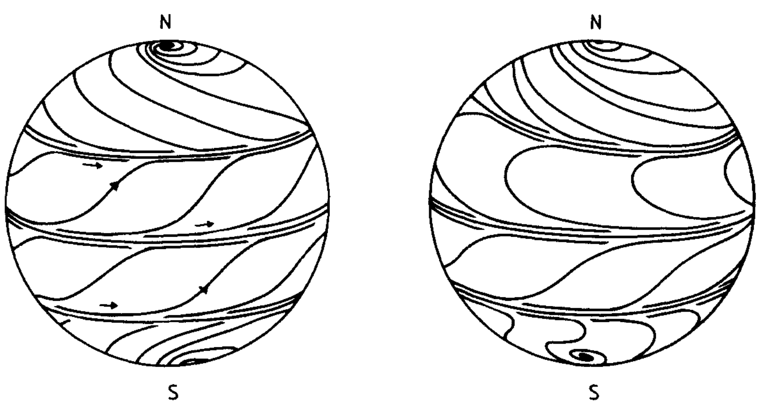

Fig. 2. Simple and almost horizontal foliations

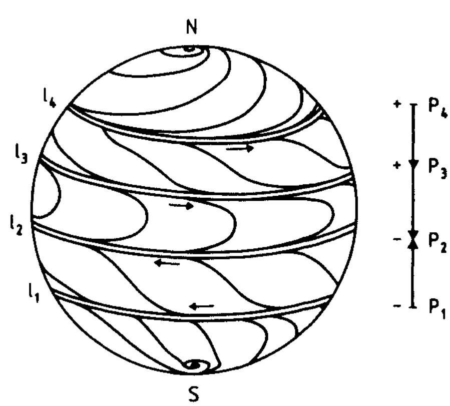

Fig. 3. A simple foliation and its diagram

2.1.3.1. *The topological type of simple foliation $\mathscr{F}$ is uniquely defined by its diagram $\mathscr{D}(\mathscr{F})$.*

2.1.4. Connected sums of simple foliations. For two simple foliations $\mathscr{F}$ on S and $\mathscr{F}'$ on S' we define the connected sum $\mathscr{F}\#\mathscr{F}'$ on $S\# S'$ as follows. Take small discs $\mathscr{D}\subset S$ on $\mathscr{D}'\subset S'$ which contain, respectively, the north pole of $\mathscr{F}$ and the south pole of $\mathscr{F}'$ whose boundaries $\partial\mathscr{D}$ and $\partial\mathscr{D}'$ are transverse to $\mathscr{F}$ and $\mathscr{F}'$. Now attach $S'\backslash\mathscr{D}'$ to $S\backslash\mathscr{D}$ along $\partial\mathscr{D}$ and $\partial\mathscr{D}'$ and smooth the resulting foliation on $S\# S'$. The procedure defines $\mathscr{F}\#\mathscr{F}'$ uniquely up to a homeomorphism.

2.1.5. Extendability of contact structures

Lemma 2.1.5.1. *Let ξ be a simple contact structure near the boundary $S=\partial B$ of a 3-ball B. The extendability of ξ as a contact structure to B depends only on the topological type of the foliation S_ξ.*

Proof. Let ξ' be another contact structure near S. Without loss of generality we can think that ξ and ξ' have the same poles and limit cycles and that ξ and ξ' are spherical near poles (i.e. contactomorphic to round spheres in the standard contact $(\mathbb{R}^3,\zeta_0)$. Let t and t' be closed transversal to ξ which is close to poles of S. Let L be the union of limit cycles of ξ (and ξ') and transversals t and t'. Denote by N a small tubular neighborhood of L. There exists a diffeomorphism $g\colon S\backslash N\to S\backslash N$ which moves $\xi'|_N$ into $\xi|_N$. We can extend g to a diffeomorphism $g'\colon S\to S$ which is constant on L and such that the foliation $\mathscr{F}=g'(S_{\xi'})$ is C^1-close to S_ξ (C^1-closeness near poles depends on how close to the poles the transversals t and t' are chosen). Let $\tilde{B}$, $\tilde{B}\supset B$, be a larger ball on which the structure ξ is still defined. Let us show that there exists an embedding $\tilde{g}\colon S\to\tilde{B}$ which is C^0-close to the inclusion $S\subsetneq B$, is the identity on L and outside a small neighborhood of L and such that the foliation $(\tilde{g}(S))_\xi$ is diffeomorphic to $\mathscr{F}$. Near each limit cycle ℓ the contact structure ξ is equivalent to the standard overtwisted structure ζ_1 near the circle $C=\{\rho=1, z=0\}$. The S in coordinates (ρ,σ,z) is transverse to the planes $\{z=\text{const}\}$ and the semiplanes $\{\rho=\text{const}\}$ near the circle C. The leaf of the foliation $\mathscr{F}$ through a given point $p=(\rho_0,\sigma_0,z_0)\in S$ can be defined by an equation $z=f(\phi)$. If $|f'(\phi_0)|$ is sufficiently small (and this is true near C) then the equation $\rho\,tg\,\rho=f'(\phi_0)$ has the unique solution $h(P)$ near $\rho=\pi$. The desired embedding $\tilde{g}$ is defined near C by the formula $\tilde{g}(P)=(h(P),\phi_0,z_0)$. Similarly we can construct $\tilde{g}$ near transversals t and t'. It is easy to see that $\tilde{g}$ induces the foliation $\mathscr{F}$ on S and is the identity where $\mathscr{F}$ coincides with S_ξ. To finish the proof, note that the diffeomorphism of foliations $\tilde{g}g'_{\xi'}\colon S'\to(\tilde{g}_\xi(S))$ can be covered by a contact diffeomorphism of germs of ξ' and ξ on S and $\tilde{g}(S)$. Because the structure $\xi|_{\tilde{g}(S)}$ is already extended to the ball, the same is true for $\xi'|_S$.

2.1.6. The contact structure near the standard overtwisted disc

Let B be the ball of radius $5\pi/4$ with the center at the origin in $\mathbb{R}^3$ provided with the contact structure

$$\zeta_1=\{\cos\rho\,dz+\rho\sin\rho\,d\phi=0\}\,.$$

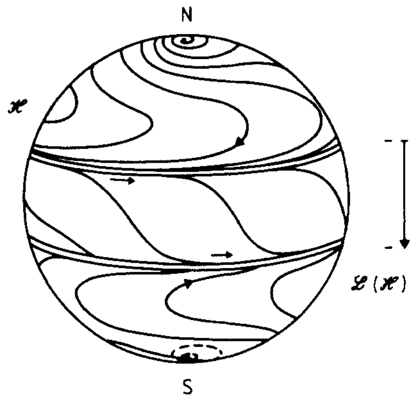

Fig. 4

Denote by $\mathscr{H}$ the foliation $(\partial B)_{\zeta_1}$ induced by ζ_1 on ∂B (see Fig. 4). It has exactly two limit cycles at the intersection of ∂B with the cylinder $\{\rho = \pi\}$. In fact, for any convex surface of revolution around the z-axis which is contained in the cylinder of radius $< 2\pi$ and contains the overtwisted disc Δ inside itself, the induced foliation on it is homeomorphic to $\mathscr{H}$. Hence for any contact embedding h of the overtwisted disc Δ into the contact manifold M, the image $h(\Delta)$ has an arbitrarily small neighborhood bounded by a sphere with the induced foliation homeomorphic to $\mathscr{H}$.

2.2. The existence of a family of transversals

Let ξ be an oriented contact structure in an oriented manifold M. An oriented transversal ℓ to ξ is said to be *positive* or *negative* according to the sign of the orientation of M defined by ξ and ℓ.

Lemma 2.2.1. *Let K be a compact space, M be an oriented 3-manifold, ξ_t, $t \in K$, be a family of oriented contact structures on M, and ϕ_t: $I \to M$, $t \in K$, be a family of embeddings. Then C^0-close to ϕ_t, $t \in K$, there exists a family of embeddings $\tilde{\phi}_t$ with $\tilde{\phi}_t|_{\partial I} = \phi_t|_{\partial I}$, $t \in K$, which consists of any of the following:* (a) *positive transversals,* (b) *negative transversals,* (c) *legendrian* (i.e. *tangent to contact structures*) *curves.*

Proof. Any family of legendrian curves can be C^∞-approximated by a family of positive as well as negative transversals (see, for example, [1]), with the same endpoints. Hence it is enough to prove the existence of the legendrian C^0-approximation. First consider the case when $M = \mathbb{R}^3$ and $\xi_t = \zeta_0$ for all $t \in K$. The structure ζ_0 can be defined in cartesian coordinates (x, y, z) by the 1-form $dz - ydx$. Denote by π and ρ projections $(x, y, z) \mapsto (x, y)$ and $(x, y, z) \mapsto z$, respectively. The desired legendrian family $\tilde{\phi}_t$: $I \to \mathbb{R}^3$, $t \in K$ will be constructed by the following procedure which we call the *legendrization*. First take a family of wave fronts γ_t, $t \in K$, which C^0-approximates the projections $\pi \circ \phi_t$ (see Fig. 5). Denote by $k_t(u)$,

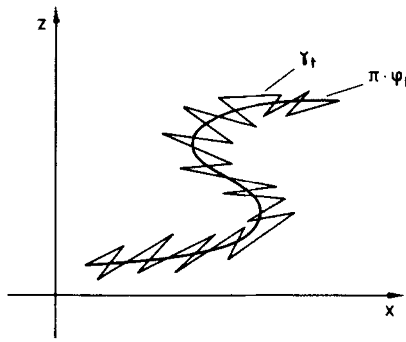

Fig. 5

$t\in K$, $u\in I$, the slope of the tangent line to γ_t at the point $\gamma_t(u)$. The family of legendrian curves $\tilde{\phi}_t\colon I\to\mathbb{R}^3$, $t\in K$, with fronts γ_t, $t\in K$, C^0-approximate ϕ_t, $t\in K$, if and only if the family of functions $k_t\colon I\to\mathbb{R}$ is C^0-close to $p\circ\phi_t$, $t\in K$. It is clear that this property can be satisfied for all $t\in K$, if the number of cusps of fronts γ_t, $t\in K$, is sufficiently large. If ϕ_t, $t\in K$, was already legendrian near ∂I then we can make $\tilde{\phi}_t$ coincide with ϕ_t near ∂I for all $t\in K$.

Consider now the case of general M. Denote by B the unit ball in $\mathbb{R}^3$ with the standard contact structure ζ_0. There exists a finite number of points $0=U_0<U_1<\ldots<U_n=1$ and a family h_t^i, $t\in K, i=1,\ldots,n$, of contact embeddings $B\to M$ such that $h_t^i(B)\supset\phi_t([U_{i-1},U_i])$, $t\in K$, $i=1,\ldots,n$. Consequently applying the legendrization procedure to $\phi_t|_{[U_{i-1},U_i]}$, $t\in K$, for $i=1,\ldots,n$ we will get the desired approximation.

2.3. Construction of a contact structure near the 2-skeleton of a general simplicial complex

A compact simplicial complex $P\subset\mathbb{R}^3$ will be called *general* if no two faces or a face and an edge with a mutual vertex are contained in one plane. For a general simplicial complex P we denote by $\alpha(P)$ the minimal angle between non-incident 1- or 2-simplices which have a mutual vertex. Denote by $d(P)$ the maximal diameter of a simplex of P and by $\delta(P)$ the minimal distance between two 0-, 1- or 2-simplices without mutual vertices.

Proposition 2.3.1. *There exists a sequence of general subdivisions P_i of P such that $d(P_i)\to 0$ while $\delta(P_i)/d(P_i)$ and $\alpha(P_i)$ are bounded below by a positive $\varepsilon>0$.*

Proof. Consider a fine cubic subdivision of P and triangulate this subdivision without creating new vertices. Now take a pattern $3\times 3\times 3$ of this subdivision, move its 64 vertices into general position and extend the perturbation periodically to the entire subdivision. It is clear that the described procedure allows us to obtain an arbitrarily fine subdivision with an à priori bounded number of shapes of simplices and of stars of vertices and hence with an à priori lower bound for α and δ/d.

Proposition 2.3.2. *Let $\mathscr{F}$ be a two-dimensional foliation on the closure $\bar{U}$ of a bounded domain $U \subset \mathbb{R}^3$. Let K be a compact space and L be its closed subset. Let ξ_t, $t \in K$, be a family of 2-plane distributions on $\bar{U}$ transversal to $\mathscr{F}$. Suppose that ξ_t is contact near a closed $A \subset U$ for all $t \in K$ and is contact everywhere for $t \in L$. Suppose that for any leaf V of $\mathscr{F}$ and any leaf ℓ of the foliation V_{ξ_t}, $t \in K$, we have $\pi_1(\ell, A \cap \ell) = 0$. Then there exists a family ξ'_t, $t \in K$, of contact structures on $\bar{U}$ such that ξ'_t coincides with ξ_t on A for all $t \in K$ and coincides with ξ everywhere for $t \in L$.*

Recall that all contact structures we consider are supposed to be positive.

Proof. For $t \in K$ let G_t be the one-dimensional tangent to ξ_t foliation on $\bar{U}$ formed by all leaves of foliations V_{ξ_t}, when V runs through all leaves of $\mathscr{F}$. In $\mathbb{R}^3$ with the standard structure ζ_0, defined by the one-form $dz - ydx$, we denote by $\mathscr{L}$ the legendrian foliation by lines parallel to the y-axis. Let us take a covering $\bigcup_{i=1}^{N} U_t^i = \bar{U}$ continuously depending on $t \in K$ and such that U_t^i, $i = 1, \ldots, N$, consists of whole leaves of G_t and there exists a (continuously depending on $t \in K$) embedding h_t^i: $U_t^i \to \mathbb{R}^3$ with $(h_t^i)^* \mathscr{L} = G_t|_{U_t^i}$. In coordinates induced by the embedding h_t^i the distribution $\xi_t|_{U_t^i}$ can be defined by a 1-form $P_t(x, y, z)\, dx + Q_t(x, y, z)\, dz$. When a point (x, y, z) moves along a leaf ℓ of G_t the point $(P_t(x, y, z), Q_t(x, y, z))$ draws a curve ℓ' in $\mathbb{R}^2$. The contactness of ξ_t means that for any $t \in K$ and any leaf ℓ of G_t the curve $\tilde{\ell}$ is nowhere tangent to rays from the origin. Because of the condition $\pi_1(\ell, A \cap \ell) = 0$, the distribution ξ_t is allowed to be perturbed at least near one end of any intervals of $\ell \setminus A$. Hence the required property of $\tilde{\ell}$ can be easily satisfied and we will get the desired family ξ'_t of contact structures by consequent perturbations of ξ_t on U_t^i for $i = 1, \ldots, N$.

A 2-plane distribution ξ on a compact $A \subset \mathbb{R}^3$ defines the Gauss mapping G_ξ: $A \to S^2$. The number $\|\xi\| = \max_{x \in A} \|dg(x)\|$ will be called the *norm* of the distribution ξ.

Note 2.3.3. Suppose that for A, B, $\varepsilon > 0$ the norm of plane distribution tangent to $\mathscr{F}$ from 2.3.2 is less than A, the minimal angle between ξ_t and $\mathscr{F}$ is greater than ε, and the diameter of U is less than B. Then the above construction allows us to construct ξ'_t, $t \in K$, to satisfy the inequality $||\xi'_t|| \leqq ||\xi_t|| + D$ where C and D depend only on A, B and ε.

Lemma 2.3.4. *Let K be a compact space and L its closed subset. Let ξ_t, $t \in K$, be a family of distributions defined near a compact $B \subset \mathbb{R}^3$ which are contact near closed $A \subset B$ for $t \in K$ and are contact everywhere for $t \in L$. Then there exists a general simplicial complex $P \supset B$ and a family of distributions ξ'_t, $t \in K$, with the following properties:*

(1) *ξ'_t is C^0-close to ξ_t, $t \in K$;*

(2) *ξ'_t coincides with ξ_t on A for $t \in K$ and everywhere for $t \in L$;*

(3) *there exists $\varepsilon > 0$ depending only on $\alpha(P)$ and $\delta(P)/d(P)$ such that ξ'_t, $t \in K$, is contact in $\varepsilon \cdot d(P)$-neighborhood of the 2-skeleton of P;*

(4) *$\|\xi'_t\| \leqq C\|\xi_t\| + D$, $t \in K$, for universal constants C and D.*

Proof. We can choose a general simplicial complex $P \supset B$ and its subcomplex $Q \supset A$ such that for all $t \in K$ the distribution ξ_t is still defined on P and contact on Q. We can suppose moreover that each simplex of $P \backslash Q$ has at most one face belonging to Q.

Let $N = \max_{t \in K} \|\xi_t\| + 1$, $\delta = \delta(P)$, $d = d(P)$, $\alpha = \alpha(P)$, $k = \delta/d$. According to 2.3.1 we can make d arbitrarily small without changing α and k. Denote by $\gamma(\pi_1, \pi_2)$ the angle between two planes or lines π_1 and π_2 in $\mathbb{R}^3$. Consider a covering $\bigcup_1^p K_i$ of K such that for each $i = 1, \ldots, p$ and for any $t, t' \in K_i$ and $x \in P$,

$$\gamma(\xi_t(x), \xi_{t'}(x)) < \frac{\alpha}{16}. \tag{*}$$

Because of the extension character of the assertion of 2.3.4, we can suppose that (*) holds for all $t, t' \in K$. Let us call a 1- or 2-simplex σ of $P \backslash Q$ *special* if $\gamma(\sigma, \xi_t(x)) < \frac{\alpha}{4}$ for some $t \in K$, $x \in \sigma$. If d is sufficiently small $\left(\text{for example, } d < \frac{\alpha}{16N}\right)$, then for any *non-special* simplex σ, any point x belonging to a 3-simplex incident to σ and any $t \in K$ we have

$$\gamma(\sigma, \xi_t(x)) > \frac{\alpha}{8}. \tag{**}$$

Near each simplex σ of $P \backslash Q$ of dimension $\leqq 2$ consider a foliation $\mathscr{F}_\sigma$ by planes which are perpendicular to $\xi_t(x)$ for some $t \in K$, $x \in \sigma$, parallel to σ if $\dim \sigma = 1$ and perpendicular to σ if $\dim \sigma = 2$.

Given foliations $\mathscr{F}_\sigma$, we will construct using 2.3.2 the required perturbation ξ'_t of ξ_t, $t \in K$, in three steps. First we change ξ_t in a neighborhood of all special simplexes and all vertices which do not belong to special simplexes. The hypotheses of 2.3.2 are satisfied because special simplexes are isolated and any simplex has at most one face belonging to Q. Consequently we now extend the perturbation to a neighborhood of non-special 1- and 2-simplexes. It is easy to see that hypotheses of 2.3.2 will be fulfilled in the εd-neighborhood of a nonspecial simplex σ if $\varepsilon < \frac{K}{2} \sin \frac{\alpha}{8}$ for $\dim \sigma = 1$ and if $\varepsilon < \frac{K}{4} \sin^2 \frac{\alpha}{8}$ for $\dim \sigma = 2$. Note that by 2.3.3 we can control the increase of the norm of ξ_t, $t \in K$, through the described three perturbations. Thus there exists a constant C such that in each step the norm of the resulting family of distributions is $\leqq CN$. Choosing d sufficiently small we can make perturbed distributions C^0-close enough to ξ_t, $t \in K$, to satisfy inequalities (*) and (**). So we can apply previous arguments at each step as well as at the beginning.

2.4. Two simple lemmas

In this section we formulate two simple assertions.

Lemma 2.4.1. *Let σ be a 3-simplex of diameter d. Then for any $\lambda > 0$ there exists an embedded ball $B \subset \sigma$ such that its boundary ∂B is contained in λ-neighborhood of $\partial\sigma$ and the normal curvatures of ∂B are everywhere $\geqq 8\lambda/(4\lambda^2 + d^2)$.*

Lemma 2.4.2. *Let $S \subset \mathbb{R}^3$ be an embedded 2-sphere with all normal curvatures $\geqq K > 0$ and let ξ be a contact structure near S with $\|\xi\| < K$. Then ξ is almost horizontal near S.*

3. Proof of 2.6.1

3.1. The extension theorem

Theorem 2.6.1 follows immediately from the following:

Theorem 3.1.1. *Let M be a compact 3-manifold and let A, $A \subset M$, be a closed subset such that $M \setminus A$ is connected. Let K be a compact space and L, $L \subset K$, a closed subspace. Let ξ_t, $t \in K$, be a family of 2-plane distributions which are contact everywhere for $t \in L$ and are contact near A for $t \in K$. Suppose there exists an embedded 2-disc $\Delta \subset M \setminus A$ such that ξ_t is contact near Δ and (Δ, ξ_t) is equivalent to the standard overtwisted disc (Δ, ζ_0) for all $t \in K$. Then there exists a family ξ'_t, $t \in K$, of contact structures on M such that ξ'_t coincides with ξ_t near A for $t \in K$ and coincides with ξ_t everywhere for $t \in L$. Moreover ξ'_t, $t \in K$ can be connected with ξ_t, $t \in K$ by a fixed on $A \times K \cup M \times L$ homotopy through families of distributions.*

The proof of 3.1.1 is contained in the next three sections.

3.2. The contactization with holes

Lemma 3.2.1. *Under the hypotheses of* 3.1.1 there *exist disjoint* 3*-balls* $B_1, \ldots, B_N \subset M \setminus (A \cup \Delta)$ *and a family of distributions* $\tilde{\xi}_t$, $t \in K$, *on M such that*
(1) $\tilde{\xi}_t$ *coincides with* ξ_t *on* $A \cup \Delta$ *for* $t \in K$ *and everywhere for* $t \in L$;
(2) $\tilde{\xi}_t$, $t \in K$, *is contact on* $M \setminus \cup \operatorname{Int} B_i$;
(3) $\tilde{\xi}_t$, $t \in K$, *is almost horizontal near* ∂B_i, $i = 1, \ldots, N$;
(4) $\tilde{\xi}_t$ *is* C^0*-close to* ξ_t, $t \in K$.
(5) $(B_i, \tilde{\xi}_t)$ *for* $t \in L$ *and* $i = 1, \ldots, N$ *is isomorphic to a convex ball in* $(\mathbb{R}^3, \zeta_0)$.

Proof. Because this is an extension problem, it is enough to consider the case when M is a compact domain in $\mathbb{R}^3$. Now we can apply 2.3.4 and find a family $\tilde{\xi}_t$, $t \in K$, defined on a general simplicial complex P containing M with the following properties:

a) $\tilde{\xi}_t$ coincides with ξ_t on a subcomplex $Q \supset A \cup \Delta$ for $t \in K$ and coincides with ξ_t everywhere for $t \in L$;
b) there exists an $\varepsilon > 0$ which does not depend on $d = d(P)$ and such that $\tilde{\xi}_t$, $t \in K$, is contact in $\varepsilon \cdot d$-neighborhood of the 2-skeleton of P;
c) $\|\tilde{\xi}_t\| < \dfrac{8\varepsilon}{d(4\varepsilon^2 + 1)}$, $t \in K$.

Let $\sigma_1, \ldots, \sigma_N$ be the sequence of all 3-simplices of $P \backslash Q$. By 2.4.1 for each $i = 1, \ldots, N$ there exists a ball $B_i \subset \sigma_i$ whose boundary ∂B_i is contained in the εd-neighborhood of $\partial\sigma$ and has all normal curvatures greater than $\dfrac{8\varepsilon}{d(4\varepsilon^2 + 1)}$. Now, applying 2.4.2, w see that $\tilde{\xi}_t$, $t \in K$, is almost horizontal near ∂B_i for $i = 1, \ldots, N$.

3.3. *Making one hole*

We now want to connect balls $B_1, \ldots, B_N$ from 3.2.1 to get contact structures on M with only one hole.

Lemma 3.3.1. *Let $\tilde{\xi}_t$, $t \in K$, be the family constructed in* 3.2.1. *Denote by $\mathscr{F}^i_t$ the foliation $(\partial B_i)_{\tilde{\xi}_t}$ induced by $\tilde{\xi}_t$ on ∂B_i $(t \in K, i = 1, \ldots, N)$. Let $\mathscr{H}$ be the foliation on the boundary of a small neighborhood of the standard overtwisted disc Δ (see* 2.16) *and let B be the 3-ball. Then there exists a family of embeddings $h_t \colon B \to M \backslash \Delta$, $t \in K$, such that for all $t \in K$, $h_t(B) \supset \bigcup_{i=1}^N B_i \cup \Delta$ and the foliation $G_t = (h_t(\partial B))_{\tilde{\xi}_t}$ induced by $\tilde{\xi}_t$ on the sphere $h_t(\partial B)$ is homeomorphic to the connected sum $\mathscr{H} \# (\#_{i=1}^N \mathscr{F}^i_t)$; for $t \in L$ the embedding h_t defines a contact isomorphism of (B, ζ_1) and $(h_t(B), \xi_t)$.*

Proof. Let $M_1 = M \backslash (\bigcup_1^N \operatorname{Int} B_i \cup \Delta)$. Let B_0, $B_0 \supset \Delta$, be the ball in M_1 such that the foliation $(\partial B_0)_{\tilde{\xi}_t}$ is homeomorphic to $\mathscr{H}$. Let $M'_1 = M_1 \backslash \operatorname{Int} B_0$. Let us orient $\tilde{\xi}_t|_{M'_1}$ if it is possible or somehow orient $\tilde{\xi}_t$ near $\bigcup_{i=0}^N \partial B_i$ if $\tilde{\xi}_t|_{M'_1}$ is not orientable. Let us connect the north pole of $\mathscr{H}$ with the south pole of $\mathscr{F}^1_t$ by an embedded curve $\ell^0_t \subset M'_1$ and then consequently for $i = 1, \ldots, N-1$ connect the north pole of $\mathscr{F}^i_t$ with the south pole of $\mathscr{F}^{i+1}_t$ by embedded disjoint curves $\ell^i_t \subset M'_1$. If $\xi_t|_{M'_1}$ is unorientable, we should take care that the orientation of $\tilde{\xi}_t|_{\bigcup_1^N \partial B_i}$ could be extendable on $\tilde{\xi}_t|_{\bigcup_1^N \partial B_i \cup \ell^i_t}$. In view of 2.2.1 we can make all ℓ^i_t, $i = 0, \ldots, N$, transverse to the contact structure $\tilde{\xi}_t$, $t \in K$. Taking the connected sum of balls B_i, $i = 0, \ldots, N$, along transversals ℓ^i_t, $i = 1, \ldots, N$, we will get the desired family of embedded balls $h_t(B) \subset M \backslash \Delta$, $t \in K$. The structure $\xi_t|_{h_t(B)}$ for $t \in L$ can be easily arranged to be standard because of the special choice of balls $B_1, \ldots, B_N$.

Note 3.3.2. Because of the foliation $\#_{i=1}^N \mathscr{F}^i_t$, $t \in K$, is almost horizontal, it can be defined up to a homeomorphism by the family of holonomy diffeomorphisms $\psi_t \colon I \to I$, $t \in K$. Hence the family of foliations G_t, $t \in K$, is homeomorphic to $\mathscr{H} \# \mathscr{F}(\psi_t)$, $t \in K$. For $t \in L$ the diffeomorphism ψ_t has no interior fixed points but for $t \in K \backslash L$ it can have isolated interior fixed points.

3.4. *The model*

Let β:$[0, 1] \to (-\pi/4, \pi/4)$ be a function with isolated zeros and with $\beta(0)$, $\beta(1) < 0$. Let γ_β be a curve in the plane with coordinates (ρ, z) as shown in Fig. 6 and $S_\beta \subset \mathbb{R}^3$ be the surface of revolution of γ_β around the z-axis. Let ζ_1 be the standard overtwisted contact structure in $\mathbb{R}^3$ (see 1.3). In cylindric coordinates

(ρ, ϕ, z) it is defined by the equation

$$\cos \rho \, dz - \rho \sin \rho \, d\phi = 0 \, .$$

Denote by $\mathscr{H}_\beta$ the foliation $(S_\beta)_{\zeta_1}$ induced on S_β by ζ_1 (see Fig. 7). If β has no zeros then $\mathscr{H}_\beta$ is homeomorphic to $\mathscr{H}$.

Proposition 3.4.1. *Let f_t: $I \to I$, $t \in K$, be a family of diffeomorphisms with isolated fixed points inside, fixed at ∂I and satisfying the condition $f(x) > x$ near ∂I. Let $g_t(x) = \pi/4\ (f_t(x) - x)$ for $t \in K$, $x \in I$. $g_t(0)$, $g_t(1) < 0$ for all $t \in K$. Then the family $\mathscr{H}_{g_t}$, $t \in K$, is topologically equivalent to the family $\mathscr{H} \# \mathscr{F}(f_t)$, $t \in K$.*

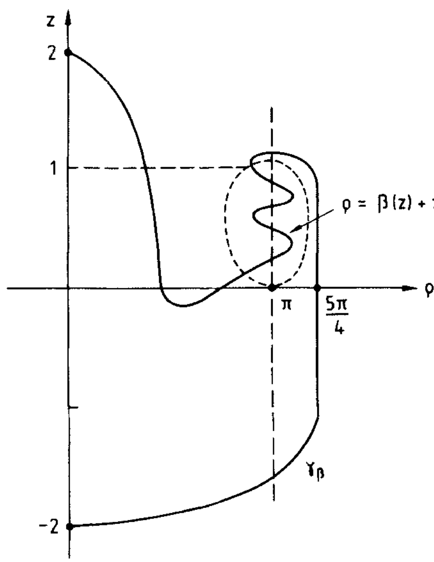

Fig. 6

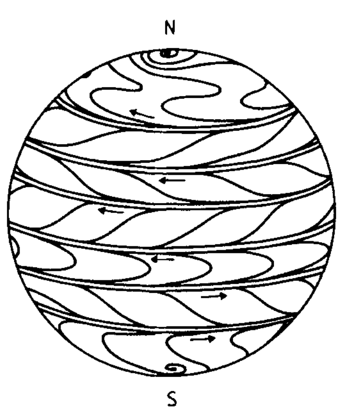

Fig. 7

According to 2.1.3.1, it is enough to show that $\mathscr{H}_{g_t}$ and $\mathscr{H} \# \mathscr{F}(f_t)$, $t \in K$, have coinciding families of diagrams. It is straightforward to check this.

3.5. *End of proof of 3.1.1*

In view of 3.2.1 and 3.3.1 there exists a family of distributions $\tilde{\xi}_t$, $t \in K$, and a family of embeddings $h_t \colon B \to M \setminus A$, $t \in K$, such that $\tilde{\xi}_t$ is C^0-close to ξ_t, $t \in K$; $\tilde{\xi}_t$ coincides with ξ_t on A for $t \in K$ and coincides with ξ_t everywhere for $t \in L$; $\tilde{\xi}_t$ is contact outside of $h_t(B) \subset M \setminus A$, $t \in K$; and the foliation $(h_t(\partial B))_{\tilde{\xi}_t}$, $t \in K$, is homeomorphic to $\mathscr{H} \# \mathscr{F}(\psi)$ where $\psi_t \colon I \to I$ is a diffeomorphism with finite number of fixed points. By 3.4.1 the family $\mathscr{H} \# \mathscr{F}(\psi_t)$, $t \in K$, is topologically equivalent to the family $\mathscr{H}_{\beta_t}$, $t \in K$, for a family of functions $\beta_t \colon [-1, 1] \to (-\pi/4, \pi/4)$, $t \in K$, with finite number of zeros. But the structure ζ_1 which induces the foliation $\mathscr{H}_{\beta_t}$ on the sphere S_{β_t} is extended to the ball bounded by S_{β_t}, $t \in K$. Hence by 2.1.5.1 we conclude that the family $\tilde{\xi}_t$, $t \in K$, of contact structures near $h_t(\partial B)$ which induces the family of foliations $(h_t(\partial B))_{\tilde{\xi}_t}$ are extendable to $h_t(\partial B)$ as a family ξ'_t, $t \in K$, of contact structures. The extension can be made to guarantee the existence of the required homotopy between families ξ_t, $t \in K$, and ξ'_t, $t \in K$. Note that in view of 3.3.1, $(h_t(B), \xi_t)$ is isomorphic for $t \in L$ to the standard overtwisted ball (B, ζ_1). But our construction provides the same property for $(h_t(B), \xi'_t)$, $t \in L$. Hence we can choose $\xi_t = \xi'_t$ for $t \in L$.

4. Discussion

In this section I discuss some related notions and open questions around the subject of the paper.

4.1. *CR structure, compatible with a contact structure*

Denote by Conv(M) the space of strictly pseudoconvex CR-structures on M. A strictly pseudoconvex CR structure on M generates the canonical positive contact structure on the oriented manifold M, by the distribution of complex tangent lines. This defines the projection $p \colon \mathrm{Conv}(M) \to \mathrm{Cont}(M)$. It was proved in [2] that p is a homotopy equivalence. In particular, any contact structure on 3-manifolds can be complexified in the above sense.

4.2. *h-fillable and s-fillable contact manifolds*

A (positive) contact structure ξ on an oriented 3-manifold M is called *holomorphically fillable* (or h-fillable) if there exists a two-dimensional complex manifold W with pseudoconvex boundary M such that the CR structure on its boundary defines the original contact structure ξ.

A (positive) contact structure on an oriented 3-manifold M is called *symplectically fillable* (or s-fillable) if there exists a four-dimensional symplectic manifold W which has M as its oriented boundary such that the symplectic form is nondegenerate on the contact distribution.

Denote by $\mathrm{Cont}^h(M)$ and $\mathrm{Cont}^s(M)$ the spaces of h-fillable and s-fillable contact structures respectively.

By a theorem of H. Grauert (see [6]), we have $\mathrm{Cont}^h(M) \subset \mathrm{Cont}^s(M)$ and by a theorem of M. Gromov and the author (see [9] and [3]) the intersection $\mathrm{Cont}^s(M) \cap \mathrm{Cont}^{ot}(M)$ is empty.

I proved recently (see [3]) that the only h-fillable contact structure on S^3 is the standard one. In [4] I showed that this is not true in higher dimensions.

4.3. *Open questions*

Question 4.3.1. It is true that $\mathrm{Cont}^s(M) = \mathrm{Cont}^h(M)$?

Question 4.3.2. Is it true that $\mathrm{Cont}(M) = \mathrm{Cont}^{ot}(M) \cup \mathrm{Cont}^s(M)$? In particular, if a contact structure violates Bennequin's inequality (see [1] and [3]), is it necessarily overtwisted?

Question 4.3.3. Which 3-manifolds admit s- or h-fillable contact structures?

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