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On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space

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1. Introduction

Let M be a symplectic manifold with symplectic form σ and let T be a torus which acts on M in a Hamiltonian way. That is, there is given a linear map $X \mapsto J_X$ from the Lie algebra $\mathfrak{t}$ of T to the space of smooth functions on M, such that

(1.1) For each $X \in \mathfrak{t}$ the infinitesimal action of X on M is given by the Hamiltonian vector field $\tilde{X}$ of the function J_X, and

(1.2) The functions J_X, $X \in \mathfrak{t}$ are in involution.

The mapping $J: M \to \mathfrak{t}^*$ defined by

$$\langle X, J(m) \rangle = J_X(m), \qquad m \in M, \ X \in \mathfrak{t} \tag{1.3}$$

is called the momentum mapping of the Hamiltonian T-action. Given (1.1), the condition (1.2) just means that T acts along the fibers of J.

For the basic definitions and properties of *non*-commutative Hamiltonian group actions, see [AM]. The results of this paper can easily be extended to Hamiltonian actions of arbitrary compact connected Lie groups, by applying our results to the action of its maximal torus and using the equivariance of the momentum mapping. For some more details, see the remarks at the end of Sect. 2.

We will assume throughout this paper that the momentum map is *proper*, that is $J^{-1}(U)$ is compact for each compact subset U of $\mathfrak{t}^*$.

Now let $\xi \in \mathfrak{t}^*$ be a regular value of J, that is $T_m J: T_m M \to \mathfrak{t}^*$ is surjective for all $m \in Y_\xi = J^{-1}(\xi)$. Then Y_ξ is a smooth submanifold of M, compact because

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J is proper, and the action of T on Y_ξ is locally free. Writing i_ξ for the inclusion $Y_\xi \to M$, the T-orbits are just the leaves of the null-foliation of $i_\xi^* \sigma$. So the orbit space $M_\xi = Y_\xi/T$, called the reduced phase space, is provided with a unique symplectic form σ_ξ such that

$$p_\xi^* \sigma_\xi = i_\xi^* \sigma \tag{1.4}$$

where p_ξ denotes the projection $Y_\xi \to M_\xi$.

However, in general the foliation of Y_ξ by T-orbits is not a fibration because the (finite) stabilizer groups

$$T_m = \{g \in T;\ gm = m\} \tag{1.5}$$

may not be locally constant[1]. As a result M_ξ may not be a smooth manifold. But its singularities are of a relatively mild nature. Let Γ_ξ be the finite subgroup of T generated by all T_m, $m \in Y_\xi$. Then we have the diagram

$$\begin{array}{ccccc} M & \xleftarrow{i_\xi} & Y_\xi & \xrightarrow{r_\xi} & Z_\xi = Y_\xi/\Gamma_\xi \\ & & & \underset{p_\xi}{\searrow} & \downarrow q_\xi \\ & & & & M_\xi = Y_\xi/T \end{array} \tag{1.6}$$

where $r_\xi\colon Y_\xi \to Z_\xi$ is a finite branched covering and $q_\xi\colon Z_\xi \to M_\xi$ is a principal T/Γ_ξ-bundle. This exhibits Z_ξ and M_ξ as V-manifolds in the sense of Satake [S]. Such manifolds, although not necessarily smooth, do carry differentiable structures like differential forms, smooth bundles, etc. In particular σ_ξ is a well-defined symplectic form on M_ξ, see [W3]. For our purpose it is also important that the de Rham theorem holds on M_ξ, that is the de Rham cohomology (defined in terms of differential forms on M_ξ) is canonically isomorphic to the Čech cohomology of M_ξ.

Now let ξ_0 be a fixed regular value of J and let ξ vary in a convex open ξ_0-neighborhood U of regular values. Introduce a T-invariant connection for the fibration $J\colon J^{-1}(U) \to U$, which can be obtained by averaging an arbitrary connection for J over T. Note that the connections for J form an affine space, and that T acts along the fibers of J. Through $m \in Y_{\xi_0}$ draw the horizontal curves lying over the straight lines through ξ_0. This defines a T-equivariant projection

$$\eta\colon J^{-1}(U) \to Y_{\xi_0} \tag{1.7}$$

such that for each $\xi \in U$ the restriction $\eta | Y_\xi\colon Y_\xi \to Y_{\xi_0}$ is a T-equivariant diffeomorphism. This induces a smooth family of diffeomorphisms $M_\xi \to M_{\xi_0}$ which allows us to identify the cohomology groups of M_ξ with the ones of the fixed space M_{ξ_0}. Because two such local trivializations differ by maps which are homotopic to the identity, this identification of $H^*(M_\xi)$ with $H^*(M_{\xi_0})$ is canonical. In particular this allows us to compare the cohomology classes $[\sigma_\xi] \in H^2(M_\xi, \mathbb{R}) \cong H^2(M_{\xi_0}, \mathbb{R})$ for various $\xi \in U$.

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[1] If T is a maximal torus in a simple compact connected Lie group G and T acts on a coadjoint orbit of G in $\mathfrak{g}^*$, then this phenomenon does actually occur unless G is of type A

On the other hand the principal T/Γ_ξ-bundle $q_\xi\colon Z_\xi \to M_\xi$ is, up to equivalence, given by an element $v_\xi \in H^1(M_\xi, \underline{T/\Gamma_\xi})$; here $\underline{V}$ denotes the sheaf of germs of continuous V-valued functions. The short exact sequence

$$0 \longrightarrow \Lambda_\xi \longrightarrow \mathfrak{t} \xrightarrow{\exp} T/\Gamma_\xi \longrightarrow 0, \tag{1.8}$$

with

$$\Lambda_\xi = \{X \in \mathfrak{t};\ \exp X \in \Gamma_\xi\} \tag{1.9}$$

being equal to the integral lattice of T/Γ_ξ, leads to the isomorphism

$$0 \longrightarrow H^1(M_\xi, \underline{T/\Gamma_\xi}) \xrightarrow{\delta} H^2(M_\xi, \Lambda_\xi) \longrightarrow 0 \tag{1.10}$$

because the sheaf $\underline{\mathfrak{t}}$ is fine and therefore $H^j(M_\xi, \underline{\mathfrak{t}}) = 0$ for $j = 1, 2$. So the fiber bundle $q_\xi\colon Z_\xi \to M_\xi$ is equally characterized by the element $c = \delta(v_\xi) \in H^2(M_\xi, \Lambda_\xi)$, called the Chern class of the bundle. As a topological class, c is constant as a function of $\xi \in U$. Indeed, the above T-equivariant trivialization of J shows that $\Lambda_\xi = \Lambda_{\xi_0}$ for all $\xi \in U$. Secondly, $\xi \mapsto \delta(v_\xi)$ is continuous and takes values in the fixed lattice

$$H^2(M_\xi, \Lambda_\xi) = H^2(M_\xi, \Lambda_{\xi_0}) \cong H^2(M_{\xi_0}, \Lambda_{\xi_0})$$

in $H^2(M_{\xi_0}, \mathfrak{t})$. We are now able to formulate our main result.

1.1 Theorem. *Let $\xi, \xi_0 \in \mathfrak{t}^*$ lie in the same connected component C of the set of regular values of the momentum map J. Then*

$$[\sigma_\xi] = [\sigma_{\xi_0}] + \langle c, \xi - \xi_0 \rangle \tag{1.11}$$

where $c \in H^2(M_\zeta, \Lambda_\zeta)$ denotes the (common) Chern class of the fibrations $q_\zeta\colon Z_\zeta \to M_\zeta$, $\zeta \in C$, and we have used the canonical identification of the $H^2(M_\zeta, \mathbb{R})$ along any ζ-path in C from ξ_0 to ξ.

The proof of Theorem 1.1 will be given in Sect. 2. In Sect. 3 we discuss its Corollary that the push forward of the canonical measure on M under the momentum map J is a measure on $\mathfrak{t}^*$ which has a *piecewise polynomial density* (assuming that J has regular points in each connected component of M). This Corollary was conjectured in some very stimulating discussions with Atiyah and Guillemin, and was the starting point for our paper. In turn, if M is compact, the property that $J_*(dm)$ has a piecewise polynomial density leads to an explicit formula (4.6) for the oscillatory integral

$$\int_M e^{i\langle X, J(m)\rangle}\, dm, \qquad X \in \mathfrak{t}. \tag{1.12}$$

In the case that T acts on a coadjoint orbit as in footnote [1] this is a well-known formula of Harish-Chandra [H]. He obtained this formula as a consequence of the computation of the radial part of the G-invariant differential operators on $\mathfrak{g}$. Conversely the formula for the radial part can be obtained from the formula (1.12) using the theory of Fourier integrals. For the relation with the characters of irreducible representations of G, see Kirillov [K]. His first proof is close to our approach, see the last remark in our Sect. 2.

Another application of Theorem 1.1 appears if M actually is a Kähler manifold with Kähler form equal to σ. Assuming that $[\sigma]\in H^2(M,\mathbb{Z})$ there is a holomorphic line bundle L over M with Chern class $[\sigma]$. Assuming that T acts holomorphically on L one gets a (finite dimensional) representation ρ of T on the space of holomorphic sections of L. Now assume that T acts freely on the inverse image under J of a connected component C of the set of regular values of J. Take ξ in C which moreover is integral in the sense that

$$t\mapsto \exp 2\pi i\langle \log t,\xi\rangle \tag{1.13}$$

defines a (1-dimensional) representation ρ_ξ of T. The main theorem of [GS2] states that in this situation $[\sigma_\xi]$ is equal to the Chern class of a holomorphic line bundle L_ξ over M_ξ and that the space of holomorphic sections of L_ξ is canonically isomorphic to the space of (ρ_ξ,ρ)-intertwining mappings. In particular its dimension is equal to the multiplicity of ρ_ξ in ρ.

On the other hand the Hirzebruch-Riemann-Roch formula[2] states that

$$\sum(-1)^j \dim H^j(M_\xi,\underline{L}_\xi)=\int_{M_\xi}\tau_\xi e^{[\sigma_\xi]} \tag{1.14}$$

where τ_ξ denotes the Todd class of the tangent bundle of M_ξ (regarded as a complex vector bundle). Being a topological class τ_ξ does not vary for ξ in C, and the conclusion is that the restriction of (1.14) to the integral values in C is a polynomial in ξ of degree less than or equal to $\dim_{\mathbb{C}}(M_\xi)$.

Under suitable positivity conditions the Kodaira vanishing theorem gives that $H^j(M_\xi,\underline{L}_\xi)=0$ for $j>0$, so in the case (1.14) is a formula for the multiplicity of ρ_ξ in ρ, which therefore is a polynomial[3] when restricted to the set of all integral $\xi\in C$.

We finally mention that the principle of Theorem 1.1 has been applied before by Weinstein [W2], [W3]. He considers the circle action of the geodesic flow on the tangent bundle of a Riemannian manifold all of whose geodesics are closed. Here $[\sigma_\xi]\to 0$ as $\xi\to 0$, so that $[\sigma_1]=c$ if we write $\mathfrak{t}^*=\mathbb{R}$. This leads to a strong conclusion about the Riemannian volume of the manifold. In this way Theorem 1.1 can be regarded as a more or less straightforward generalization of the basic idea of [W2], [W3].

2. Proof of the Theorem

For simplicity, write $Y=Y_{\xi_0}$. Using the mapping η of (1.7) we get a trivialization

$$J\times\eta\colon J^{-1}(U)\to U\times Y \tag{2.1}$$

[2] In this form the Hirzebruch-Riemann-Roch formula does not hold for Kähler V-manifolds, being the reason for the assumption that T acts freely on $J^{-1}(C)$. However, both [GS2] and the Hirzebruch-Riemann-Roch formula can probably be modified for Kähler-V-manifolds, making that we could drop the assumption that T acts freely on $J^{-1}(C)$

[3] More generally, if T acts only effectively on M then the Lefschetz fixed point formula (applied to the T-action on M) implies that the multiplicity of ρ_ξ in ρ is polynomial in ξ for those $\xi\in C$ which are congruent modulo the weight lattice of T/Γ_ξ. For the Lefschetz fixed point formula with fixed manifolds rather than fixed points see [AS]

which allows us to replace M by $U \times Y$. Because of the T-equivariance of η, T acts only in the second component in a way which does not depend on the first component. Note that in this trivialization the momentum map is equal to projection onto the first component. All $M_\xi = Y_\xi/T$ are now identified with Y/T. Writing $\Gamma = \Gamma_\xi$ (which does not depend on $\xi \in U$) all $Z_\xi = Y_\xi/\Gamma$ get identified with $Z = Y/\Gamma$, which is a principal T/Γ-bundle over Y/T. The only object which now still depends on ξ is σ_ξ, determined by

$$p^* \sigma_\xi = i_\xi^* \sigma. \tag{2.2}$$

Here p is the projection $Y \to Y/T$ and $i_\xi : y \mapsto (\xi, y)$ the embedding of Y in $U \times Y$ at the level ξ.

For $\lambda \in \mathfrak{t}^*$ we denote by ∂_λ differentiation in the direction of the constant vector field λ. Write $\tilde{\lambda} = (\lambda, 0)$ for the horizontal vector field in $U \times Y$ over λ. Then

$$\begin{aligned} p^*(\partial_\lambda \sigma_\xi) &= \partial_\lambda (p^* \sigma_\xi) = \partial_\lambda (i_\xi^* \sigma) = i_\xi^* (\mathscr{L}_{\tilde{\lambda}} \sigma) \\ &= i_\xi^* (d(\tilde{\lambda} \lrcorner \sigma)) = d(i_\xi^* (\tilde{\lambda} \lrcorner \sigma)). \end{aligned} \tag{2.3}$$

Here $\mathscr{L}_v$ denotes the Lie derivative with respect to the vector field v; the 4th identity follows because $d\sigma = 0$.

For each $m \in Y$,

$$(\alpha_\xi)_m : \lambda \mapsto i_\xi^* (\tilde{\lambda} \lrcorner \sigma)_m \tag{2.4}$$

belongs to $(\mathfrak{t}^*)^* = \mathfrak{t}$, so this defines a $\mathfrak{t}$-valued 1-form α_ξ on Y. Taking inner product with the infinitesimal action $\tilde{X}$ of $X \in \mathfrak{t}$ we get

$$\langle \tilde{X} \lrcorner \alpha_\xi, \lambda \rangle_m = (\tilde{X} \lrcorner (\tilde{\lambda} \lrcorner \sigma))_m = (dJ_X)_m (\tilde{\lambda}_m) = \langle X, \lambda \rangle \tag{2.5}$$

because $\tilde{X} \lrcorner \sigma = -dJ_x$ (this convention has to be taken if $\sum dp_j \wedge dq_j$ is the symplectic form in the (p, q)-space). Relation (2.5) then expresses that the restriction of σ_m to $\mathfrak{t} \times \mathfrak{t}^*$, considered as a subspace of $T_m M$ under the map $(X, \lambda) \mapsto (\tilde{X}_m, \tilde{\lambda}_m)$, is equal to the standard symplectic form on $\mathfrak{t} \times \mathfrak{t}^*$.

Because α_ξ is obviously T-(hence Γ-)invariant there is a unique $\mathfrak{t}$-valued 1-form β_ξ on $Z = Y/\Gamma$ such that

$$r^* \beta_\xi = \alpha_\xi, \tag{2.6}$$

here r is the projection $Y \to Y/\Gamma$. But now (2.5) exhibits β_ξ as a connection form for the principal T/Γ-fibration q: $Z \to Y/T$. As a consequence

$$d\beta_\xi = q^* \Omega_\xi, \quad d\alpha_\xi = p^* \Omega_\xi \tag{2.7}$$

for a uniquely determined closed $\mathfrak{t}$-valued 2-form Ω_ξ on the V-manifold Y/T, called the *curvature* of the $\mathrm{Ker}(\beta)$-connection in the fiber bundle q. Collecting (2.3), (2.4) and (2.7) we get

$$\{\lambda \mapsto p^*(\partial_\lambda \sigma_\xi)\} = \{\lambda \mapsto \langle p^* \Omega_\xi, \lambda \rangle\} \tag{2.8}$$

and using that p is a submersion this in turn implies that

(2.9) $$\{\lambda \mapsto \partial_\lambda \sigma_\xi\} = \{\lambda \mapsto \langle \Omega_\xi \lambda \rangle\}.$$

On the other hand the general theory of connections (see for instance [L]) tells that the cohomology class of the curvature of the connection is equal to the Chern class of the bundle, hence

(2.10) $$\partial_\lambda [\sigma_\xi] = \langle c, \lambda \rangle, \quad \text{independent of } \xi.$$

This proves Theorem 1.1.

Remarks. In the case of a Hamiltonian action of a non-abelian compact connected Lie group G on a symplectic manifold $\tilde{M}$, with momentum mapping $\tilde{J}$: $M \to \mathfrak{g}^*$ we can make the following reduction. By means of an inner product, which is invariant under the adjoint action, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ intertwining the adjoint with the coadjoint action. Then, if ξ is a regular value of $\tilde{J}$, its centralizer in G is a maximal torus T in G. The Lie algebra $\mathfrak{t}$ of T is a local cross section for the adjoint action. Therefore $M = J^{-1}(\mathfrak{t}_{\text{reg}})$ is a symplectic submanifold of $\tilde{M}$, invariant under the T-action. The T-action on M is Hamiltonian with momentum mapping J equal to $\tilde{J}|M$. So the reduced phase space at ξ for the G-action on $\tilde{M}$ is equal to the reduced phase space at ξ for the T-action on M. The cohomology class of its symplectic form therefore depends linearly on ξ when we restrict ξ to a connected component of the regular set in the Lie algebra of a maximal torus. On the other hand the equivariance of the momentum mapping makes it constant along the (co-)adjoint orbits. Put together this gives a complete descrition on the regular set in $\mathfrak{g}^*$ in the non-abelian case as well.

As pointed out to us by Alan Weinstein, it is interesting to apply this to the left action of G on $\tilde{M} = T^*G$, the cotangent bundle of G. Identifying T^*G with $G \times \mathfrak{g}^*$ using the left-trivialization, the momentum mapping $\tilde{J}$: $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is equal to

(2.11) $$\tilde{J}: (g, \xi) \mapsto {}^t\mathrm{Ad}\, g^{-1}(\xi).$$

The reduced phase spaces are the complex flag manifolds G/G_ξ with G_ξ the stabilizer of ξ for the coadjoint action. If ξ is regular then G_ξ is a maximal torus T in G and $[\sigma_\xi] \in H^2(G/T)$ is equal to the image of $[\xi] \in H^1(T)$ via the "transgression": $H^1(T) \to H^2(G/T)$ for the fibration $G \to G/T$. This example of our Theorem 1.1 occurs already for instance in the exposition of Serre [Se] of the Borel-Weil theorem.

3. The Push Forward of the Liouville Measure

In this section we assume that M is connected, which is not a serious restriction because we can simply restrict the discussion to connected components of M. Secondly we may replace T by T/T_M where

(3.1) $$T_M = \bigcap_{m \in M} T_m$$

denotes the common stabilizer of all elements of M. The effect of this is that now

$$\bigcap_{m\in M} T_m=\{1\}, \tag{3.2}$$

that is T acts effectively on M.

Recall that locally there are only finitely many possibilities for the T_m, $m\in M$. For any closed subgroup S of T the fixed point set $\mathrm{Fix}(S)$ is a closed symplectic submanifold of M. This can be seen using the equivariant Darboux lemma, see [W1]. Collecting the $\mathrm{Fix}(S)$'s which are not open we get a locally finite collection of closed submanifolds of codimension $\geqq 2$, whose complement we denote by M'. If on the other hand $\mathrm{Fix}(S)$ is open then, being also closed and M being connected, $\mathrm{Fix}(S)=M$, that is $S=\{1\}$ in view of (3.2). So $m\in M'$ means that $m\notin\mathrm{Fix}(S)$ for all $S\neq\{1\}$, or $T_m=\{1\}$. We have proved

3.1 Lemma. *The assumptions that M is connected and that T acts effectively on M imply that the set M' on which T acts freely is equal to the complement of a locally finite union of closed symplectic submanifolds of codimension $\geqq 2$. In particular M' is open, connected, dense, and $M\setminus M'$ has measure 0. Also T_mJ is surjective for all $m\in M'$.*

After these preparatory remarks we consider now the following measures

(3.3)
- dm is the Liouville measure of (M,σ), defined by the volume form $\frac{1}{n!}\sigma^n$, $2n=\dim M$.
- dt is the normalized Haar measure on T; write $l=\dim T$.
- dX is the corresponding Lebesgue measure on $\mathfrak{t}$.
- $d\xi$ is the dual Lebesgue measure on $\mathfrak{t}^*$.

The assumption that J is proper implies that the push forward $J_*(dm)$ of dm under J is a measure in $\mathfrak{t}^*$ (Regarding measures as continuous linear forms on the space of compactly supported continuous functions, J_* is equal to the transposed of J^*, which is the pull back of functions by J). The additional assumptions that M is connected and T acts effectively on M imply in view of Lemma 3.1 that

$$J_*(dm)=f\,d\xi \tag{3.4}$$

for a locally integrable function f on $\mathfrak{t}^*$. Of course f is smooth at the regular values ξ of J and using the Fubini theorem one gets that $f(\xi)$ is equal to the volume of $Y_\xi=J^{-1}(\xi)$ with respect to the quotient of dm by $J^*d\xi$. In turn, $dm/J^*d\xi$ is locally given by the $2n-l=2(n-l)+l$-form

$$\frac{1}{(n-l)!}(i_\xi^*\sigma)^{n-l}\wedge\omega \tag{3.5}$$

(signs are irrelevant for measures), where ω is an l-form which on the T-orbits takes the value ± 1 on an l-tuple $(\tilde X_1,\dots,\tilde X_l)$ such that $dX(\tilde X_1,\dots,\tilde X_l)=1$.

However, the trivialization (2.1) showed that the T-action locally does not depend on ξ which shows that $p_\xi(M'\cap Y_\xi)=M'_\xi$ is equal to the complement of a finite union of closed symplectic (V-)submanifolds of M_ξ of codimension $\geqq 2$. Because $p_\xi\colon M'\cap Y_\xi \to M'_\xi$ is a principal T-fibration we get in view of (3.5) and the convention that $\mathrm{vol}(T)=1$ that the volume of $M'\cap Y_\xi$ is equal to the volume of M'_ξ with respect to its Liouville measure. Because the complements have measure zero we have proved

3.2 Proposition. *Let M be connected and T act effectively on M. Then the density f of $J_*(dm)$ satisfies*

$$f(\xi)=\int_{M_\xi} \frac{1}{(n-l)!}(\sigma_\xi)^{n-l} = \left\langle [M_\xi], \frac{1}{(n-l)!}[\sigma_\xi]^{n-l}\right\rangle \tag{3.6}$$

for each regular value ξ of J. Here M_ξ has been given the orientation of $(\sigma_\xi)^{n-l}$ and $[M_\xi]$ denotes the corresponding orientation class in $H_{2(n-l)}(M_\xi, \mathbb{Z})$.

Combining (3.6) with (1.11) now immediately yields

3.3 Corollary. *f is a polynomial (of degree $\leqq n-l$) on each connected component of the set of regular values of the momentum map.*

4. An Oscillatory Integral

In this section we assume that M is connected, T acts effectively on M, and moreover that M is *compact*. As already observed in the introduction of Sect. 3, $\mathrm{Fix}(T)$ is a finite union of compact connected symplectic submanifolds M_j of even *co*dimension $2n_j>0$. More precisely, near each $m_0\in M_j$ there is a canonical system of coordinates $(p_1,\dots,p_n,q_1,\dots,q_n)$ in which the T-action is linear. That means that we can arrange these coordinates such that the Hamilton functions J_X have the following standard quadratic form

$$J_X(m)=J_X(m_0)+\sum_{k=1}^{m} \omega_{jk}(X)\,(p_k^2+q_k^2)/2. \tag{4.1}$$

Here the coefficients $\omega_{jk}(X)$ depend linearly on $X\in\mathfrak{t}$, that is $\omega_{jk}\in\mathfrak{t}^*$. Note that these objects are well-defined and constant along each M_j.

The fact that the corresponding Hamilton vector fields generate an action of T implies that necessarily

$$\omega_{jk}(X)\in 2\pi\mathbb{Z} \quad \text{if } X\in \mathrm{Ker}\exp, \tag{4.2}$$

that is

$$\lambda_{jk}=\frac{1}{i}\,\omega_{jk} \tag{4.3}$$

is a weight of T. In fact, providing the real (p,q)-space with a multiplication by i, called J, such that $\sigma(J\cdot,\cdot)$ is an inner product (for the standard symplectic form $\sum dp_k\wedge dq_k$ this means that $q_k=p_k\circ J$), the λ_{jk} are the weights for the complex linear representation of T on the tangent space of M at m_0. It is also obvious that M_j is the set of (p,q) such that $p_k=q_k=0$ if $\lambda_{jk}\neq 0$, so

$$n_j=\#\{k;\ \lambda_{jk}\neq 0\}. \tag{4.4}$$

We will enumerate the weights such that

$$\lambda_{jk} \neq 0 \quad \text{for } k=1,\dots,n_j. \tag{4.5}$$

With these conventions made we can now state the

4.1 Theorem[4]. *The inverse Fourier transform of $J_*(dm)$ is given by*

$$\int_M e^{i\langle X, J(m)\rangle}\, dm = \sum_j \frac{\operatorname{vol}(M_j)\, e^{i\langle X, J(M_j)\rangle}}{\prod_{k=1}^{n_j} (\langle X, \lambda_{jk}\rangle / 2\pi)} \tag{4.6}$$

for $X \in \mathfrak{t}$ such that

$$\langle X, \lambda_{jk}\rangle \neq 0 \quad \textit{for all } k=1,\dots,n_j \ (\textit{all } j). \tag{4.7}$$

Here $J(M_j)$ denotes the common value of J on M_j.

Proof. Application of the method of stationary phase (see for instance [Hö], Sect. 3.2) yields

$$\int_M e^{it\langle X, J(m)\rangle}\, dm = \sum_j \frac{\operatorname{vol}(M_j)\, e^{it\langle X, J(M_j)\rangle}}{\left(\frac{t}{2\pi i}\right)^{n_j} \prod_{k=1}^{n_j} \langle X, \omega_{jk}\rangle} + f(t), \tag{4.8}$$

where t is a real parameter and f coincides on $\mathbb{R}\setminus\{0\}$ with a Schwartz function. Observe that all higher terms in the asymptotic expansion vanish, because in the local coordinates introduced before the phase function is quadratic and the amplitude (being the Jacobi determinant of the local coordinates) is equal to 1.

Now choose $X \in \mathfrak{t}$ such that (4.7) holds and such that $\{\exp tX;\ t \in \mathbb{R}\}$ is a closed subgroup of T (isomorphic to a circle). The set of these X is dense in $\mathfrak{t}$, and since both left and right hand side of (4.6) are smooth in X (on the set of X satisfying (4.7)) it is sufficient to prove (4.6) for such X.

Now Corollary 3.3 for circle actions yields that the left hand side in (4.8), as a function of t, is equal to the inverse Fourier transform of a compactly supported piecewise polynomial functions on $\mathbb{R}$. Multiplying by $(it)^N$, N sufficiently large, the first summand in the right hand side becomes smooth at the origin. The left hand side stays smooth, so $(it)^N f(t)$ is a smooth Schwartz function on $\mathbb{R}$.

However, on the Fourier transform side multiplication by $(it)^N$ acts as $\left(\frac{d}{dt}\right)^N$ and we get that the Fourier transform of $(it)^N f(t)$ is equal to a linear combination of derivatives of δ-functions (situated at the points $\langle X, J(M_j)\rangle$). Combining with the smoothness we get $\mathcal{F}((it)^N f(t))=0$, hence $(it)^N f(t)\equiv 0$, or $f(t)=0$ for $t\neq 0$. Reading (4.8) now for $t=1$ completes the proof.

Remarks. Clearly the right hand side of (4.6) extends to a smooth function of X on $\mathfrak{t}$, and even to a complex analytic function on $\mathfrak{t}\otimes\mathbb{C}$, because the left hand side does. Also the identity (4.6) extends to $X \in \mathfrak{t}\otimes\mathbb{C}$.

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[4] See "Note added in proof" on page 268

Secondly, (4.6) shows that $J_*(dm)$ is completely determined by the weights λ_{jk}, the volumes of the fixed point manifolds M_j and the values of J on these. The formulas (1.11) and (3.6) for the density of $J_*(dm)$ suggest a relation between these data and the topological properties of the fibration q_ξ in (1.6).

Finally, from (4.6) or directly, one can derive that the locus of singularities of $J_*(dm)$ (being the set of singular values of J) is a union of pieces of affine hyperplanes, going through the points $J(M_j)$ and spanned by $l-1$ of the ω_{jk}. (The pieces themselves have piecewise linear boundaries.) This fits in with the description of the image of J as the convex hull of the points $J(M_j)$ in $\mathfrak{t}^*$, see [A] and [GS1].

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Note Added in Proof

Formula (4.6) is only correct if the fixed points are isolated. In the general case, for n_j strictly less than n, the factor in front of $e^{i\langle X, J(M_j)\rangle}$ gets additional terms, which are rational in X and homogeneous of degree $-(n_j+1)$ up to $-n$. We hope to give a more precise determination of these terms at another occasion.