# A Vanishing Theorem for Piecewise Constant Curvature Spaces

by

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Let $X^n$ be a triangulated closed normal pseudomanifold (see [GM] for definitions) equipped with a metric of piecewise constant curvature. Recall that $X^n$ can be described as follows. Start with a collection of simplices, $\{\sigma^n\}$, whose interiors have a metric of (fixed) constant curvature, $K$, and whose faces are all totally geodesic. Then identify various faces by isometries in such a way that underlying topological space so obtained is a normal pseudomanifold (see [CMS], $[C_1]$-$[C_4]$, which provide the general background for this paper).

Associated to $X^n$ is a natural stratification, $X^n = \bigcup_j \mathfrak{S}^i$, where $\mathfrak{S}^j$ is a smooth manifold of codimension $i$ and constant curvature $K$ ; $\mathfrak{S}^1$ is empty. If $\mathfrak{C}$ is any piecewise constant curvature triangulation as above, then the j-skeleton, $\Sigma^j$ of $\mathfrak{C}$, contains $\mathfrak{S}^{n-j}$. The <u>link</u>, $L(\mathfrak{S}^j,p)$, of $\mathfrak{S}^j$ at $p$ $\mathfrak{S}^j$ is, by definition, the <u>cross section</u> (or <u>base</u>) of the <u>normal cone</u>, $C^{\perp}(\mathfrak{S}^j,p)$, to $\mathfrak{S}^j$ at $p$. The link, $L(\Sigma^{n-j},p)$, at $p \in \Sigma^j$ is defined similarly.

We say that $X^n$ has <u>positive curvature at its singularities</u> if for each $p \in \mathfrak{S}^2$, the link, $L(\mathfrak{S}^2,p)$ is a circle of length $< 2\pi$. If, in addition $K > 0$ (respectively $K = 0$) we say that $X^n$ has <u>positive curvature</u> (respectively <u>nonnegative curvature</u>).

<u>1. Observation</u>. If $K = 0$ and the triangulation $\mathfrak{C}$ can be chosen such that $\mathfrak{S}^2 = \Sigma^{n-2}$ and the curvature at the singularities is positive, then $X^n$ admits a metric of positive curvature.

Proof : Choose small $K' > 0$ and replace each (totally geodesic flat), simplex of $\mathfrak{C}$ by the (totally geodesic) simplex of constant curvature $K'$, which has the same edge lengths. For $K'$ sufficiently small, the curvature at the singularities will still be positive.

2. Example. If $X^n$ is the surface of a tetrahedron in $R^{n+1}$, then $X^n$ admits a metric of positive curvature. Moreover, by "rounding the corners", this metric can be approximated (in an obvious sense) by a smooth metric of positive curvature on the n-sphere, $S^n$.

For some time, it was assumed that the condition that $X^n$ has positive or negative curvature in the above sense, was the analog of the corresponding condition for the sectional curvature in the smooth case. However (as M. Gromov pointed out) in view of the following result, it may be more accurate to think of these conditions as replacing conditions on the curvature operator (compare [Gal, Mey]).

3. Theorem. Let $X^n$ be a closed normal pseudomanifold with piecewise constant curvature metric.

i) If $X^n$ has nonnegative curvature then $X^n$ is a real homology manifold. Moreover,

$$b^i(X^n) \leq \binom{n}{i}. \tag{4}$$

ii) If $X^n$ has positive curvature then it is a real homology sphere.

Theorem 3 was discovered in 1977 and announced in $[C_2]$, $[C_3]$. Here we will indicate the proof, but some of the more technical analytical details will be omitted.

5. Remark. It is conceivable that an even stronger version of Theorem 3 could be proved by other means, perhaps even by a direct geometric argument.

6. Remark. M. Gromov has suggested that the method of [Ham] might eventually be brought to bear on our situation.

In proving Theorem 3, essentially, one attempts to repeat the argument of the Bochner Vanishing Theorem in the smooth case. If the curvature at the singularities is positive this goes through.

To fix ideas, first consider the case in which $X^n$ is actually a piecewise flat real homology manifold of nonnegative curvature. Let $\bar{\mathfrak{S}}^2 = \bigcup_{j\geq 2} \mathfrak{S}^j$ and let $\mathcal{H}^i$ denote the space of $L_2$-harmonic forms on $X^n \setminus \bar{\mathfrak{S}}^2$ which are closed and coclosed. According to the Hodge Theorem proved in $[C_3]$, we have

(7) $$\dim \mathcal{H}^i = b^i(X^n).$$

Let $h \in \mathcal{H}^i$ and let $\{e_i\}$ be a local orthonormal frame field near $x \in X^n \setminus \bar{\mathfrak{S}}^2$ satisfying $\nabla e_i = 0$ at $x$. The standard local computation at $x$ gives

(8) $$\begin{aligned} 0 &= \langle (d\delta+\delta d)h,h\rangle , \\ &= \langle -\sum_i \nabla_{e_i} \nabla_{e_i} h,h\rangle , \\ &= -\tfrac{1}{2} \operatorname{div} (\operatorname{grad} \|h\|^2) + \langle \nabla h,\nabla h\rangle , \end{aligned}$$

(where we have used $K = 0$ in going from the first line to the second). Assuming for the moment that the integrals exist, we have

(9) $$0 = -\frac{1}{2} \int_{X^n \setminus \bar{\mathfrak{S}}^2} \operatorname{div} (\operatorname{grad} \|h\|^2) + \int_{X^n \setminus \bar{\mathfrak{S}}^2} \|\nabla h\|^2 .$$

If $X^n$ were actually smooth we could replace the domain of integration in the first integral by $X^n$ and conclude by the divergence theorem that this integral vanishes. Then (11) would imply that $\nabla h \equiv 0$ giving (4). Since $X^n$ is not smooth, we take a suitable tubular neighborhood $T_\varepsilon(\bar{\mathfrak{S}}^{n-2})$ (as in [$C_3$], [$C_4$]) and by Stokes' Theorem, write

(10) $$\begin{aligned} &\pm\frac{1}{2} \int_{\partial T_\varepsilon(\bar{\mathfrak{S}}^2)} *d(\|h\|^2) \\ &= -\int_{X^n \setminus T_\varepsilon(\bar{\mathfrak{S}}^2)} \|\nabla h\|^2 . \end{aligned}$$

We claim that the condition that $X^n$ has positive curvature implies that in the limit as $\varepsilon \to 0$, the left hand side of (10) vanishes. This yields (4).

We begin by deriving an analytic condition on the links which implies the above vanishing and then show how positive curvature guarantees that this condition holds. Observe that near $p \in \mathfrak{S}^j$, the geometry of $X^n$ is locally a product, $U^{n-j} \times C^{\perp}_{0,\varepsilon}(\mathfrak{S}^j,p)$. Here $U^{n-j} \subset \mathfrak{S}^{n-j}$ is flat and $C^{\perp}_{0,\varepsilon}(\mathfrak{S}^j,p) \subset C^{\perp}(\mathfrak{S}^j,p)$ denotes the set of points whose radial polar coordinate, $r$, satisfies $r < \varepsilon$. One can show that on $U^{n-j} \times C^{\perp}_{0,\varepsilon}(\mathfrak{S}^j,p)$, a closed and coclosed $L_2$-harmonic $i$-form, $h$, can be written as a convergent series of products

(11) $$h = \sum_k h_{1,k} \wedge h_{2,k},$$

where $h_{1,k}$ is a closed and coclosed harmonic form on $U^{n-j}$ and $h_{2,k}$ is a closed and coclosed $L_2$-harmonic form on $C^{\perp}(\mathcal{S}^j,p)$ ($\deg h_{1,k}+\deg h_{2,k}=i$). Since the forms $h_{1,k}$ are smooth and the dimension of the fibre of $\partial T_\varepsilon(\mathcal{S}^j)$ is $j-1$, in order for the left hand side of (10) to vanish in the limit, we must have*

$$(12) \qquad \|{*d}(\|h\|^2)\| = o(\varepsilon^{-(j-1)}),$$

or equivalently, for all $k$,

$$(13) \qquad \|{*d}(\|h_{2,k}\|^2)\| = o(\varepsilon^{-(j-1)}).$$

To see the meaning of (13), we recall the representation of the forms $h_{2,k}$ in polar coordinates $(r,y)$ on $C^{\perp}(\mathcal{S}^j,p)$ (see [$C_1$], [$C_4$] for details). Put $m = j-1 = \dim L(\mathcal{S}^j,p)$. It is a consequence of the method of separation of variables that the closed and coclosed harmonic $(i+1)$-forms on $C^{\perp}(\mathcal{S}^j,p)$ can be written as convergent sums of forms with the following description. Let $\phi$ be a coexact eigen $i$-form of the Laplacian, $\tilde{\Delta}$, on $L(\mathcal{S}^{m+1},p)$, with eigenvalue $\mu > 0$ (see [$C_4$] for a discussion of anlaysis on $L(\mathcal{S}^{m+1},p)$). Put

$$(14) \qquad \alpha = \frac{1+2i-m}{2}.$$

$$(15) \qquad \nu = \sqrt{\alpha^2+\mu}$$

$$(16) \qquad a^+ = \alpha+\nu.$$

Then corresponding to $\phi$, we have the closed and coclosed $L_2$-harmonic $(i+1)$-form on $C^{\perp}(\mathcal{S}^{m+1},p)$,

$$(17) \qquad r^{a^+}d\phi + a^+r^{a^+-1}dr\wedge\phi.$$

For the case in which $X^n$ is a manifold, we must also include the constant function $h_{2,0} \equiv 1$ and its dual, $*1$. Although these are not of the above type, they can be ignored since they satisfy

$$(18) \qquad {*d}(\|1\|^2) = {*d}(\|{*1}\|^2) \equiv 0.$$

* It is no essential loss of generality to assume $X^n$ is oriented.

If $X^n$ is not a real homology manifold, in general, there are analogous exceptional closed and coclosed $L_2$-harmonic forms corresponding to the $L_2$-cohomology of $L(\mathfrak{S}^{n+1},p)$ $(d\phi=\tilde{\delta}\phi=0,\mu=0)$ whose pointwise norms are not constant. But the inductive argument below shows that for the case in which $X^n$ has nonnegative curvature, the possible existence of these forms can be ruled out before they need be considered (i.e. positive curvature implies $X^n$ is a real homology manifold).

We now examine (13) for the form in (17). The pointwise norms of the forms $\phi(y)$, $d\phi(y)$ in (17) satisfy

$$\|\phi\| = O(r^{-2i}), \tag{19}$$

$$\|d\phi\| = O(r^{-2(i+1)}) \; ; \tag{20}$$

see $[C_2]$. Thus,

$$\|r^{a^+}+a^+r^{a^+-1}dr\wedge\phi\| = O(r^{2a^+-2(i+1)}), \tag{21}$$

$$\begin{aligned}\|{*d}(\|r^{a^+}d\phi+a^+r^{a^+-1}dr\wedge\phi\|)\| &= O(r^{2a^+-2i-3}) \\ &= O(r^{2(\nu-1)-m})\end{aligned} \tag{22}$$

Thus, we must verify that $\nu > 1$. In view of (14) and the condition $\mu > 0$, this is automatic unless

$$\alpha = 0, \tag{23}$$

$$\alpha = 1/2, \tag{24}$$

in which case,

$$i = (m-1)/2, \tag{25}$$

respectively,

$$i = m/2. \tag{26}$$

In these cases, we still have $\nu > 1$, provided,

$$\mu > 1, \tag{27}$$

respectively,

$$\mu > 3/4. \tag{28}$$

Here, the hypothesis that $X^n$ has positive curvature at its singularities will intervene.

Suppose for the moment that $L(\mathcal{S}^{m+1},p)$ is actually <u>smooth</u> (of curvature $\equiv 1$). Then for i-forms on $L(\mathcal{S}^{m+1},p)$, the Weitzenbock formula is

$$\tilde{\Delta}\phi = -\nabla^2\phi + i(m-i)\phi. \tag{29}$$

The same integration by parts argument whose validity we are investigating for $X^n$, shows that $-\nabla^2$ is a positive semidefinite operator and we immediately obtain

$$\mu \geq i(m-i). \tag{30}$$

For the case, (23), this gives

$$\mu \geq \frac{(m-1)^2}{2}, \tag{31}$$

$m$ odd, while for (24), it gives

$$\mu \geq (\tfrac{m}{2})^2. \tag{32}$$

$m \geq 2$, even.

By (30) we get $\mu \geq 1 \geq 3/4$. The remaining cases for (31) are $m = 3$, $i = 1$ and $m = 1$, $i = 0$. In the former case, (29) still yields $\mu > 1$ unless $\nabla\phi \equiv 0$. Since, by de Rham's decomposition theorem, a space of curvature $K \equiv 1$ admits no parallel vector field (and hence, no parallel 1-form) even locally, we obtain, $\mu > 1$ in this case.

Finally, suppose $m = 1$, $i = 0$. Here, $L(\mathcal{S}^2,p)$ is a circle and the hypothesis of positive curvature at the singularities of $X^n$ says precisely that the length of this circle is $> 2\pi$. Thus, the smallest nonzero eigenvalue, $\mu$, of $\tilde{\Delta} = \frac{-d^2}{dy^2}$, satisfies $\mu > 1$.

It remains to remove the hypotheses that $X^n$ is a rational homology manifold and that $L(\mathcal{S}^m,p)$ is smooth. For this, we point out the obvious fact that for the natural stratification of $L(\mathcal{S}^m,p)$, every link is isometric to the link of some stratum of $X^n$, which contains $p$ in its closure. Similarly, links of links on $L(\mathcal{S}^m,p)$ and in fact, all such iterated links, are isometric to links of strata of $X^n$. Thus, such iterated links are actually spaces of positive curvature (in our sense) and have dimension strictly smaller than that of $X^n$.

An analysis like that just performed for the space $X^n$ (and which we omit) shows that $\tilde{\nu} > 1$ is the condition justifies the integration by parts argument needed to show $\langle -\nabla^2\phi,\phi\rangle \geq 0$ in (29). Here $\tilde{\nu}$ is defined as in (15) but $\mu$ in (15) is replaced by an eigenvalue of the Laplacian $-\tilde{\tilde{\nabla}}^2$ on a link of a stratum of $L(\mathcal{S}^{m+1},p)$.

Now an obvious inductive argument shows that for all spaces of positive curvature, $Y^\ell$ (and in particular all iterated links above) the smallest nonzero eigenvalue of the Laplacian on forms is nonzero (except for the zero eigenvalues in dimensions $0,\ell$) and that the integration by parts argument $(\langle -\nabla^2\phi,\phi\rangle \geq 0)$ is valid. By the Hodge-de Rham Theory, of $[C_3]$, the spaces, $Y^\ell$, are real homology spheres. Thus, $X^n$ is a real homology manifold and in the same way, the inequality (4) follows. q.e.d.

<u>33 Remark</u>. For general piecewise constant curvature pseudomanifolds the Laplacian on $C_0^\infty(\Lambda^+)$ need not be essentially self adjoint and one must choose "ideal boundary conditions". Even if the Laplacian is essentially self adjoint the closed and coclosed harmonic forms represent the $L_2$-cohomology of $X^n$ (or equivalently, the middle intersection cohomology) which in general is different from the simplicial cohomology of $X^n$ (see $[C_3]$, [GM]). The hypothesis that $X^n$ is a normal pseudomanifold rules out these possibilities via the inductive argument.

<u>34 Remark</u>. If we allow 1-dimensional links consisting of several circles, each of length $<2\pi$, then our conclusions apply to the $L_2$-cohomology of the (non-normal) space $X^n$, which coincides with the simplicial cohomology of an associated normal space $\hat{X}^n$ called the normalization. As an example, let $X^2$ be a tetrahedron $\hat{X}^2$ with all vertices identified to a point.

<u>35 Remark</u>. One can start with a smooth manifold $M^n$ of nonnegative curvature and form a piecewise flat approximation as in [CMS]. By Rauch comparison one obtains a piecewise flat space for which all links, L, with their natural induced triangulations, have the following property. There is a combinatorially isomorphic totally geodesic triangulation of the unit sphere such that all corresponding edge lengths are at most equal to the corresponding edge lengths on L. It would be interesting to know if any useful information for the smooth case can be derived from this condition (which coincides with our definition nonnegative curvature only in dimension 2).

## References

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