A GENERAL SCHWARZ LEMMA FOR KAHLER MANIFOLDS.

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Introduction. The classical Schwarz-Pick lemma states that any holomorphic map of the unit disk into itself decreases the Poincaré metric. Later Ahlfors generalized this lemma to holomorphic mappings between two Riemann surfaces where curvatures of these Riemann surfaces were used in a very explicit way. More recently, Chern initiated the study of holomorphic mappings between higher-dimensional complex manifold by generalizing the Ahlfors lemma to these spaces. Then this lemma was further extended by Kobayashi, Griffiths, Wu and others. It plays a very important role in their theory. In this note, we shall prove the following generalization of the Schwarz lemma.

THEOREM *Let M be a complete Kähler manifold with Ricci curvature bounded from below by a constant, and N be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant. Then any holomorphic mapping from M into N decreases distances up to a constant depending only on the curvatures of M and N.*

The main point here is that the domain M is a very general manifold. The best known result in this direction is due to Kobayashi [3], who assumed that M is a bounded domain whose Bergman kernel function should behave well near the boundary. The method employed previously in proving the Schwarz lemma depends largely on a nice exhaustion of the manifold M. This is assumed in order to assure the existence of a maximal point of a certain function. In this note, we eliminate these hypotheses by applying a method that we developed in [5].

We would like to thank Professor H. Wu for his interest and encouragement in this work. Professor H. Royden informed us that he is able to improve the estimate in our main theorem. Namely, he is able to replace K_2 by the upper bound of the holomorphic sectional curvature of N.

1. Notation and Formulas for Hermitian Manifolds. Let M^m be a Hermitian manifold of dimension m. Let $e_1, e_2, \ldots, e_m$ be a unitary frame field in an open set of M^m. Let $\theta_1, \theta_2, \ldots, \theta_m$ be its coframe field. Then there are

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complex-valued linear differential forms of type $(1,0)$ such that the Hermitian metric is given by

$$dS_M^2 = \sum_{i=1}^{m} \theta_i \bar{\theta}_i. \tag{1}$$

It is known that there are connection forms θ_{ij} such that

$$d\theta_i = \sum_{j=1}^{m} \theta_j \wedge \theta_{ij} + \Theta_i \tag{2}$$

with

$$\theta_{ij} + \bar{\theta}_{ji} = 0 \quad \text{and} \quad \Theta_i = \frac{1}{2} \sum_{j,k}^{m} T_{ijk} \theta_j \wedge \theta_k. \tag{3}$$

The tensor T_{ijk} is called the torsion tensor. The curvature forms Θ_{ij} are defined by

$$\Theta_{ij} = d\theta_{ij} + \sum_k \theta_{ik} \wedge \theta_{kj}, \tag{5}$$

and we have

$$\Theta_{ij} = -\bar{\Theta}_{ji} = \frac{1}{2} \sum_{k,l}^{m} R_{ijkl} \theta_k \wedge \bar{\theta}_l. \tag{6}$$

The skew-Hermitian symmetry of Θ_{ij} expressed by the first equation of (6) is equivalent to

$$R_{ijkl} = \bar{R}_{jilk}. \tag{7}$$

If $\xi = \sum_i \xi_i e_i$ and $\eta = \sum_i \eta_i e_i$ are two tangent vectors, then the holomorphic bisectional curvature determined by ξ and η is defined [3] by

$$\frac{\sum_{i,j,k,l} R_{ijkl} \bar{\xi}_i \xi_j \bar{\eta}_k \eta_l}{\left(\sum_i \xi_i \bar{\xi}_i\right)\left(\sum_i \eta_i \bar{\eta}_i\right)}.$$

If $\xi = \eta$, the above quantity is called the holomorphic sectional curvature in direction ξ. The Ricci tensor is defined as

$$R_{kl} = \sum_{i=1}^{m} R_{iikl} = \bar{R}_{lk}, \tag{8}$$

and the scalar curvature is defined as

$$R=\sum_k R_{kk}. \tag{9}$$

Let N^n be another Hermitian manifold with dimension n. Then we can define the corresponding frame field ω_α, curvature tensor $S_{\alpha\beta\gamma\delta}$, Ricci tensor $S_{\alpha\beta}$ and scalar curvature S.

Let $f: M^m \to N^n$ be any holomorphic mapping. Then we can define

$$f^* \omega_\alpha = \sum_{i=1}^{m} a_{\alpha i}\theta_i \tag{10}$$

and

$$u=\sum_{\alpha,i} a_{\alpha i}\bar{a}_{\alpha i}. \tag{11}$$

Clearly, we have

$$f^* dS_N^2 \leqslant u dS_M^2. \tag{12}$$

In order to relate things to curvature, one has to compute the Laplacian of u. It is defined as follows. Let

$$du=\sum_i \left(u_i\theta_i+\bar{u}_i\bar{\theta}_i\right), \tag{13}$$

$$-d\left(\sum_i u_i\theta_i\right)=\sum_{i,j} u_{i\bar{j}}\theta_i\wedge\bar{\theta}_j. \tag{14}$$

Then the Laplacian of u is

$$\Delta u=\sum_i u_{i\bar{i}}. \tag{15}$$

The Chern-Lu formula [1] states the following:

$$\tfrac{1}{2}\Delta u \geqslant \sum_{\alpha,i,j} R_{j\bar{i}}a_{\alpha i}\bar{a}_{\alpha j}-\sum_{i,k}\sum_{\alpha,\beta,\gamma,\eta}\bar{a}_{\alpha i}a_{\beta i}\bar{a}_{\gamma k}a_{\eta k}S_{\alpha\beta\gamma\eta}. \tag{16}$$

[In applying (16), one also has to use the elementary fact that the bisectional curvature of a complex submanifold is not greater than that of the ambient manifold.]

2. Schwarz Lemma for General Complex Manifolds. We shall apply the following theorem of [5] and [6].

THEOREM 1. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from below on M. Then for all $\varepsilon > 0$, there exists a p in M such that at p,*

$$|\mathrm{grad} f| < \varepsilon, \qquad \Delta f > -\varepsilon \quad \text{and} \quad f(p) < \inf f + \varepsilon. \tag{17}$$

Now consider the function u defined in Section 1. Let c be any positive number. Then direct computation shows

$$\Delta\left(\frac{1}{\sqrt{u+c}}\right) = \frac{-\Delta u}{2(u+c)^{3/2}} + \frac{3}{(u+c)^{5/2}} \sum_i |u_i|^2. \tag{18}$$

Hence applying (16), we have

$$\Delta\left(\frac{1}{\sqrt{u+c}}\right) \leqslant \frac{3}{(u+c)^{5/2}} \sum_i |u_i|^2 - \frac{1}{2(u+c)^{3/2}}$$
$$\times\left[\sum_{\alpha,i,j} \bar{a}_{\alpha i} a_{\alpha j} R_{ji} - \sum_{i,k} \sum_{\alpha,\beta,\gamma,\eta} a_{\alpha i} \bar{a}_{\beta i} a_{\gamma k} \bar{a}_{\eta k} S_{\alpha\beta\gamma\eta}\right]. \tag{19}$$

Let $\varepsilon > 0$ be any number. Then, by Theorem 1, there is a point p such that at p,

$$\frac{1}{4(u+c)^3} \sum_i |u_i|^2 < \varepsilon,$$
$$\Delta\left(\frac{1}{\sqrt{u+c}}\right) > -\varepsilon, \tag{20}$$
$$\frac{1}{\sqrt{u+c}} < \inf \frac{1}{\sqrt{u+c}} + \varepsilon.$$

Dividing (20) by $\sqrt{u+c}$ and comparing it with (21)8 we obtain

$$\frac{-1}{2(u+c)^2}\left[\sum_{\alpha,i,j} \bar{a}_{\alpha i} a_{\alpha j} R_{ji} - \sum_{i,k} \sum_{\alpha,\beta,\gamma,\eta} a_{\alpha i} \bar{a}_{\beta i} a_{\gamma k} \bar{a}_{\eta k} S_{\alpha\beta\gamma\eta}\right]$$
$$\geqslant -\varepsilon\left(\inf \frac{1}{\sqrt{u+c}} + \varepsilon\right) - 12\varepsilon. \tag{21}$$

Let K_1 be the greatest lower bound of the Ricci curvature of M, and K_2 be the least upper bound of the holomorphic bisectional curvature of N. Then it follows from (21) that

$$\frac{-u}{(u+c)^2}K_1+\frac{u^2}{(u+c)^2}K_2 \geqslant -2\varepsilon\left(\inf\frac{1}{\sqrt{u+c}}+\varepsilon\right)-24\varepsilon. \tag{22}$$

When $\varepsilon \to 0$, $1/\sqrt{u+c}$ goes to its infimum and u goes to its supremum. Therefore, if K_2 is negative and u is not identically zero, then K_1 is non-positive and

$$0 \leqslant \sup u \leqslant \frac{K_1}{K_2}. \tag{23}$$

THEOREM 2. *Let M be a complete Kähler manifold with Ricci curvature bounded from below by K_1. Let N be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant K_2. Then if there is a non-constant holomorphic mapping f from M into N, we have $K_1 \leqslant 0$ and*

$$f * dS_N^2 \leqslant \frac{K_1}{K_2} dS_M^2. \tag{24}$$

In particular, if $K_1 \geqslant 0$, every holomorphic mapping from M into N is constant.

Since the unit disk has a Kähler metric with constant negative holomorphic sectional curvature, we have the following

COROLLARY. *Let M be a complete Kähler manifold with non-negative Ricci curvature. Then M does not admit any bounded holomorphic function.*

In case $\dim M = 1$, one can weaken the hypothesis on N.

THEOREM 2′. *Let M be a complete Riemann surface with curvature bounded from below by a constant K_1. Let N be another Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant K_2. Then for any constant holomorphic mapping f from M into N, (24) holds.*

3. Other Generalizations. Instead of taking the trace of the tensor $f * dS_N^2$, we can also consider the other elementary function of this tensor. Since formulas corresponding to (16) still exist [4], one can derive corresponding properties for these elementary functions. For simplicity, we shall only state the following

THEOREM 3. *Let M be a complex Kähler manifold with scalar curvature bounded from below by K_1. Let N be another Hermitian manifold with Ricci curvature bounded from above by a negative constant K_2. Suppose the Ricci curvature of M is bounded from below and* $\dim M = \dim N$. *Then the existence of a non-degenerate holomorphic map f from M into N implies that $K_1 \leqslant 0$ and*

$$f_* dV_N \leqslant \frac{K_1}{K_2} dV_M, \tag{25}$$

where dV_M, dV_N are volume elements of M and N respectively.

We can partially generalize Theorem 3 in the following sense: Let dV_N be a non-negative top-dimensional form on a complex manifold N such that the Ricci curvature of dV_N is bounded from above by a negative constant K_2. Let P be the ball whose Poincaré metric has scalar curvature K_1 and whose dimension is equal to $\dim N$. Let f be a meromorphic map (in the sense of Remmert) mapping the polydisk P into N. Then we have $f_* dV_N \leqslant (K_1/K_2) dV_P$.

The proof of this assertion follows from the fact that f is holomorphic outside a subvariety of codimension two, so that $f_* dV_N$ can be extended through this subvariety. The standard proof of the Ahlfors lemma can be applied to prove our claim.

Finally, we remark that Eells and Sampson [2] have studied harmonic mappings between two Riemannian manifolds. One can also deduce a formula similar to (16) for this class of mappings. However, in order to draw a useful conclusion, it seems that one has to assume the mapping is quasi-conformal.

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