Compactification of arithmetic quotients of bounded symmetric domains

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Introduction

This paper is chiefly concerned with a bounded symmetric domain X and an arithmetically defined discontinuous group Γ of automorphisms of X. Its main goals are to construct a compactification V^* of the quotient space $V = X/\Gamma$, in which V is open and everywhere dense, to show that V^* may be endowed with a structure of normal analytic space which extends the natural one on V, and to establish, using automorphic forms, an isomorphism of V^* onto a normally projective variety, which maps V onto a Zariski-open subset of the latter.

We now proceed to a synopsis of the contents and methods of this paper, making, for convenience in this introduction, the following assumptions, which are no essential loss in generality: $X = K\backslash G_{\mathbf{R}}$ is the quotient by a maximal compact subgroup K of the group $G_{\mathbf{R}}$ of real points of a connected algebraic matric group defined over $\mathbf{Q}$, *simple over* $\mathbf{Q}$, and Γ is an arithmetic subgroup of G (i.e., Γ is commensurable with the group $G_{\mathbf{Z}}$ of integral matrices in G,

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see 3.1). Our program may be roughly divided into three parts:

I. Construction and properties of the compactification V^* of V as a topological space.

II. Study of certain automorphic forms, and of their behavior under a Φ-operator.

III. Analytic structure on V^*, projective embedding.

Part I is covered in §§1–4. The first paragraph deals with the *natural compactification* of X; i.e., the closure $\bar{D}$ of the Harish-Chandra realization of X as a bounded domain D [22]. We recall that $\bar{D} - D$ is the union of locally closed analytic subsets of the ambient vector space, which are themselves (equivalent to) bounded symmetric domains in a smaller number of dimensions, called the *boundary components of* $\bar{D}$. The *normalizer* $N(F) = \{g \in G^0_{\mathbf{R}} \mid F \cdot g = F\}$ of the boundary component F is a maximal parabolic subgroup of the topological identity component $G^0_{\mathbf{R}}$ of $G_{\mathbf{R}}$, and conversely. It contains as a normal subgroup the *centralizer* $Z(F) = \{g \in G^0_{\mathbf{R}} \mid x \cdot g = x (x \in F)\}$ of F. To F there is associated a(n essentially) canonical unbounded realization S_F of X, and a complex analytic mapping σ_F of X onto F, whose fibres are affine subspaces of the ambient vector space, and are the orbits of $Z(F)^0$. These results, due to Pyateckii-Shapiro for the classical domains [30], were extended to the general case by Korányi-Wolf [27]. In §1 we review those facts which are needed later, establish some properties of functional determinants, and some technical lemmas for later use.

Our case of interest is when X/Γ is not compact. This implies that G has a non-trivial maximal $\mathbf{Q}$-split torus, and a non-trivial system ${}_{\mathbf{Q}}\Phi$ of $\mathbf{Q}$-roots (see 2.1). Section 2 is mainly devoted to the study of the natural restriction map from $\mathbf{R}$-roots to $\mathbf{Q}$-roots. This will show notably that ${}_{\mathbf{Q}}\Phi$ is of one of the two types occurring for the systems of $\mathbf{R}$-roots of irreducible bounded symmetric domains (2. 9).

Section 3 introduces the notion of rational boundary component F by means of two conditions:

(i) $U(F)/(U(F) \cap \Gamma)$ is compact, if $U(F)$ is the unipotent radical of $N(F)$,

(ii) $\Gamma(F) = (N(F) \cap \Gamma)/(Z(F) \cap \Gamma)$ is discontinuous on F.

The condition (i) is equivalent to $N(F)_{\mathbf{C}}$ being defined over $\mathbf{Q}$. The main result of § 3 shows that, in our case, this in fact implies (ii), or rather more precisely implies that $\Gamma(F)$ is of arithmetic type. The map $F \mapsto N(F)_{\mathbf{C}}$ induces then a bijection of the set of rational boundary components onto the set of proper maximal parabolic $\mathbf{Q}$-groups (3.7). If X is not the unit disc, then $\dim_{\mathbf{C}} F \leqq \dim_{\mathbf{C}} X - 2$, (3.15).

Section 4 is devoted to the construction of V^*, following the pattern of

Satake's paper [33]: the union X^* of X and its rational boundary components is endowed with a topology, defined by means of a suitable fundamental set in X, such that each $g \in G_{\mathbf{Q}} \cap G^0_{\mathbf{R}}$ operates continuously on X^* and such that X^*/Γ, supplied with the quotient topology, is a compact Hausdorff space. This is the sought-for compactification V^* of V. It is the union of V and of the quotients $F_i/\Gamma(F_i) = V_i$, where F_i runs through a set of representatives of the different Γ-orbits of rational boundary components (4.9, 4.11). It is shown that every $x \in V^*$ has a basis of open neighborhoods $\{U_\alpha\}$ such that each $U_\alpha \cap V$ is connected (4.15).

Sections 5–8 are devoted to automorphic forms, and in particular to those which are called here Poincaré-Eisenstein series (P-E series for short); they generalize simultaneously Poincaré series and Eisenstein series. They are first introduced in § 6 in a general setting, suggested by results of Harish-Chandra and Godement on Poincaré series and Eisenstein series, proved or stated in §§ 5, 6. In § 7 we turn to the more special P-E series which play a central role in our paper. (On the generalized upper half-plane, they are different from, although related to, the series introduced by Maass [28], under the name of Poincaré series.) They are defined as follows: let F be a rational boundary component and σ_F be the canonical projection of X onto F (see *supra*). A P-E series adapted to F, of weight m, is a series of the form

$$E(x) = \sum\nolimits_{\Gamma/\Gamma_0} \varphi\big(\sigma_F(x\cdot\gamma)\big)\cdot J(x, \gamma)^m ,$$

where φ is a polynomial on F, (in the coordinates of the canonical bounded realization of F), J the functional determinant in the unbounded realization of X associated to F, and Γ_0 a suitable subgroup of Γ. The convergence of these series follows from the results of § 6. Their behavior at rational boundary components is studied in §§ 7, 8, where an operator similar to the Φ-operator of Maass is developed, at least for P-E series. The main idea is to prove the existence of normal (absolute) majorants of the above series in certain sufficiently big sets, which are parts of Siegel domains, so that it becomes possible to deal with such series termwise in such sets. This majorant is constructed by means of a suitable rational representation of G, and to discuss the behavior of an individual term, we use mainly the Bruhat decomposition of $G_{\mathbf{Q}}$ and some properties of weights of representations (7.6, 7.8). Our main result is that a P-E series (adapted to F) has, in a suitable sense, a holomorphic limit $\Phi_{F'}E$ as we approach any rational boundary component F'; the image of E under Φ is by definition the collection of the limits $\Phi_{F'}E$; if $\dim F' \leqq \dim F$ and $F' \not\subset F\cdot\Gamma$, then $\Phi_{F'}E = 0$; moreover, the image of Φ_F contains the module of all Poincaré series of F with respect to $\Gamma(F)$ for infinitely many weights (8.5).

Part III consists of §§ 9, 10. In the latter, we endow V^* with the sheaf $\mathcal{A}$ of germs of continuous functions whose restrictions to the V_i's are analytic. Section 9 proves a prolongation theorem of analytic structure, similar in spirit to those of [2, 18, 35], which, combined with the results of § 8 on P-E series, and known facts on Poincaré series [19], allows us to prove that $(V^*, \mathcal{A})$ is an irreducible normal analytic space (10.4). The existence of a projective embedding of V^* by means of automorphic forms, whose image is projectively normal, follows then in the usual manner (10.11).

Let $\dim G > 3$. Then we have $\dim_{\mathbf{C}}(V^* - V) \leqq \dim_{\mathbf{C}} V - 2$. Standard facts about normal spaces imply therefore that every Γ-automorphic function, i.e., every meromorphic function on V, extends to a meromorphic function on V^*; hence the field of Γ-automorphic functions is an algebraic function field, each element of which is the quotient of two automorphic forms of the same weight (10.12). Also, an extension theorem of Serre [36] shows then that every automorphic form of the classical type extends to a holomorphic cross section of an algebraic coherent sheaf on V^* (10.14); this generalizes Koecher's principle.

Finally, an appendix (§ 11) contains some remarks on the full groups of isometries and of automorphisms of X.

The main results of this paper were announced in [7], and are also described in [5]. Similar theorems have been stated independently, with sketches of some proofs, by Pyateckii-Shapiro [31]. Earlier special cases may be found notably in [2, 3, 30, 35]. These are mostly connected with families of abelian varieties, and the construction of the compactification gives a concrete realization, in many cases, of the variety of moduli of such varieties. In this paper, we leave untouched the question of the minimal field of definition for a projective model of V^*, and of the possible connection of V^* with moduli of algebraic structures. For the known results in that direction, we refer to [37] where other references to related work are also given.

0. Notation and conventions

In this paragraph, we collect some notation to be used frequently in this paper without further reference.

0.1. As is usual, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote respectively the ring of integers, and the fields of rational, real, and complex numbers. If A is a commutative ring, $\mathbf{GL}(n, A)$ or $\mathbf{GL}_{n,A}$ is the group of $n \times n$ matrices with coefficients in A whose determinant is a unit of A, and $\mathbf{SL}(n, A)$ or $\mathbf{SL}_{n,A}$, the group of elements of determinant one in $\mathbf{GL}_{n,A}$. The group of units of a ring B is denoted by B^*.

0.2. If G is a group, and M a non-empty subset of G, then $N(M)$ or $N_G(M)$

(resp. $Z(M)$ or $Z_G(M)$) is the normalizer (resp. centralizer) of M in G. Thus

$$N(M) = \{g \in G \mid g \cdot M \cdot g^{-1} = M\} ,$$
$$Z(M) = \{g \in G \mid g \cdot m \cdot g^{-1} = m \,(m \in M)\} = \bigcap_{m \in M} N(m) .$$

The inner automorphism $h \mapsto g \cdot h \cdot g^{-1} (h \in G)$ is denoted $\operatorname{Int} g$. Often, we write ${}^gM$ for $\operatorname{Int} g(M)$, and M^g for $\operatorname{Int} g^{-1}(M)$.

0.3. As regards algebraic groups, we follow in general the notation of [14]. However, our *universal field* is $\mathbf{C}$, and so, in this paper algebraic group stands for complex linear algebraic group. An algebraic group here may always be (and will tacitly be whenever convenient) identified with an algebraic subgroup of $\mathbf{GL}(n, \mathbf{C})$. Algebraic group defined over k and k-group will be used synonymously. For a subring A of $\mathbf{C}$, we put $G_A = G \cap \mathbf{GL}(n, A)$. The algebraic group G will be identified with $G_{\mathbf{C}}$.

The Lie algebra of an algebraic group, or of a Lie group, $G, H, \cdots$ will usually be denoted by the corresponding lower case German letter. If G is algebraic, defined over k, then $\mathfrak{g} = \mathfrak{g}_k \otimes \mathbf{C}$, where $\mathfrak{g}_k$ is a uniquely determined Lie algebra over k. If k' is an overfield of k, then $\mathfrak{g}_{k'} = \mathfrak{g}_k \otimes_k k'$.

In both the algebraic and Lie group cases, Ad denotes the adjoint representation of G into $\mathfrak{g}$, where $\operatorname{Ad} g (g \in G)$ is the differential of $\operatorname{Int} g$ at e. The restriction of $\operatorname{Ad} g$ to a subspace $\mathfrak{v}$ is denoted $\operatorname{Ad}_{\mathfrak{v}} g$.

0.4. Let G be a k-group. Unless otherwise said, a *character* of G is a rational character, i.e., a morphism of algebraic groups of G into $\mathbf{GL}(1, \mathbf{C})$. The characters of G form a finitely generated commutative group, denoted $X(G)$, which is free if G is connected. The subgroup of elements of $X(G)$ which are defined over k is denoted by $X(G)_k$.

The value of $a \in X(G)$ on $g \in G$ will be written $a(g)$, or more often g^a. In the latter case, it is implied that the group operation in $X(G)$ is written additively, and that usually no notational distinction is made between a and its differential, which is a linear form on $\mathfrak{g}$. In particular, we have, by convention, $g^a = \exp a(X)$ $(X \in \mathfrak{g}, g = \exp X)$.

0.5. An algebraic group G is a *torus* if it is isomorphic to a product of groups $\mathbf{C}^*$; a torus *splits over* k, or is k-trivial, if it is moreover defined over k and isomorphic over k to a product of groups $\mathbf{C}^*$.

Let G be a k-group. Its *radical* $R(G)$ (resp. *unipotent radical* $R_u(G)$, resp. *split radical*) is the greatest connected normal solvable subgroup of G (resp. normal unipotent subgroup of G, resp. the normal subgroup generated by $R_u(G)$ and the k-split tori of $R(G)$). G is *reductive* (resp. *semi-simple*) if $R_u(G) = \{e\}$ (resp. $R(G) = \{e\}$). G is *simple over* k (resp. *almost simple* *over* k) if it has no (resp. connected) proper normal k-subgroup. G is an *almost direct product* of normal subgroups G_i if it is the quotient by a finite group of the product of the G_i's.

0.6. The identity component of a topological group H is denoted by H^0. We recall that if G is algebraic, then $G_{\mathbf{C}}$ is connected as a topological group if and only if it is connected as an algebraic group (i.e. the underlying algebraic variety is irreducible). However, if G is connected, defined over $\mathbf{R}$, the group $G_{\mathbf{R}}$, viewed as a real Lie group may have more than one connected component, but will always have only finitely many connected components.

0.7. Let A be a set. A function on A, with values in a locally compact space, is *bounded* if its range is relatively compact. A function with values in the space $\mathbf{R}^+$ of *strictly* positive real numbers is *multiplicatively bounded* if there are strictly positive constants c, c' such that $c \leqq f(a) \leqq c'(a \in A)$.

Let u, v be functions on A with values in the set of positive real numbers. We write $u \prec v$ if there exists a strictly positive constant c such that

$$u(a) \leqq c \cdot v(a) , \qquad (a \in A)$$

and $u \succ v$ (resp. $u \asymp v$) if $v \prec u$ (resp. $u \prec v$ and $v \prec u$).

I. The compactification V^* as a topological space

1. Natural compactification and Cayley transforms of a bounded symmetric domain.

1.1. The following notation will be used in this section.

G is a connected reductive algebraic group defined over $\mathbf{R}$ which has no non-trivial character defined over $\mathbf{R}$. Thus $G^0_{\mathbf{R}}$ is a connected Lie group with reductive Lie algebra and compact center.[1] We denote by $\mathfrak{g}$ the Lie algebra of $G^0_{\mathbf{R}}$.

K is a maximal compact subgroup of $G^0_{\mathbf{R}}$. The symmetric space $X = K\backslash G^0_{\mathbf{R}}$ is assumed to carry an invariant complex structure. It is then equivalent to a bounded symmetric domain [22, 24], and is hermitian symmetric.

$\mathfrak{p}$ is the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{g}$ with respect to the Killing form, hence $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Since X is hermitian symmetric, we have the direct sum decomposition

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- \qquad (\mathfrak{p}^+ \oplus \mathfrak{p}^- = \mathfrak{p}_{\mathbf{C}}) ,$$

where $\mathfrak{p}^\pm$ is a commutative subalgebra normalized by $\mathfrak{k}_{\mathbf{C}}$.

[1] Essentially, it would suffice to consider the case where G is semi-simple, without compact factors. However, it is more convenient for future references in this paper to start from a slightly more general assumption.

$\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{k}$, and therefore also of $\mathfrak{g}$, in view of our assumption on X, and $\Phi = \Phi(\mathfrak{h}_C, \mathfrak{g}_C)$ is the set of roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$. We let $E_\mu (\mu \in \Phi)$ be root vectors, and H_μ be elements of $\mathfrak{h}_C$ verifying

$$[E_\mu, E_{-\mu}] = H_\mu , \qquad \nu(H_\mu) = 2(\nu, \mu)\cdot(\mu, \mu)^{-1} \qquad (\mu, \nu \in \Phi) ,$$

where (,) is the restriction of the Killing form to $\mathfrak{h}_C$, and such that the complex conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$ permutes E_μ and $E_{-\mu}$ whenever $E_\mu \in \mathfrak{p}^\pm$. Let $\pi^\pm = \{\mu \in \Phi \mid E_\mu \in \mathfrak{p}^\pm\}$. The elements $E_\mu(\mu \in \pi^\pm)$ form a basis of $\mathfrak{p}^\pm$, and the elements

$$X_\mu = E_\mu + E_{-\mu} , \qquad Y_\mu = i(E_\mu - E_{-\mu}) \qquad (\mu \in \pi^+)$$

form a basis (over $\mathbf{R}$) of $\mathfrak{p}$.

Two independent roots μ, ν are said to be strongly orthogonal if neither $\mu + \nu$ nor $\mu - \nu$ are roots. We fix once for all a maximal set $(\mu_1, \cdots, \mu_t)$ of strongly orthogonal roots in π^+, as in [24], and write $H_i, E_i, E_{-i}, X_i, Y_i$ for $H_{\mu_i}, E_{\mu_i}, E_{-\mu_i}, X_{\mu_i}, Y_{\mu_i}$.

1.2. *The system of* $\mathbf{R}$*-roots.* We let $\mathfrak{a}$ be the subalgebra of $\mathfrak{p}$ spanned by $X_1, \cdots, X_t$, and ${}_{\mathbf{R}}\Phi = {}_{\mathbf{R}}\Phi(\mathfrak{a}, \mathfrak{g})$, the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, to be called the $\mathbf{R}$-roots of $\mathfrak{g}$. The algebra $\mathfrak{a}$ is a maximal commutative subalgebra of $\mathfrak{p}$, and is maximal among the subalgebras of $\mathfrak{g}$ which can be diagonalized in the adjoint representation. $\mathfrak{g}$ is the direct sum of the centralizer $\mathfrak{z}(\mathfrak{a})$ of $\mathfrak{a}$ and of the root spaces

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [a, x] = \alpha(a)\cdot x, a \in \mathfrak{a}\} \qquad (\alpha \in {}_{\mathbf{R}}\Phi) .$$

Assume X to be irreducible. Then ${}_{\mathbf{R}}\Phi$ is known to be of one of two types, to be denoted by C_t and BC_t. If (γ_i) are coordinates with respect to the basis $((1/2)X_i)$, then C_t consists of the roots $\pm(\gamma_i \pm \gamma_j)/2$, $(1 \leqq i < j \leqq t)$, $\pm\gamma_i (1 \leqq i \leqq t)$ and BC_t is the union of C_t and of the set of elements $\pm\gamma_i/2\,(1 \leqq i \leqq t)$. In both cases we always take as ordering the lexicographic ordering defined by the basis (X_i). The set ${}_{\mathbf{R}}\Delta$ of simple $\mathbf{R}$-roots consists then of

$$\alpha_i = (\gamma_i - \gamma_{i+1})/2 \qquad (1 \leqq i < t) ,$$

and of $\alpha_t = \gamma_t$ (resp. $\alpha_t = \gamma_t/2$) if ${}_{\mathbf{R}}\Phi$ is of type C_t (resp. BC_t).

The numbering of the simple $\mathbf{R}$-roots thus defined will be referred to as the *canonical numbering.*

1.3. *Maximal parabolic subgroups.* A *parabolic subgroup* of $G^0_{\mathbf{R}}$ is the intersection of $G^0_{\mathbf{R}}$ with an algebraic subgroup P of G which is parabolic, (i.e., such that G/P is a projective variety) and defined over $\mathbf{R}$. The description of the parabolic subgroups of an algebraic group is recalled, in a more general setting, in 2.2. Here we introduce the minimal ones and the maximal ones, which will play a fundamental role in this paper.

Let $\mathfrak{n}$ be the sum of the $\mathfrak{g}_\alpha$ $(\alpha > 0)$, and $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. These are closed subgroups, with N unipotent, normalized by A, and $A \cdot N$ is maximal among the connected subgroups of $G^0_{\mathbf{R}}$ which can be put in triangular form over $\mathbf{R}$. The normalizer P of N is equal to the semi-direct product $P = Z(A) \cdot N$ and $Z(A) = M \times A$ with $M = Z(A) \cap K$. The group P is generated by P^0 and a commutative subgroup of type $(2, 2, \cdots, 2)$ of M, which can be described as $K \cap \exp i \cdot \mathfrak{a}$, as follows from [14, 14.4]. Every minimal parabolic subgroup of $G^0_{\mathbf{R}}$ is conjugate to P.

Assume, for convenience, X to be irreducible. We let $\mathfrak{a}_b$ $(1 \leqq b \leqq t)$ be the one-dimensional subspace on which all simple $\mathbf{R}$-roots but α_b are zero, and $A_b = \exp \mathfrak{a}_b$. The space $\mathfrak{a}_b$ is spanned by $X_1 + \cdots + X_b$. We let P_b be the subgroup generated by $Z(A_b)$ and N, and V_b be its unipotent radical. The group P_b is the semi-direct product of V_b by $Z(A_b)$. The Lie algebra $\mathfrak{v}_b$ is the sum of the root spaces $\mathfrak{g}_\alpha$ where α is >0 and not zero on $\mathfrak{a}_b$. Therefore, α runs through the roots

$$(\gamma_i \pm \gamma_j)/2, (1 \leqq i \leqq b < j \leqq t) , \qquad (\gamma_i + \gamma_j)/2, (1 \leqq i \leqq j \leqq b) ,$$

and the roots $\gamma_i/2$ $(1 \leqq i \leqq b)$ in the case BC_t. Let $\mathfrak{l}_b$ (resp. $\mathfrak{l}'_b$) be the sum of the subspaces $\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, where α runs through the $\mathbf{R}$-roots which are linear combinations of $\alpha_{b+1}, \cdots, \alpha_t$ (resp. $\alpha_1, \cdots, \alpha_{b-1}$). These are two simple ideals of $\mathfrak{z}(\mathfrak{a}_b)$, clearly normalized by the Lie algebra $\mathfrak{m}$ of M, and $\mathfrak{z}(\mathfrak{a}_b)$ is the direct sum of $\mathfrak{l}_b$, $\mathfrak{l}'_b$, $\mathfrak{a}_b$ and of an ideal $\mathfrak{m}_b$ of $\mathfrak{m}$. The group $Z(A_b)$ is generated by the analytic groups L_b, L'_b, A_b, with Lie algebras $\mathfrak{l}_b$, $\mathfrak{l}'_b$, $\mathfrak{a}_b$, and by M. Let $\mathfrak{z}_b = \mathfrak{m}_b \oplus \mathfrak{a}_b \oplus \mathfrak{l}'_b \oplus \mathfrak{v}_b$. It is an ideal of $\mathfrak{p}_b$ such that $\mathfrak{p}_b = \mathfrak{l}_b + \mathfrak{z}_b$. Denote by Z^0_b the analytic subgroup of $G^0_{\mathbf{R}}$ with Lie algebra $\mathfrak{z}_b$. We let Z_b be the inverse image in P_b of the centralizer of $(P_b/Z^0_b)^0$ in P_b/Z^0_b. It is a closed normal subgroup of P_b, with Lie algebra $\mathfrak{z}_b$, whose intersection with L_b is the center of L_b. It contains every normal subgroup of P_b with Lie algebra $\mathfrak{z}_b$: in fact, the image in P_b/Z^0_b of such a subgroup is a finite normal subgroup, and therefore centralizes $(P_b/Z^0_b)^0$. In particular, Z_b contains $(Z^0_b)_{\mathbf{C}} \cap G^0_{\mathbf{R}}$, whence

$$(1) \qquad Z_b = (Z_b)_{\mathbf{C}} \cap G^0_{\mathbf{R}} ,$$

where $(Z_b)_{\mathbf{C}}$ and $(Z^0_b)_{\mathbf{C}}$ denote the smallest algebraic subgroups of G containing Z_b and Z^0_b respectively.

By the general conjugacy theorems on parabolic groups (2.2 below), every maximal proper parabolic subgroup of $G^0_{\mathbf{R}}$ is conjugate to one and only one of the groups P_b. It will sometimes be convenient to extend the definition of P_b to $b = 0$, by putting $P_0 = L_0 = G^0_{\mathbf{R}}$, $\mathfrak{l}'_0 = 0$; then $\mathfrak{z}_0 = \mathfrak{v}_0 = 0$ and Z_0 is the center of $G^0_{\mathbf{R}}$.

1.4. *The natural compactification.* Let $P^{\pm} = \exp \mathfrak{p}^{\pm}$ and $K_{\mathbf{C}}$ be the analytic subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k}_{\mathbf{C}}$. These are closed subgroups, and the semi-direct product $K_{\mathbf{C}} \cdot P^{\pm}$ is a parabolic subgroup of $G_{\mathbf{C}}$. The map $(x, k, y) \mapsto e^x \cdot k \cdot e^y$ is a biregular map of $\mathfrak{p}^- \times K_{\mathbf{C}} \times \mathfrak{p}^+$ onto a Zariski-open subset $\Omega = P^- \cdot K_{\mathbf{C}} \cdot P^+$ of $G_{\mathbf{C}}$, which contains $G^0_{\mathbf{R}}$ [24]. An element $g \in \Omega$ will often be written

$$g = g_- \cdot g_0 \cdot g_+ \qquad (g_0 \in K_{\mathbf{C}};\ g_{\pm} \in P^{\pm}) ,$$

and the map $g \mapsto \log g_+$ of Ω onto $\mathfrak{p}^+$ will be denoted ζ. It is known [22], [24] that ζ induces an isomorphism of $X = K \backslash G^0_{\mathbf{R}}$ onto $\zeta(G) = D$, and that D is a bounded domain in $\mathfrak{p}^+$. This is the Harish-Chandra realization of X as a bounded domain. Its closure $\bar{D}$ is therefore compact, and will be called the *natural compactification* of X. The action of $G^0_{\mathbf{R}}$ on D is defined by right translations; i.e., by

$$(1) \qquad p \cdot g = \zeta(e^p \cdot g) \qquad (p \in D,\ g \in G^0_{\mathbf{R}})$$

and is known to extend to a continuous action on $\bar{D}$. Then (1) is true with $p \in \bar{D}$.

1.5. *Boundary components* (see [27], [29], [30]). (i) Assume first X to be irreducible. We use the notation of 1.3. We have the direct sum decomposition

$$\mathfrak{l}_{b,\mathbf{C}} = \mathfrak{k}_{b,\mathbf{C}} \oplus \mathfrak{p}_b^+ \oplus \mathfrak{p}_b^- \qquad (\mathfrak{p}_b^{\pm} = \mathfrak{l}_{b,\mathbf{C}} \cap \mathfrak{p}^{\pm}) ,$$

the space $X_b = K_b \backslash L_b$ is hermitian symmetric, and the restriction of ζ to L_b yields the Harish-Chandra realization D_b of X_b as a bounded domain.

Let $o_b = -(E_1 + \cdots + E_b)$ $(1 \leqq b \leqq t)$, and put $o_0 = o$. Then

$$\bar{D} = \bigcup\nolimits_{0 \leqq b \leqq t} o_b \cdot G^0_{\mathbf{R}} .$$

Moreover, if $g \in L_b$, then $o_b \cdot g = o_b + \zeta(g)$. Therefore, the orbit F_b of o_b under L_b is just $o_b + D_b$, and is contained in an affine subspace of $\mathfrak{p}^+$. The transforms of the F_b's by elements of G are the *boundary components* of $\bar{D}$. We allow here b to be equal to zero, and view D itself as a boundary component (sometimes called the *improper boundary component* of $\bar{D}$).

If X is not irreducible, then it is a product of irreducible hermitian symmetric spaces X_i corresponding to the different semi-simple, simple, non-compact ideals of $\mathfrak{g}$, D is the product of the Harish-Chandra realizations D_i of the X_i, and $\bar{D}$ the product of the $\bar{D}_i$. A boundary component is a product of boundary components of the different factors. The F_b's or, if X is not irreducible, the products of components F_b's corresponding to the different irreducible factors of X, are the *standard boundary components*.

The above construction is *hereditary*: if F is a boundary component then its closure $\bar{F}$ in $\bar{D}$ may be identified with the natural compactification of F, and its boundary components are also boundary components of X. More specifically, if X is irreducible and $F = F_b$, then $\bar{F}_b = \bar{D}_b + o_b$ and the standard boundary components of $\bar{D}_b$ may be identified with the F_c's ($b \leqq c \leqq t$); in fact F_c would have $c - b$ as index in the canonical numbering for X_b. The groups L_c and $P_c \cap L_b$ are in the same relationship to L_b as L_b and P_b are to $G^0_{\mathbf{R}}$. This is clear from the construction.

We refer to [30] for various more geometric definitions of the boundary components in the natural compactification and to [29] for a proof of their equivalence.

For every boundary component F, we put

$$N(F) = \{g \in G^0_{\mathbf{R}} \mid F \cdot g = F\} ,$$
$$Z(F) = \{g \in G^0_{\mathbf{R}} \mid f \cdot g = f (f \in F)\} ,$$
$$G(F) = N(F)/Z(F) ,$$

and let $U(F)$ be the unipotent radical of $N(F)$. The group $N(F)$ is the *normalizer*, and $Z(F)$ the *centralizer*, of F. If X is irreducible, we have, in the notation of 1.3,

$$(1) \qquad N(F_b) = P_b , \qquad U(F_b) = V_b .$$

Moreover

$$(2) \qquad Z(F_b) = Z_b .$$

In fact, $Z(F_b)$ is a normal subgroup of P_b with Lie algebra $\mathfrak{z}_b$, hence $Z(F_b) \subset Z_b$ by 1.3; on the other hand, the image in $G(F_b)$ of an element $z \in Z_b$ centralizes the image L''_b of L_b, and therefore the maximal compact subgroups of L''_b, hence it acts trivially on F_b, and $Z_b \subset Z(F_b)$.

Returning to the general case, we see, by applying 1.3 (1) and 1.5 (2) to each irreducible factor of X, that $Z(F)$ is the intersection of $G^0_{\mathbf{R}}$ with an $\mathbf{R}$-subgroup of G. Furthermore, by 11.2, each element of $N(F)$ induces a complex analytic homeomorphism of F, hence (11.6), $G(F)$ is connected, with center reduced to $\{e\}$; equivalently, if X is irreducible, we have $N(F_b) = L_b \cdot Z_b$.

(ii) If F and F' are two boundary components such that $F' \subset \bar{F}$, then there exists $g \in G^0_{\mathbf{R}}$ such that $F \cdot g$ and $F' \cdot g$ are both standard boundary components. To see this, we may assume X to be irreducible. Let then b, c be the indices such that $F \subset F_b \cdot G^0_{\mathbf{R}}$, $F' \subset F_c \cdot G^0_{\mathbf{R}}$, and let $u \in G^0_{\mathbf{R}}$ be such that $F \cdot u = F_b$. Then $F' \cdot u$ and F_c are both boundary components of F_b, of the same dimension. Consequently, there exists $v \in L(F_b)$ such that $F' \cdot u \cdot v$ is standard. $F' \cdot u \cdot v$ is then equal to F_c; hence, $g = u \cdot v$ verifies our condition.

(iii) *If X is irreducible, and* $\dim_{\mathbf{C}} X \geqq 2$, *then* $\dim_{\mathbf{C}} X \geqq \dim_{\mathbf{C}} F + 2$ *for every proper boundary component F of X.*

To see this, we may assume that $F = F_b$ $(1 \leqq b \leqq t)$. If $b = t$, F_b is a point, and there is nothing to prove. So assume $b \neq t$. Then $t \geqq 2$, and $\mathfrak{n}$ contains at least three root spaces $\mathfrak{g}_\alpha$, whose sum intersects $\mathfrak{n} \cap \mathfrak{l}_b$ only at the origin, namely those corresponding to $\alpha = (\gamma_b \pm \gamma_t)/2$, γ_b, hence,

$$\dim \mathfrak{n} - \dim (\mathfrak{l}_b \cap \mathfrak{n}) \geqq 3 .$$

On the other hand $\dim_{\mathbf{R}} X = \dim \mathfrak{a} + \dim \mathfrak{n}$, $\dim_{\mathbf{R}} F_b = \dim (\mathfrak{a} \cap \mathfrak{l}_b) + \dim (\mathfrak{l}_b \cap \mathfrak{n})$, and $\dim \mathfrak{a} - \dim (\mathfrak{a} \cap \mathfrak{l}_b) \geqq 1$, whence our assertion.

1.6. *The Cayley transforms of X.* The space X also admits certain unbounded realizations, introduced by Pyateckii-Shapiro [40] in the classical cases under the name of Siegel domains of type I, II or III, and discussed in general by Korányi and Wolf [27]. In this and the next section, we summarize only the results which are used in the sequel. We assume again X to be irreducible.

The *Cayley transform* c_b is, by definition,

$$c_b = \prod\nolimits_{1 \leqq i \leqq b} \exp (\pi/4) \cdot (E_{-i} - E_i), \ (1 \leqq b \leqq t); \qquad c_0 = e .$$

It verifies

$$(1) \qquad \operatorname{Ad} c_b(H_i) = X_i , \qquad \operatorname{Ad} c_b(X_i) = -H_i , \qquad (1 \leqq i \leqq b)$$

$$(2) \qquad \operatorname{Ad} c_b(H_i) = H_i , \qquad \operatorname{Ad} c_b(X_i) = X_i , \qquad (b < i \leqq t)$$

and is transformed into its inverse by the complex conjugation of $G_{\mathbf{C}}$ with respect to $G_{\mathbf{R}}$. Moreover

$$(3) \qquad c_b \cdot g = g \cdot c_b , \qquad (g \in L_b) ,$$

$$(4) \qquad G \cdot c_b \subset P^- \cdot K_{\mathbf{C}} \cdot P^+ .$$

We put then $S_b = \zeta(G \cdot c_b)$, and let G act on S_b by

$$(5) \qquad s \cdot g = \zeta(e^s \cdot c_b^{-1} \cdot g \cdot c_b) .$$

The map $g \mapsto g \cdot c_b$ induces then an isomorphism ν_b of X onto S_b; by definition $S_0 = D$, and S_0 is just the bounded realization. Often, we shall denote also by o the fixed point of K in S_b.

In the next proposition, we denote by $\mathfrak{q}_b$ the subspace of $\mathfrak{p}^+$ spanned by the vectors $E_\mu (\mu \in \pi^+, \mu(H_i) \neq 0$ for at least one $i \leqq b)$. Thus $\mathfrak{p}^+ = \mathfrak{p}_b^+ \oplus \mathfrak{q}_b$.

1.7. PROPOSITION. *We keep the preceding notation. We have* $c_b^{-1} \cdot Z_b \cdot c_b \subset K_{\mathbf{C}} \cdot P^+$, *and the action of Z_b (resp. V_b) on S_b extends to an action of Z_b on $\mathfrak{p}^+$ by means of affine transformations (resp. affine transformations with unipotent linear homogeneous parts) which leave $\mathfrak{q}_b$ stable and induce the* *identity on* $\mathfrak{p}^+/\mathfrak{q}_b$. *The projection* σ_b *of* $\mathfrak{p}^+$ *onto* $\mathfrak{p}_b^+$ *with kernel* $\mathfrak{q}_b$ *maps* S_b *onto* D_b, *and its fibres in* S_b *are the orbits of* Z_b^0 $(0 \leqq b \leqq t)$. *Its restriction to* S_b *commutes with* $N(F_b)$.

This is contained in the more precise results of [27, § 7]. Let $z \in Z_b$. We have then $z' = c_b^{-1}\cdot z\cdot c_b = z'_0\cdot z'_+$ $(z'_0 \in K_{\mathbf{C}},\ z'_+ \in P^+)$. The action of z on S_b or $\mathfrak{p}^+$ is therefore given by

$$(1) \qquad s\cdot z = \operatorname{Ad} z_0'^{-1}(s) + \log z'_+ \qquad (s \in \mathfrak{p}^+)\ .$$

Since $\mathfrak{p}^+$ is commutative, we can replace z'_0 by z', whence

$$(1') \qquad s\cdot z = \operatorname{Ad} c_b^{-1}\cdot z\cdot c_b(s) + \log (c_b^{-1}\cdot z\cdot c_b)_+\ , \qquad (s \in \mathfrak{p}^+,\ z \in Z_b)\ .$$

If $g \in L_b$, then it commutes with c_b, therefore 1.6 (5) becomes

$$(2) \qquad s\cdot g = \zeta(e^s\cdot g) \qquad (s \in S_b,\ g \in L_b)\ .$$

In particular

$$(3) \qquad s\cdot g = \operatorname{Ad} g^{-1}(s) \qquad (s \in S_b,\ g \in K_b)\ .$$

REMARK. Let F be a boundary component, and $g \in G_{\mathbf{R}}^0$ be such that $F\cdot g = F_b$. Then $x \mapsto \sigma_b(x\cdot g)\cdot g^{-1}$ is a holomorphic map of X onto F. If g' is such that $F\cdot g' = F_b$, then $g' = g\cdot n$ $(n \in N(F_b))$; since translation by n commutes with σ_b, we get $\sigma_b(x\cdot g)\cdot g^{-1} = \sigma_b(x\cdot g')\cdot g'^{-1}$ $(x \in X)$. We have thus defined a canonical holomorphic projection of X onto F, equivariant with respect to $N(F)$, to be denoted σ_F. If $F' \subset \bar{F}$, then we have a factorization

$$\sigma_{F'} = \sigma_{F'F} \circ \sigma_F\ ,$$

where $\sigma_{F'F}$ is the canonical projection of F on its boundary component F'. In fact, there exists by 1.5 an element $g \in G_{\mathbf{R}}^0$ such that $F\cdot g = F_b$ and $F'\cdot g = F_c$ $(b \leqq c)$, and it is clear from Proposition 1.7 that $\sigma_c = \sigma_{c,b} \circ \sigma_b$ where $\sigma_{c,b}$ is the canonical projection of F_b onto the standard boundary component F_c.

The remark extends obviously to non-irreducible bounded symmetric domains.

1.8. *Automorphy factors, functional determinants.* Let M be a complex manifold, H a group of automorphisms of M, and Q a complex Lie group. We recall that a (holomorphic) automorphy factor for H on M, with values in Q, is a map $\mu\colon M \times H \to Q$ which, for fixed $h \in H$, is holomorphic in $x \in M$, and which verifies the identity

$$(1) \qquad \mu(x, h\cdot h') = \mu(x, h)\cdot \mu(x\cdot h, h') \qquad (x \in M;\ h, h' \in H)\ ,$$

to be referred to as the *cocycle formula*; it implies

$$(2) \qquad \mu(x, h\cdot h'\cdot h'') = \mu(x, h)\cdot \mu(x\cdot h, h')\cdot \mu(x\cdot h\cdot h', h'') \qquad (x \in M;\ h, h', h'' \in H)\ .$$

It follows immediately from (1) that the set R of elements $h \in H$ for which $\mu(x, h) = \rho(h)$ is independent of x is a subgroup, and that

$$(3) \qquad \mu(x, h\cdot r) = \mu(x, h)\cdot\rho(r) \qquad (x \in M;\ h \in H;\ r \in R)\ .$$

If M is a domain in $\mathbf{C}^n$, then the jacobian $\mathrm{Jac}\,(x, h)$ which associates to $h \in H$ and $x \in M$ the differential of h at x, is an automorphy factor with values in $\mathbf{GL}(n, \mathbf{C})$, and $J(x, h) = \det \mathrm{Jac}\,(x, h)$ is an automorphy factor with values in $\mathbf{C}^*$.

Let $M = X$ be an irreducible bounded symmetric domain. We shall denote by $\mathrm{Jac}_b\,(x, g)$ the jacobian of $g \in G^0_{\mathbf{R}}$ at $x \in S_b$, in the unbounded realization associated to F_b, by $J_b(x, g)$ its determinant, and by $j_b(x, g)$ the functional determinant of $g \in P_b$ at $x \in D_b$ $(0 \leqq b \leqq t)$. Our next aim is to obtain some information on $J_b(x, g)$ when $g \in P_b$, which will be used in studying Poincaré-Eisenstein series.

It is immediate that $\mu_b(x, g) = (e^x\cdot c_b\cdot g\cdot c_b^{-1})_0$ $(x \in S_b,\ g \in G^0_{\mathbf{R}})$ is a holomorphic automorphy factor, with values in $K_{\mathbf{C}}$. It is called the *canonical automorphy factor* for the unbounded realization S_b. The automorphy factors usually considered in the theory of holomorphic automorphic forms are of the form $\rho(\mu_b(x, g))$ where $\rho\colon K_{\mathbf{C}} \mapsto \mathbf{GL}(m, \mathbf{C})$ is a holomorphic representation. The following lemma asserts that Jac_b is of this type. It is well-known for the bounded realization; the proof is essentially the same in the general case, and is included for the sake of completeness.

1.9. LEMMA. *We keep the notation of* 1.8, *and identify the tangent space to a point* $x \in \mathfrak{p}^+$ *with* $\mathfrak{p}^+$ *by translation. Then*

$$\mathrm{Jac}_b\,(x, g) = \mathrm{Ad}_{\mathfrak{p}^+}\, h_0^{-1} \qquad \bigl(x \in S_b,\ g \in G^0_{\mathbf{R}};\ h = (e^x\cdot c_b\cdot g\cdot c_b^{-1})\bigr)\ .$$

Let $X \in \mathfrak{p}^+$. Then $x + X$ goes under the differential dg of the automorphism of S_b defined by g onto an element $Y + x\cdot g$; we have to prove that

$$(1) \qquad Y = \mathrm{Ad}\, h_0^{-1}(X)\ .$$

Write g' for $c_b\cdot g\cdot c_b^{-1}$. By definition

$$(2) \qquad x\cdot g = \zeta(e^x\cdot g') = \log\,(e^x\cdot g')_+ = \log h_+\ ,$$

and

$$Y + x\cdot g = d\zeta\Bigl\{\frac{d}{dt}(e^{tX}\cdot e^x\cdot g')\Big|_{t=0}\Bigr\}\,.$$

Clearly

$$e^{tX}\cdot e^x\cdot g' = h_-\cdot e^{tU}\cdot h_0\cdot h_+ \qquad \bigl(U = \mathrm{Ad}\, h_-^{-1}(X)\bigr)\ .$$

But the bracket relations

$$(3) \qquad [\mathfrak{k}_{\mathbf{C}}, \mathfrak{p}^{\pm}] \subset \mathfrak{p}^{\pm} , \qquad [\mathfrak{p}^{+}, \mathfrak{p}^{-}] \subset \mathfrak{k}_{\mathbf{C}} ,$$

imply readily that if $p \in \mathfrak{p}^-$ and $X \in \mathfrak{p}^+$, then $\mathrm{Ad}\exp p(X) \equiv X$, modulo $\mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^-$. We may therefore write $U = X + Z$ ($Z \in \mathfrak{k}_{\mathbf{C}} \oplus p^-$), whence

$$e^{tX} \cdot h = h_- \cdot h_0 \cdot e^{t(X'+Z')} \cdot h_+ \qquad \left(X' = \mathrm{Ad}\, h_0^{-1}(X),\, Z' = \mathrm{Ad}\, h_0^{-1}(Z)\right) .$$

We have then

$$\frac{d}{dt}(e^{tX} \cdot h)\Big|_{t=0} = h_- \cdot h_0 \cdot (X' + Z') \cdot h_+ .$$

Since Z' also belongs to $\mathfrak{k}_{\mathbf{C}} + \mathfrak{p}^-$, the image of the right-hand side under $d\zeta$ is $X' + \log h_+$, which, in view of (2), proves our contention.

1.10. LEMMA. *Let β_b ($0 \leqq b \leqq t$) be the sum of roots $\mu \in \pi^+$ such that $E_\mu \in \mathfrak{p}_b^+$. Then $m_b = \beta_b(H_i)$ is a strictly positive integer independent of i ($b < i \leqq t$), and $m_b > m_c$ if $0 \leqq b < c \leqq t$.*

In view of the "hereditary" character of the natural compactification (1.5), it is enough to prove this when $b = 0$, $G_{\mathbf{R}}^0 = L_b$.

Let $\mathfrak{h}_t = \mathrm{Ad}\, c_t^{-1}(\mathfrak{a}_{\mathbf{C}})$. It is the subalgebra of $\mathfrak{h}$ spanned by the vectors H_i ($1 \leqq i \leqq t$). We denote the coordinates with respect to the basis $(H_i/2)$ also by γ_i. We choose an ordering on Φ verifying the following conditions:

The elements of π^+ are positive, the restrictions to $\mathfrak{h}_t$ of the elements of π^+ are the linear forms $(\gamma_i + \gamma_j)/2$ ($1 \leqq i \leqq j \leqq t$), and also the forms $\gamma_i/2$ in the case BC_t; the positive roots of $\mathfrak{k}_{\mathbf{C}}$ restrict to the differences $(\gamma_i - \gamma_j)/2$ ($1 \leqq i < j \leqq t$), and also to $\gamma_i/2$ in the case BC_t.

This is always possible [22, § 6]. Let $\Delta = \{\nu_1, \cdots, \nu_l\}$ be the corresponding set of simple roots. It is known that we may assume $\theta = \{\nu_1, \cdots, \nu_{l-1}\}$ to be the set of simple roots of $\mathfrak{k}_{\mathbf{C}}$, and that the elements of π^+ are the roots which are congruent to ν_l modulo a linear combination of elements in θ, [22]. Moreover, since $\mathfrak{k}_{\mathbf{C}}$ normalizes $\mathfrak{p}^+$, its Weyl group permutes the elements of π^+ and leaves β_0 invariant. In particular, β_0 is left fixed by the fundamental reflections r_ν ($\nu \in \theta$), whence

$$(1) \qquad (\beta_0, \nu_i) = 0 \qquad (1 \leqq i < l) .$$

The sum of two elements in π^+ is never a root, hence

$$(2) \qquad (\mu, \nu) \geqq 0 \qquad (\mu, \nu \in \pi^+) .$$

We have therefore

$$(3) \qquad (\beta_0, \mu) = (\beta_0, \nu_l) \geqq (\nu_l, \nu_l) > 0 \qquad (\mu \in \pi^+) .$$

But $\nu(H_i) = 2(\nu, \gamma_i) \cdot (\gamma_i, \gamma_i)^{-1}$ is an integer for every $\nu \in \Phi$, and each i; therefore,

$$(4) \qquad \beta_0(H_i) = 2 \cdot (\beta_0, \gamma_i) \cdot (\gamma_i, \gamma_i)^{-1} \in \mathbf{Z},\ \beta_0(H_i) > 0 .$$

The relative Weyl group of $\mathrm{Ad}\, c_t^{-1}(\mathfrak{g})$ with respect to $\mathrm{Ad}\, c_t^{-1}(\mathfrak{a})$ contains the permutations of the γ_i. But every such transformation is induced by an element of the Weyl group of $\mathfrak{g}_{\mathrm{C}}$ with respect to $\mathfrak{h}_{\mathrm{C}}$ (see e.g. [14, 5.5]). It follows then that (γ_i, γ_i) is independent of i, whence our first assertion.

The difference $m_0 - m_c$ ($c \geqq 1$) is the sum of the numbers $\mu(H_t)$ where μ runs through the elements of π^+ such that $E_\mu \notin \mathfrak{p}_c^+$; these numbers are all $\geqq 0$ by the above. But there is at least one such μ, for instance one which restricts to $(\gamma_1 + \gamma_t)/2$, for which $\mu(H_t) \neq 0$, which ends the proof.

1.11. PROPOSITION. *Let X be irreducible. Let J_b be the functional determinant function for G_{R}^0 acting on S_b, and j_b the functional determinant function for L_b acting on D_b. Then*

(i) *The function $J_b(x, g)$ is constant along the fibres of the projection $\sigma_b\colon S_b \to D_b$ of 1.7 if $g \in P_b$, is independent of x if $g \in Z_b$, and is equal to one if g is a unipotent element of Z_b. The restriction η_b of J_b to Z_b is a rational character.*

(ii) *If $g \in L_b$, we have $J_b(x, g)^{m_b} = j_b(\sigma_b(x), g)^{m_0}$, with m_0, m_b as in* 1.10.

PROOF OF (i). If $g \in Z_b$ (resp. $g \in Z_b$ and is unipotent), then g acts on S_b by means of an affine transformation (resp. with unipotent linear part) in $\mathfrak{p}^+$; therefore, $J_b(x, g) = \eta_b(g)$ is independent of x (resp. is equal to one); then η_b is a rational character by 1.7 (1′).

Write $g = l \cdot u$ ($l \in L_b$, $u \in Z_b$), and let $z \in Z_b^0$. Using the cocycle formula, we have

$$J_b(x \cdot z, l \cdot u) = J_b(x \cdot z, l) \cdot \eta_b(u) = J_b(x, z \cdot l) \cdot J_b(x, z)^{-1} \cdot \eta_b(u) ,$$
$$J_b(x \cdot z, l \cdot u) = J_b(x, l) \cdot \eta_b(z') \cdot \eta_b(z)^{-1} \cdot \eta_b(u) ,$$

where $z' = l^{-1} \cdot z \cdot l$. But Z_b^0 is the semi-direct product of V_b by a reductive group which centralizes L_b (see 1.3); therefore, $\eta_b(z) = \eta_b(z')$ and

$$J_b(x \cdot z, l \cdot u) = J_b(x, l \cdot u) .$$

Since the fibres of σ_b are the orbits of Z_b^0, this ends the proof of (i).

PROOF OF (ii). For every element $g \in K_{b,\mathrm{C}}$, let us put

$$\Psi(g) = \det(\mathrm{Ad}_{\mathfrak{p}^+} g^{-1}) , \qquad \psi(g) = \det(\mathrm{Ad}_{\mathfrak{p}_b^+} g^{-1}) .$$

We want to prove

$$\Psi(g)^{m_b} = \psi(g)^{m_0} \qquad (g \in K_{b,\mathrm{C}}) . \tag{4}$$

Assume first that $g = \exp(\lambda_{b+1} H_{b+1} + \cdots + \lambda_t H_t)$. In this case, $\Psi(g)$ (resp. $\psi(g)$) is the product of the numbers $\exp \mu(-\log g)$ where μ runs through the roots μ such that $E_\mu \subset \mathfrak{p}^+$ (resp. $E_\mu \subset \mathfrak{p}_b^+$). Using 1.10, we get

$$\Psi(g) = \textstyle\prod_{b < i \leqq t} \exp -\lambda_i \cdot m_0 , \qquad \psi(g) = \textstyle\prod_{b < i \leqq t} \exp -\lambda_i \cdot m_b , \tag{5}$$

which proves our contention in this case. It is also clear from (5) that Ψ and ψ are not identically equal to one on the subgroup just considered. The group $K_{b,\mathrm{C}}$ is generated by its derived group, on which both Ψ and ψ are equal to one, and by its one-dimensional center. It is therefore also generated by its derived group and the group of elements considered in (5), which proves (4).

Any element $g \in L_b$ commutes with the Cayley transform c_b; therefore, we have by 1.9, applied to L_b operating on S_b and on D_b:

$$(6) \qquad J_b(x, g) = \Psi(g) \,, \qquad j_b(y, g) = \psi(g) \,, \qquad (x \in S_b;\, y \in D_b;\, g \in K_b)$$

$$(7) \qquad J_b(o, g) = \Psi(g_0) \,, \quad j_b(o_b, g) = \psi(g_0) \,, \qquad (g \in L_b) \,.$$

Given $x \in S_b$, there exists $l \in L_b$ such that $\sigma_b(x) = o_b \cdot l$. The points x and $o_b \cdot l$ belong to the same fibre of σ_b, hence $J_b(x, g) = J_b(o_b \cdot l, g)$ by (i), and the latter functional determinant has to be compared with $j_b(o_b \cdot l, g)$. The desired relationship then follows from the cocycle formula and what has already been proved.

1.12. PROPOSITION. *Let $a = \exp(\lambda_1 X_1 + \cdots + \lambda_t X_t)$ be an element of A. Then*

$$J_b(o, a) = \prod\nolimits_{1 \leqq i \leqq b} e^{-\lambda_i \cdot m_0} \cdot \prod\nolimits_{b < i \leqq t} (\cosh \lambda_i)^{-m_0} \,.$$

Write $a = u \cdot v$ ($u = \exp(\lambda_1 X_1 + \cdots + \lambda_b X_b)$; $v = \exp(\lambda_{b+1} X_{b+1} + \cdots + \lambda_t X_t)$). We have then $u \in Z_b$, $v \in L_b$, and therefore, by 1.11,

$$J_b(o, a) = J_b(o, u) \cdot J_b(o, v) \,.$$

Since $c_b^{-1} \cdot u \cdot c_b = u' = \exp(\lambda_1 H_1 + \cdots + \lambda_b H_b) \in K_{\mathrm{C}}$, the action of u on S_b is given by

$$s \cdot u = \operatorname{Ad} u'^{-1}(s) \,, \qquad (s \in S_b) \,,$$

hence

$$(1) \qquad J_b(x, u) = \Psi(c_b^{-1} \cdot u \cdot c_b) = \prod\nolimits_{1 \leqq i \leqq b} e^{-\lambda_i \cdot m_0} \,.$$

On the other hand, a standard computation on the three-dimensional simple group (see e.g. [24, p. 316]) shows that the K_{C}-component v_0 of v is

$$v_0 = \prod\nolimits_{i \leqq t} \exp \log \cosh \lambda_i \cdot H_i \,,$$

and our assertion now follows from 1.11 (5), (6).

1.13. COROLLARY. *Let $h_c(\lambda) = \exp \lambda(X_1 + \cdots + X_c)$ $(1 \leqq c \leqq t, \lambda \in \mathbf{R})$. Then $J_c(o, h_c(\lambda)) = J_b(o, h_c(\lambda))$ if $c \leqq b$, and $J_b(o, h_c(\lambda)) \cdot J_c(o, h_c(\lambda))^{-1}$ tends monotonically to zero as $\lambda \to -\infty$ if $c > b$.*

By 1.11, we have

$$J_c(o, h_c(\lambda)) = \exp -\lambda \cdot m_0 \cdot c \,,$$

and

$$J_b(o, h_c(\lambda)) = \exp -\lambda\cdot m_0\cdot c\ , \qquad (c \leqq b)\ ,$$
$$J_b(o, h_c(\lambda)) = \exp -\lambda\cdot m_0\cdot b(\cosh\lambda)^{-m_0(c-b)}\ , \qquad (c > b)\ ,$$

whence our assertion.

1.14. *Remark on the Bergman kernel function.* Let $K_b(z, w)$ be the Bergman kernel function in S_b. We have therefore

$$K_b(z\cdot g, w\cdot g) = K_b(z, w)\cdot|J_b(z, g)|\cdot|J_b(w, g)|\ , \qquad (z, w \in S_b; g \in G^0_{\mathbf{R}})\ .$$
$$K_b(z\cdot k, w\cdot k) = K_b(z, w) \qquad (z, w \in S_b; k \in K)\ .$$

Since $G = K\cdot A\cdot K$ and o is fixed under K, this shows that $K_b(z, z)$ is completely determined by $K_b(o\cdot a, o\cdot a)$, $(a \in A)$, which is given by

$$K_b(o\cdot a, o\cdot a) = K_b(o, o)\cdot|J_b(o, a)|^2\ .$$

We may then insert the expression of $J_b(o, a)$ given by 1.12; the formula thus obtained in the two extreme cases $b = 0$, $S_b = D$, and $b = t$ have been given, in a slightly different form, and with the value of the constant $K_b(o, o)$, by Bott-Korányi [27, 5.7] and Korányi [27, 5.5] respectively.

Our next aim is to relate $J_b(x, g)$ to the determinant of $\mathrm{Ad}\, g$ in $\mathfrak{v}_b$ (cf. 1.3) when $g \in Z_b$. For this, we need the following lemma:

1.15. LEMMA. *Let X be irreducible. Let u (resp. v) be the multiplicity of the $\mathbf{R}$-roots $(\gamma_i \pm \gamma_j)/2$ $(i \neq j)$ (resp. $\gamma_i/2$ in the case BC_t). Let ν_b be the restriction of α_b to $\mathfrak{a}_b$ $(1 \leqq b \leqq t)$. Then the weights of $\mathfrak{a}_b$ in $\mathfrak{g}$, for the adjoint representation are 0, $\pm\nu_b$, and possibly $\pm 2\cdot\nu_b$. Let p_b (resp. q_b) be the multiplicity of ν_b (resp. $2\cdot\nu_b$):*

(i) *if ${}_{\mathbf{R}}\Phi$ is of type C_t and $b = t$, then $p_b = t + u\cdot\binom{t}{2}$, $q_b = 0$.*
(ii) *if ${}_{\mathbf{R}}\Phi$ is of type C_t and $b \neq t$, then $p_b = 2\cdot u\cdot b\cdot(t-b)$, $q_b = b + u\cdot\binom{b}{2}$.*
(iii) *if ${}_{\mathbf{R}}\Phi$ is of type BC_t, then $p_b = v\cdot b + 2\cdot u\cdot b\cdot(t-b)$, $q_b = b + u\binom{b}{2}$.*

The $\mathbf{R}$-roots are linear combinations of the simple ones with coefficients $0, \pm 1, \pm 2$. Since $\mathfrak{a}_b$ annihilates all the simple $\mathbf{R}$-roots except α_b, this proves the first assertion.

We have $\nu_b = \gamma_t$ in the case (i), and $\nu_b = \gamma_b/2$ in the other cases. In the case (i), p_b is the sum of the multiplicities of the $\mathbf{R}$-roots γ_i $(i \leqq t)$, which are all equal to one, and of the $\mathbf{R}$-roots $(\gamma_i + \gamma_j)/2$ $(1 \leqq i < j \leqq t)$, while $q_b = 0$. In cases (ii) and (iii), q_b is the sum of the multiplicities of the $\mathbf{R}$-roots γ_i $(i \leqq b)$ and $(\gamma_i + \gamma_j)/2$ $(1 \leqq i < j \leqq b)$. In case (ii), p_b is the sum of the multiplicities of the roots $(\gamma_i \pm \gamma_j)/2$ $(1 \leqq i \leqq b < j \leqq t)$, and in case (iii), we have to add also the multiplicities of the roots $\gamma_i/2$ $(1 \leqq i \leqq b)$, whence the lemma.

In the next proposition, the important point is not the explicit value of

n_b, but rather the fact that it is >0 and completely determined by p_b and q_b. This will play an important role in our discussion of Eisenstein series.

1.16. PROPOSITION. *Let η_b be the restriction to Z_b of the functional determinant J_b, and let $\chi_b(g) = \det \mathrm{Ad}_{\mathfrak{v}_b}\, g$ $(g \in P_b)$. Then $\eta_b(g) = \chi_b(g)^{-n_b}$ if $g \in A_b \cdot V_b$, and $|\eta_b(g)| = |\chi_b(g)|^{-n_b}$ if $g \in Z_b$, where $n_b = 1$ in case* (i) *of* 1.15, *and $n_b = (p_b + 4q_b)\cdot(2p_b + 4q_b)^{-1}$ in the cases* (ii), (iii) *of* 1.15.

By 1.7 (1′), we have

$$(1)\qquad \begin{aligned} J_b(x, g) &= \eta_b(g) \\ &= \det(\mathrm{Ad}_{\mathfrak{p}^+}\, c_b^{-1}\cdot g^{-1}\cdot c_b) = \Psi_b(c_b^{-1}\cdot g\cdot c_b) \qquad (x \in S_b;\, g \in Z_b)\,. \end{aligned}$$

Both χ_b and η_b are rational characters of Z_b. They are therefore equal to one on V_b and on the derived group of Z_b. On the compact subgroup $K \cap Z_b$ they are both of modulus one. Since Z_b is generated by its intersection with K, a semi-simple subgroup L_b', its unipotent radical V_b, and A_b (see 1.3), it remains to check 1.16 on A_b.

The group A_b belongs to the center of the maximal reductive subgroup $Z(A_b)$ of the parabolic subgroup P_b; hence, the weights of A_b in $\mathfrak{v}_b$ are the restrictions of the positive **R**-roots which are not equal to one on A_b and therefore

$$(2)\qquad \chi_b(g) = \det \mathrm{Ad}_{\mathfrak{v}_b}\, g = \nu_b(g)^{p_b+2q_b} \qquad (g \in A_b)\,.$$

We may write $a \in A_b$ in the form $a = \exp \lambda(X_1 + \cdots + X_b)$. Therefore (1.6) we have $c_b^{-1}\cdot a\cdot c_b = \exp \lambda(H_1 + \cdots + H_b)$. Let ν_b' be the image of ν_b under $\mathrm{Int}\, c_b^{-1}$. Its value on $c_b^{-1}\cdot a\cdot c_b$ is again equal to the restriction of γ_b in case (i), of $\gamma_b/2$ in cases (ii), (iii), where γ_i are now coordinates in $\mathfrak{h}_t$ with respect to the base $(H_i/2)$. By definition $\eta_b(a^{-1})$ is equal to the product of the values on $c_b^{-1}\cdot a\cdot c_b$ of the roots $\mu \in \pi^+$. Therefore $\eta_b(a^{-1}) = \nu_b'(a)^{r_b}$ where the exponent r_b is in case (i),

the number of elements of π^+, which is equal to p_b;

in case (ii),

the number of elements of π^+ restricting to one of $(\gamma_i + \gamma_j)/2$, $(1 \leqq i \leqq b < j \leqq t)$, plus twice the number of elements in π^+ restricting to one of γ_i $(1 \leqq i \leqq b)$, or of $(\gamma_i + \gamma_j)/2$ $(1 \leqq i < j \leqq c)$, which gives

$$(3)\qquad r_b = u\cdot b(t - b) + 2b + u\cdot b(b - 1) = p_b/2 + 2q_b\,;$$

in case (iii),

it is the same as in case (ii) plus the number of $\mu \in \pi^+$ restricting to one of $\gamma_i/2$ $(1 \leqq i \leqq b)$. According to Lemma 14 in [22], this last number is *half* the multiplicity of the **R**-root $\gamma_i/2$, which gives

$$(4) \qquad r_b = u \cdot b(t-b) + 2b + u \cdot b(b-1) + v \cdot b/2 = p_b/2 + 2q_b ,$$

and our assertion follows from (2), (3) and (4).

1.17. PROPOSITION. *We keep the notation of* 1.11, 1.16, *and let* $q_b = m_0/m_b$. *Then*

$$|J_b(x, g)| = |j_b(\sigma_b(x), g)|^{q_b} \cdot |\chi_b(g)|^{-n_b} , \qquad (x \in S_b;\ g \in N(F_b)) .$$

In view of 1.3, we may write $g = l \cdot z$ $(l \in L(F_b), z \in Z(F_b))$. By the cocycle formula

$$J_b(x, g) = J_b(x, l) \cdot J_b(x \cdot l, z) .$$

Since z acts trivially on F_b, we have

$$j_b(y, l) = j_b(y, l \cdot z) , \qquad (y \in F_b) .$$

The proposition follows then from 1.11, 1.16.

1.18. PROPOSITION. *Let* $1 \leqq b < d \leqq t$, *and* $\nu_{b,d} = \nu_d \circ \nu_b^{-1}: S_b \rightarrow S_d$, *where* $\nu_b: X \rightarrow S_b$ *is as in* 1.6. *Then*

(i) $J_b(x, g) = J_d(\nu_{b,d}(x), g)$, $(g \in Z(F_b)^0;\ x \in S_b)$.

(ii) *the functional determinant* $j(x, \nu_{b,d})$ *of* $\nu_{b,d}$ *is constant along the fibres of the canonical projection* $\sigma_b: S_b \rightarrow F_b$.

The group $Z(F_b)^0$ is the semi-direct product of a reductive group R_b with Lie algebra $\mathfrak{m}_b + \mathfrak{a}_b + \mathfrak{l}'_b$, in the notation of 1.3, by the unipotent radical V_b of $P_b = N(F_b)$, and it is contained in $Z(F_d)$. Both $J_b(x,\ )$ and $J_d(x,\ )$ are equal to one if $z \in V_b$ by 1.11. Therefore, it suffices to prove (i) when $g \in R_b$. By the definition of the Cayley transform, $c_b^{-1} \cdot c_d \in L_{b,\mathbf{C}}$; therefore, $c_b^{-1} \cdot c_d$ centralizes R_b, and (i) follows from 1.7 (1′).

By the composition rule for functional determinants, we have

$$j(x, \nu_{b,d}) \cdot J_d(\nu_{b,d}(x), g) = J_b(x, g) \cdot j(x \cdot g, \nu_{b,d}) \qquad (x \in S_b,\ g \in G^0_{\mathbf{R}}) ;$$

therefore, (ii) follows from (i) and from the transitivity of $Z(F_b)^0$ on the fibres of σ_b, (1.7).

We end this section with a result which will be used in discussing rational boundary components. The following lemma will be needed.

1.19. LEMMA. *Let X be irreducible. Then*

(i) $[x, \mathfrak{g}_{(\gamma_i \mp \gamma_j)/2}] \neq \{0\}$ $(x \in \mathfrak{g}_{(\gamma_i \pm \gamma_j)/2} - \{0\};\ 1 \leqq i < j \leqq t)$,

(ii) *if* ${}_{\mathbf{R}}\Phi$ *is of type* BC_t, $[x, \mathfrak{g}_{\gamma_i/2}] \neq \{0\}$ $(x \in \mathfrak{g}_{\gamma_i/2} - \{0\};\ 1 \leqq i \leqq t)$.

PROOF OF (i). By Lemmas 13, 15 of [22], the roots $\mu \in \Phi$ restricting on $\mathfrak{h}_t$ to $(-\gamma_i + \gamma_j)/2$ $(i \neq j)$ are compact, those restricting to $(\gamma_i + \gamma_j)/2$ are in π^+, and $\mu \mapsto \mu + \gamma_i$ is a bijective map of the first set C_{ij} onto the second one P_{ij}. Let $\mathfrak{c}_{ij}$ (resp. $\mathfrak{p}_{ij}$) be the C-subspace of $\mathfrak{g}_{\mathbf{C}}$ spanned by the vectors E_μ, $(\mu \in C_{ij}$, resp. $\mu \in P_{ij}$). With c_t being as in 1.6, we have

$$(1) \qquad \mathrm{Ad}\, c_t(\mathfrak{c}_{ij}) \cap \mathfrak{g} = \mathfrak{g}_{(\gamma_j - \gamma_i)/2} , \qquad \mathrm{Ad}\, c_t(\mathfrak{p}_{ij}) \cap \mathfrak{g} = \mathfrak{g}_{(\gamma_i + \gamma_j)/2} .$$

Furthermore, $\mathfrak{g}_{\gamma_i, \mathbf{C}}$ is one-dimensional and spanned by $\mathrm{Ad}\, c_t(E_{\gamma_i})$. From the result just quoted, and the standard fact $[E_\mu, E_\nu] \neq 0$ if $\mu + \nu$ is a root, we get then

$$(2) \qquad [g_{\gamma_i}, \mathfrak{g}_{(-\gamma_i \pm \gamma_j)/2}] = \mathfrak{g}_{(\gamma_i \pm \gamma_j)/2} ,$$

in particular, we may write

$$x = [y, u] \qquad (y \in \mathfrak{g}_{(-\gamma_i \pm \gamma_j)/2}, u \in \mathfrak{g}_{\gamma_i}) .$$

It is well-known that, given $v \in \mathfrak{g}_\alpha$ ($\alpha \in {}_{\mathbf{R}}\Phi$), there exists $v' \in \mathfrak{g}_{-\alpha}$ such that $[v, v']$ is a non-zero multiple of the element $h_\alpha \in \mathfrak{a}$ such that $\beta(h_\alpha) = (\beta, \alpha)$ ($\beta \in {}_{\mathbf{R}}\Phi$). (One may take, for instance, for v' the transform of v under a suitable Cartan involution.) There exists therefore $z \in \mathfrak{g}_{(\gamma_i \mp \gamma_j)/2}$ such that $[z, y] = c \cdot h_{(\gamma_i \mp \gamma_j)/2}$ ($c \neq 0$). We have then

$$[[z, y], u] = c \cdot \gamma_i(h_{(\gamma_i \mp \gamma_j)/2}) \cdot u \neq 0 .$$

Since $[z, u] = 0$, because $3\gamma_i/2 \pm \gamma_j/2$ is not an $\mathbf{R}$-root, the Jacobi identity shows that $[x, z] = [[y, u], z] \neq 0$, which proves (i).

PROOF OF (ii). Let C'_i (resp. P_i) be the set of compact (resp. non-compact) roots which restrict on $\mathfrak{h}_t$ to $\gamma_i/2$ and $\mathfrak{c}'_i$ (resp. $\mathfrak{p}_i$) the space spanned by the vectors E_μ ($\mu \in C'_i$, resp. $\mu \in P_i$). We have

$$\mathrm{Ad}\, c_t(\mathfrak{c}'_i + \mathfrak{p}_i) \cap \mathfrak{g} = \mathfrak{g}_{\gamma_i/2} ,$$

therefore (ii) is equivalent to

$$(3) \qquad \mathfrak{z}(\mathfrak{c}'_i + \mathfrak{p}_i) \cap (\mathfrak{c}'_i + \mathfrak{p}_i) = \{0\} .$$

By Lemma 14 of [22], the map $\alpha \mapsto \alpha + \gamma_i$ is a bijection of $C_i = -C'_i$ onto P_i. Hence, given $\mu \in C'_i$ (resp. $\mu \in P_i$), there exists $\nu \in P_i$ (resp. $\nu \in C'_i$) such that $\mu + \nu = \gamma_i$. Then $[E_\mu, E_\nu] \neq 0$. Since the left hand side of (3), being stable under $\mathfrak{h}_{\mathbf{C}}$, is spanned by root vectors, this proves (3).

1.20. PROPOSITION. *Let X be irreducible. Let W_b be the center of V_b ($1 \leqq b \leqq t$), and C be the connected centralizer of W_b in P_b. Then $\mathfrak{c}$ is the direct sum of $\mathfrak{l}_b + \mathfrak{v}_b$, which is an ideal of $\mathfrak{c}$, and of its intersection with $\mathfrak{m} = \mathfrak{z}(\mathfrak{a}) \cap \mathfrak{k}$. In particular, $C/(L_b \cdot V_b)$ is compact.*

The ideal $\mathfrak{v}_b$ is the sum of the root spaces $\mathfrak{g}_\alpha$ where α runs through the $\mathbf{R}$-roots of the form

$$(\gamma \ \pm \gamma_j)/2 \ (1 \leqq i \leqq b < j \leqq t) , \qquad (\gamma_i + \gamma_j)/2 \ (1 \leqq i < j \leqq b) ,$$
$$\gamma_i \ (1 \leqq i \leqq b) ,$$

together with $\gamma_i/2$ $(1 \leqq i \leqq b)$ in the case BC_t. We want to prove that $\mathfrak{w}_b$ is the sum of the root spaces g_α, where α runs through the roots

$$(\gamma_i + \gamma_j)/2 \ , \qquad (1 \leqq i \leqq j \leqq b) \ .$$

The relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and the structure of ${}_{\mathbf{R}}\Phi$, show that $\mathfrak{w}_b$ contains the root spaces just listed. Of course $\mathfrak{w}_b$ is stable under $\mathfrak{a}$, hence is the sum of its intersections with the $\mathfrak{g}_\alpha$. In order to prove our assertion, it is therefore enough to show that

$$\mathfrak{w}_b \cap \mathfrak{g}_{(\gamma_i \pm \gamma_j)/2} = 0 \qquad (1 \leqq i \leqq b < j \leqq t) \ ,$$

and that

$$\mathfrak{w}_b \cap \mathfrak{g}_{\gamma_i/2} = \{0\} \ , \qquad (1 \leqq i \leqq b) \ ,$$

if ${}_{\mathbf{R}}\Phi$ is of type BC_t, but this follows from 1.19.

The Lie algebra $\mathfrak{c}$ is also stable under $\mathfrak{a}$. It is obvious that it contains the $\mathfrak{g}_\alpha \subset \mathfrak{l}_b$ and that $\mathfrak{c} \cap \mathfrak{a} \cap \mathfrak{z}_b = 0$. Lemma 1.19 shows moreover that, if $\mathfrak{g}_\alpha \subset \mathfrak{l}'_b$, then $\mathfrak{g}_\alpha \cap \mathfrak{c} = 0$. The proposition follows then from the facts about $\mathfrak{p}_b$ and $\mathfrak{z}_b$ recalled in 1.3.

REMARK. Proposition 1.20 was suggested by a statement in [31, §3.3] which becomes essentially equivalent to 1.20, if the word *normalizer* there is replaced by *centralizer*.

2. Relative root systems

For most of the facts recalled below, we refer to [14]. As was already pointed out in 0.3, the ground fields may be assumed to be contained in $\mathbf{C}$, which is then our *universal field*, although the results of 2.1, 2.2 are valid in greater generality.

2.1. *Relative roots.* Let G be a connected reductive k-group. Its maximal k-split tori are conjugate over k and their common dimension is the k-rank $\mathrm{rk}_k\,(G)$ of G. Let S be a maximal k-split torus of G. The k-roots, or *roots relative to k*, or *restricted roots* are the non-trivial characters of S in the adjoint representation of G, and the relative Weyl group ${}_kW = {}_kW(G)$ is the quotient $N(S)/Z(S)$. We denote by ${}_k\Phi$ or ${}_k\Phi(G)$ the set $\Phi(S, G)$ of k-roots. It is a *root system* in $X^*(T) \otimes \mathbf{R} = V$. This means in particular that, with respect to a scalar product $(\ ,\ )$ on V invariant under ${}_kW$, the group ${}_kW$ is generated by the reflections in the hyperplanes orthogonal to the k-roots, leaves ${}_k\Phi$ stable, and that $2(\alpha, \beta)\cdot(\beta, \beta)^{-1} \in \mathbf{Z}$ for all $\alpha, \beta \in {}_k\Phi$. For every $\alpha \in {}_k\Phi$ we put

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}\, s(x) = s^\alpha \cdot x (s \in S)\} \ .$$

Then $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}_\alpha$ ($\alpha \in {}_k\Phi$) and of the Lie algebra $\mathfrak{z}(S)$ of the centralizer of S.

Given an ordering in $X^*(S)$, we denote by ${}_k\Delta$ the set of simple k-roots. A subset of ${}_k\Delta$ is *connected* if it is not the union of two non-empty disjoint subsets which are mutually orthogonal.

2.2. *Parabolic k-subgroups.* An algebraic subgroup P of G is *parabolic* if the quotient space G/P is a projective variety. P is then connected, equal to its normalizer, and is the normalizer of its unipotent radical.

Let U be the subgroup normalized by S whose Lie algebra $\mathfrak{u}$ is the sum of the $\mathfrak{g}_\alpha$, where α runs through the positive k-roots (for some fixed ordering). Then U is a unipotent k-subgroup, normalized by $Z(S)$.

For every subset θ of ${}_k\Delta$, let $S_\theta = (\bigcap_{\alpha\in\theta} \ker \alpha)^0$. We let ${}_kP_\theta$ be the subgroup generated by $Z(S_\theta)$ and U; it is the semi-direct product of $Z(S_\theta)$ and of its unipotent radical $U_\theta \subset U$. The split radical (0.5) of ${}_kP_\theta$ is the semi-direct product $S_\theta \cdot U_\theta$. Every parabolic k-subgroup of G is conjugate over k to one and only one ${}_kP_\theta$. Moreover, two parabolic k-subgroups are conjugate in G if and only if they are conjugate over k. The groups ${}_kP_\theta$ are the *standard parabolic k-subgroups* (for a given choice of S and U). If $\theta = \emptyset$, then ${}_kP_\theta = {}_kP = Z(S)\cdot U$ is the minimal standard parabolic k-subgroup. We can write uniquely

$$Z(S) = M\cdot S \qquad\qquad (M \text{ normal } k\text{-subgroup, } M \cap S \text{ finite})\ ,$$

and M is anisotropic over k, i.e., $\mathrm{rk}_k\,(M) = 0$.

For $\theta \subset {}_k\Delta$, we let $[\theta]$ be the set of k-roots which are linear combinations of elements in θ, and let ${}_kL_\theta$ be the smallest connected k-subgroup normalized by $Z(S)$ whose Lie algebra ${}_k\mathfrak{l}_\theta$ contains the subspaces $\mathfrak{g}_\alpha$ ($\alpha \in [\theta]$). It is easily seen that $[\theta] = {}_k\Phi({}_kL_\theta)$, that $S \cap {}_kL_\theta$ is a maximal k-split torus of ${}_kL_\theta$, and that ${}_kL_\theta$ is semi-simple with Lie algebra

$${}_k\mathfrak{l}_\theta = \textstyle\sum_{\alpha\in[\theta]} \mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]\ .$$

Moreover, we have ${}_kP_\theta = M_\theta \cdot {}_kL_\theta \cdot S_\theta \cdot U_\theta$, where M_θ is the identity component of $M \cap Z({}_kL_\theta)$. *If θ is connected, then ${}_kL_\theta$ is almost k-simple*, since otherwise, by [14, 5.11, 8.5] ${}_kL_\theta$ would be the almost direct product of a k-group without k-rational unipotent elements $\neq e$, and of a k-group containing all unipotent elements rational over k of ${}_kL_\theta$, and ${}_kL_\theta$ could not be generated by unipotent k-subgroups.

We shall sometimes denote by ${}_kT$ a maximal k-split torus and by ${}_kU$ the unipotent radical of a minimal parabolic k-subgroup.

We recall finally the Bruhat decomposition $G_k = P_k \cdot N(S)_k \cdot P_k$ [14, 5.15].

More precisely, let n_w be a representative in $N(S)_k$ of $w \in {}_kW$. Then G_k is the disjoint union of the double classes

$$P_k \cdot n_w \cdot P_k = U_k \cdot n_w \cdot Z(S)_k \cdot U_k \ .$$

If G is connected, but not reductive, it is the semi-direct product of its unipotent radical $R_u(G)$ by a reductive k-subgroup (we are in characteristic zero), and $R_u(G)$ is contained in every parabolic subgroup. Since $N(S) \cap R_u(G) = Z(S) \cap R_u(G)$, it follows that the above decomposition is still valid, with P a minimal parabolic k-subgroup, and $N(S)$ being the normalizer of S either in G or in a maximal reductive k-subgroup containing S.

2.3. *Fundamental highest weights relative to k.* There is a basis of $X(S) \otimes \mathbf{Q}$ over $\mathbf{Q}$ consisting of elements $d_\alpha \in X(P)(\alpha \in {}_k\Delta)$ such that $(d_\alpha, \beta) = c_\alpha \delta_{\alpha\beta}(\alpha, \beta \in {}_k\Delta)$ where c_α are positive integers. The restriction to S_θ of the elements $d_\alpha(\alpha \in \theta' = {}_{\mathbf{Q}}\Delta - \theta)$ form a basis of $X(S_\theta) \otimes \mathbf{Q}$. Let $d \in X(S)$ be a linear combination of elements $d_\alpha(\alpha \in \theta')$ with strictly positive integral coefficients. Then there exists an absolutely irreducible representation $\rho: G \longrightarrow \mathbf{GL}(V)$ defined over k, and a unique one-dimensional subspace $V' \subset V$ stable under P_θ and on which $g \in P_\theta$ acts *via* multiplication by $d(g)$ [14, 12.2, 12.13]. The characters $d_\alpha(\alpha \in \theta')$ will be called *fundamental highest weights* for P_θ. Let

$$\chi_0(p) = \det(\mathrm{Ad}_{\mathfrak{u}_\theta}\, p) \qquad (p \in P_\theta) \ .$$

We have then

$$\chi_0 = \sum\nolimits_{\alpha \in \theta'} e_\alpha d_\alpha \qquad (e_\alpha \in \mathbf{Q};\ e_\alpha > 0) \ .$$

In fact, by definition, χ_0 is the sum of the weights of S in $\mathfrak{u}_\theta$. These are the positive roots which involve at least one of the elements of θ', each root being of course counted with its multiplicity. χ_0 is stable under $N(S) \cap P_\theta$, hence under the fundamental symmetries $s_\beta(\beta \in \theta)$; therefore it is orthogonal to θ, and is a linear combination of the elements $d_\alpha(\alpha \in \theta')$. The coefficient e_α of d_α is equal to $(\chi_0, \alpha) \cdot c_\alpha^{-1}$. Let χ_1 be the sum of the positive elements in $[\theta]$, counted with their multiplicities, and $\chi = \chi_0 + \chi_1$. Then $(\chi_1, \alpha) \leqq 0$ for $\alpha \in \theta'$ and, by a standard argument $(\chi, \alpha) > 0$, (in fact it is equal to $(c + 2d) \cdot (\alpha, \alpha)$, where c and d are the multiplicities of α and 2α), hence $(\chi_0, \alpha) > 0$ and $e_\alpha > 0$.

2.4. *Restriction of relative roots.* Let K be an overfield of k, T a maximal K-split torus of G containing S and $r: X(T) \longrightarrow X(S)$ the restriction homomorphism. Two orderings of $X(T)$ and $X(S)$ are *compatible* if $\alpha > 0$, $r(\alpha) \neq 0$ imply $r(\alpha) > 0$ $(\alpha \in X(T))$.

The existence of an ordering on $X(T)$ compatible with a given ordering on $X(S)$ is immediate [14, 3.1]. Let ${}_K\Delta$ and ${}_k\Delta$ be the sets of simple roots for compatible orderings. Then we have

(i) ${}_k\Delta \subset r({}_K\Delta) \subset {}_k\Delta \cup \{0\}$.

(ii) *Let* $\theta \subset {}_k\Delta$, $\psi \subset {}_K\Delta$ *and* $\alpha \in {}_K\Delta \cap r^{-1}(\theta)$. *If* ψ *is connected, then* $r(\psi) \cap {}_k\Delta$ *is connected. If* θ *is connected, there exists a connected subset* θ' *of* ${}_K\Delta$ *containing* α *such that*

$$\theta \subset r(\theta') \subset \theta \cup \{0\}\ .$$

For the proofs, see [14, 6.8, 6.15, 6.16]. A simple K-root will be said to be *critical* if it restricts onto a simple k-root. Thus a simple K-root either is critical or restricts to zero.

(iii) *Let* θ *be a connected subset of* ${}_k\Delta$, *and assume there exists a unique greatest connected subset* ψ *of* ${}_K\Delta$ *such that* $r(\psi) \cap {}_k\Delta = \theta$. *Then* ${}_kL_\theta = {}_KL_\psi$. *In particular,* ${}_KL_\psi$ *is defined over* k.

PROOF. Let $\beta \in {}_k\Phi$. Then the space $\mathfrak{g}_\beta$ is the direct sum of the eigenspaces $\mathfrak{g}_\alpha$ of T, where α runs through the K-roots whose restriction to S is equal to β. It is a standard fact about root systems that, if a root α is expressed as linear combination of simple roots, then the set of simple roots which occur in α with a non-zero coefficient is connected. Therefore if $\beta \in [\theta]$ and $\alpha \in {}_K\Phi$ restricts to β, then $\alpha \in [\psi]$. This implies that ${}_kL_\theta \subset {}_KL_\psi$. Moreover, $Z(S)$ normalizes ${}_kL_\theta$, and it is clear from the definitions that ${}_KL_\psi \subset Z(S)\cdot {}_kL_\theta$. Consequently, ${}_kL_\theta$ is a normal subgroup of ${}_KL_\psi$. However the latter is almost K-simple (2.2), whence our assertion.

2.5. PROPOSITION. *Let* k *be an algebraic number field,* k_v *its completion with respect to an archimedean valuation* v, *and* G *a connected reductive* k-*group. Then every maximal torus defined over* k_v *of* G *is conjugate over* k_v *to a maximal torus defined over* k.

(i) We show first that, if L is a connected k-group, then L_k is dense in $(L_{k_v})^0$ in the usual topology. By [32, p. 41], there exists a generically surjective rational map of an affine space into L which is defined over k. In other words, we may find a Zariski k-open subset U of an affine space, and a k-morphism $f\colon U \to L$ whose image contains a non-empty Zariski k-open subset V of L. Since we are in characteristic zero, f is separable, and there exists a non-empty Zariski k-open subset U' of U such that $f\colon U'_{k_v} \to L_{k_v}$ is open. Of course $f(U'_k) \subset L_k$ and U'_k is dense in U'_{k_v}. Thus, L_k is dense in a non-empty open subset of L_{k_v}, hence in $(L_{k_v})^0$.

(ii) If $k_v = \mathbf{C}$, then all maximal tori of G_{k_v} are conjugate, and 2.5 amounts to the existence of a maximal torus defined over k, which is known [32, p. 45]. Assume now that $k_v = \mathbf{R}$. Let T be a maximal torus of G defined over $\mathbf{R}$ and H' the set of regular elements of $(T_{\mathbf{R}})^0$. It is well known that the set C of conjugates of elements of H' by elements of $(G_{\mathbf{R}})^0$ is an open subset of $(G_{\mathbf{R}})^0$. By (i), it contains an element x rational over k. Then $Z(x)^0$ is the desired maximal torus.

2.6. COROLLARY. *Let k be a subfield of* $\mathbf{R}$. *Then G has a maximal torus defined over k which contains a maximal* $\mathbf{R}$*-split torus of G.*

2.7. REMARKS. (1) Proposition 2.5 is also valid for an arbitrary connected k-group G. In fact, let U be the unipotent radical of G and $\pi: G \rightarrow G' = G/U$ the canonical projection. Let T be a maximal torus defined over k_v of G. Then $\pi(T)$ is a maximal torus of G' [9, § 22], obviously defined over k_v. By 2.5, it is conjugate over k_v to a maximal torus T' defined over k of G'. Since U is unipotent, and k is perfect, the map $G_{k_v} \rightarrow G'_{k_v}$ is surjective by Rosenlicht's cross-section theorem, whence the existence of $x \in G_{k_v}$ such that ${}^xT \subset \pi^{-1}(T') = Q$. The group Q is a connected solvable k-group, hence its maximal tori defined over k_v, are conjugate over k_v and one of them is defined over k (see e.g. [14, 11.4]; or, in characteristic zero, Borel-Mostow, Annals of Math. 61 (1955), 389–405).

Proposition 2.5 is then of course also true if maximal tori are replaced by Cartan subgroups, since the latter are the centralizers of the former.

(2) Although this will not be needed in this paper, we point out that, if G is a connected k-group, G_k is dense in G_{k_v}, not only in $(G_{k_v})^0$, as was shown in (i). If $k_v = \mathbf{C}$, then G_{k_v} is connected, and there is nothing new to prove. If $k_v = \mathbf{R}$, there remains to show that G_k meets every connected component of $G_{\mathbf{R}}$. By 2.6, applied to a maximal connected reductive k-subgroup of G, (or by remark (1) above), there exists a maximal torus T of G defined over k and containing a maximal $\mathbf{R}$-split torus of G. By [14, 14.4], each connected component of $G_{\mathbf{R}}$ contains one of $T_{\mathbf{R}}$, so that we are reduced to the case of a torus, where our assertion follows from a result of Serre quoted in [25, 5.1].

In the terminology of [25], this means essentially that G has the weak approximation property for archimedean valuations. As a matter of fact, it has been checked here only for one such valuation, but the case of several is easily reduced to that of one by considering the group $R_{k/\mathbf{Q}}G$.

2.8. Let k be a subfield of $\mathbf{R}$ and G a connected semi-simple and absolutely simple k-group. Let ${}_{\mathbf{R}}T$ be a maximal $\mathbf{R}$-split torus of G containing a maximal k-split torus ${}_kT$. We let $r: X({}_{\mathbf{R}}T) \rightarrow X({}_kT)$ be the restriction map, endow $X({}_{\mathbf{R}}T)$ and $X({}_kT)$ with compatible orderings, and denote by ${}_k\Delta$, ${}_{\mathbf{R}}\Delta$ the corresponding sets of simple relative roots.

We assume further that the riemannian symmetric space $K\backslash G_{\mathbf{R}}$, where K is a maximal compact subgroup of $G_{\mathbf{R}}$, is a bounded symmetric domain, necessarily irreducible since $G_{\mathbf{R}}$ is simple. Then ${}_{\mathbf{R}}\Phi$ is either of type C_t or of type BC_t (1.2). In fact, the following proposition, and its proof are valid under that last assumption. On ${}_{\mathbf{R}}\Phi = \{\alpha_1, \cdots, \alpha_t\}$, we use the canonical numbering of 1.2. For each $\beta \in {}_k\Delta$, let $m(\beta)$ be the greatest value of the index i such that $r(\alpha_i) = \beta$. We number the elements $\beta_1, \cdots, \beta_s$ of ${}_k\Delta$ in such a way that $i < j$ if and only if $m(\beta_i) < m(\beta_j)$, and then write $m(j)$ for $m(\beta_j)$.

2.9. PROPOSITION. *We keep the notation and assumptions of* 2.8. *We assume that* $\dim {}_kT > 0$. *Then*

(a) ${}_k\Phi$ *is of type* BC_s *if either* ${}_{\mathbf{R}}\Phi$ *is of type* BC_t *or* ${}_{\mathbf{R}}\Phi$ *is of type* C_t *and* $r(\alpha_t) = 0$, *and is of type* C_s *otherwise. The numbering of* ${}_k\Delta$ *defined in* 2.8 *is the canonical one.*

(b) *Each* $\beta \in {}_k\Delta$ *is the restriction of one and only one simple* **R**-*root.*

By our choice of the numberings, any final segment $(\beta_i, \cdots, \beta_s)$ in ${}_k\Delta$ (possibly with zero added), is the restriction of a final segment of ${}_{\mathbf{R}}\Delta$, hence is connected (2.4). Conversely, any connected subset θ of ${}_k\Delta$ containing β_a, β_b $(a < b)$ contains β_i for every i between a and b. In fact, there exists by 2.4 (ii) a connected subset θ' of ${}_{\mathbf{R}}\Delta$ containing $\alpha_{m(b)}$ such that $\theta \subset r(\theta') \subset \theta \cup \{0\}$. The set θ' contains then at least one simple **R**-root α_c with $c \leqq m(a)$. In view of the structure of ${}_{\mathbf{R}}\Delta$, the set θ' must then contain all simple **R**-roots with index i between c and $m(b)$, hence in particular all **R**-roots $\alpha_{m(j)}$ $(a \leqq j \leqq b)$, whence our contention.

This shows that the graph of ${}_k\Delta$ is a chain (no branch point). ${}_k\Delta$ is therefore of one of the types A_s, B_s, C_s, G_2, F_4, BC_s, where the first five symbols refer to the standard Cartan-Killing classification. We now distinguish some cases.

(i) ${}_{\mathbf{R}}\Phi$ *is of type* BC_t. In this case, the set of $\alpha \in {}_{\mathbf{R}}\Delta$ whose double is a root spans $X({}_{\mathbf{R}}T) \otimes \mathbf{R}$, hence contains at least one element γ whose restriction is not zero. Thus $r(\gamma)$ and $2 \cdot r(\gamma)$ are k-roots, and ${}_k\Delta$ is of type BC_s. Moreover, the highest root in ${}_{\mathbf{R}}\Delta$ is $\gamma_1 = 2(\alpha_1 + \cdots + \alpha_t)$; therefore, if $\beta \in {}_k\Delta$ is the restriction of at least two simple **R**-roots, then β has a coefficient $\geqq 4$ in the highest k-root, which is impossible in type BC_s. Also, $\alpha_{m(s)} + \cdots + \alpha_t = \gamma_{m(s)}/2$ and its double are roots, hence β_s and $2\beta_s$ are k-roots; and, by the above, the connected subsets of ${}_k\Delta$ containing β_s are the final segments. This shows that our numbering is the canonical one, and ends the proof of the proposition in this case.

From now on, ${}_{\mathbf{R}}\Delta$ is of type C_t. Its highest root is then

$$2(\alpha_1 + \cdots + \alpha_{t-1}) + \alpha_t \ .$$

Let c_i denote the number of simple **R**-root restricting onto β_i.

(ii) Assume that $r(\alpha_t) \neq 0$. Then the highest k-root is

$$\delta = 2c_1 \cdot \beta_1 + \cdots + 2 \cdot c_{s-1} \cdot \beta_{s-1} + \big(2(c_s - 1) + 1\big)\beta_s \,.$$

If $c_i \geqq 2$ for some $i < s$, then there is a coefficient $\geqq 4$ in δ, and ${}_k\Phi$ is of type F_4. There is in this case in δ a coefficient 3, which must then be the coefficient of β_s. However, in F_4 the simple root with coefficient 3 in the highest root is not an end point of the graph. Therefore $c_i = 1$ for $i < s$.

Assume now that $\beta_s = r(\alpha_t) = r(\alpha_j)$ for some $j < t$. (There is at most one such j since no simple root has a coefficient $\geqq 5$ in δ in the systems under consideration here). Then β_s has a coefficient $\geqq 3$ and all other simple k-roots have coefficient 2. This occurs only in G_2. If $m(1) < j$, then

$$3 \cdot \beta_2 = r\big(2(\alpha_j + \cdots + \alpha_{t-1}) + \alpha_t\big)$$

is a k-root, which is absurd. If $j < m(1)$, then

$$r\Big(\frac{1}{2}(\gamma_{m(1)} + \gamma_j)\Big) = 2(\beta_1 + \beta_2)$$

is a k-root, which is absurd because $\beta_1 + \beta_2$ is a k-root, and G_2 has no root whose double is a root. This proves that $c_i = 1$ $(1 \leqq i \leqq s)$, hence that

$$\delta = 2(\beta_1 + \cdots + \beta_{s-1}) + \beta_s \,.$$

Therefore ${}_k\Phi$ is of type C_t, and the numbering is the canonical one.

(iii) Assume that $r(\alpha_t) = 0$. In this case the highest k-root is

$$\delta = 2 \cdot c_1 \cdot \beta_1 + \cdots + 2 \cdot c_s \cdot \beta_s \,.$$

By the classification of root systems, this implies that ${}_k\Phi$ is of type BC_s and that $c_i = 1$ $(1 \leqq i \leqq s)$. Furthermore, $2\beta_s = r(\gamma_{m(s)})$ is a k-root, so that again the numbering is the canonical one. This completes the proof of the proposition.

2.10. COROLLARY. (a) *The proper maximal parabolic k-subgroups of G are also proper maximal among parabolic* $\mathbf{R}$*-subgroups.*

(b) *Let ψ be an initial (resp. a final) segment of ${}_{\mathbf{R}}\Delta$ consisting of all roots which come before (resp. after) a critical root and $\theta = r(\psi) \cap {}_k\Delta$. Then ${}_{\mathbf{R}}L_\psi = {}_kL_\theta$ and is defined over k.*

PROOF OF (a). Let P be a proper maximal parabolic k-subgroup of G. It follows from 2.2 that there exists a simple k-root β such that P is conjugate over k to the group ${}_kP_\theta$ $\big(\theta = {}_k\Delta - \{\beta\}\big)$. Let α be the unique critical simple $\mathbf{R}$-root which restricts onto β and $\psi = {}_{\mathbf{R}}\Delta - \{\alpha\}$. Then $T_\psi = S_\theta$ by 2.9. Furthermore, since the given orderings are compatible, we have ${}_{\mathbf{R}}U \subset Z({}_kT) \cdot {}_kU$, and therefore ${}_{\mathbf{R}}P_\psi \subset {}_kP_\theta$. Since ${}_{\mathbf{R}}P_\psi$ is a proper maximal parabolic $\mathbf{R}$-group, we have ${}_{\mathbf{R}}P_\psi = {}_kP_\theta$, which proves (a).

PROOF OF (b). By 2.9, θ is an initial or final segment of ${}_k\Delta$, and ψ is the greatest connected subset of ${}_K\Delta$ such that $\theta = {}_k\Delta \cap r(\psi)$. Therefore (b) follows from 2.4 (iii).

3. Rational boundary components

3.1. Let G be an algebraic group defined over $\mathbf{Q}$. A subgroup Γ of $G_{\mathbf{Q}}$ is *arithmetic* if for one (and hence for every [13, 6.3]) faithful $\mathbf{Q}$-morphism $\rho: G \longrightarrow \mathbf{GL}_m$, the group $\rho(\Gamma)$ is commensurable with $\rho(G)_{\mathbf{Z}}$.

We recall that, if $f: G \longrightarrow G'$ is a surjective $\mathbf{Q}$-morphism of G onto a $\mathbf{Q}$-group G', and Γ is an arithmetic subgroup of G, then $f(\Gamma)$ is also arithmetic. (See [13, 6.11] for isogenies, [11, Th. 6] for the generalization to surjective morphisms.) Since we are interested in automorphism groups of symmetric spaces, we may, without restricting generality, limit ourselves to centerless groups whenever convenient. Moreover, it follows from [13, 6.11] that, if G is an almost direct (or a semi-direct) product of two $\mathbf{Q}$-subgroups G_1, G_2 and Γ is an arithmetic subgroup of G, then $(\Gamma \cap G_i)$ is an arithmetic subgroup of G_i $(i = 1, 2)$ and $(\Gamma \cap G_1)\cdot(\Gamma \cap G_2)$ is commensurable with Γ.

Let G be simple over $\mathbf{Q}$. Then there exists an algebraic number field k and an absolutely simple k-group G' such that $G = R_{k/\mathbf{Q}}G'$ [14, 6.21 (ii)], where $R_{k/\mathbf{Q}}$ is the functor of restriction of the ground field [38, Ch. I], from k to $\mathbf{Q}$.

3.2. LEMMA. *Let k be an algebraic number field. G' a connected semi-simple and absolutely simple k-group, and $G = R_{k/\mathbf{Q}}G'$. Let K be a maximal compact subgroup of $G_{\mathbf{R}}$, $X = K\backslash G_{\mathbf{R}}$, and Γ be an arithmetic subgroup of G.*

(a) *If K has the same rank as G, in particular if X is a bounded domain, then k is totally real.*

(b) *If X/Γ is not compact, then $G^0_{\mathbf{R}}$ has no compact factor $\neq \{e\}$, and* $\mathrm{rk}_k(G') \neq 0$.

Let V be the set of normalized archimedean valuations of k, and k_v the completion of k with respect to $v \in V$. Then

$$G_{\mathbf{R}} \cong \textstyle\prod_{v\in V} G'_{k_v}\,, \qquad [38, 1.3.2]$$

hence

$$X = \textstyle\prod_{v\in V} X_v \qquad (X_v = (K \cap G'_{k_v})\backslash G'_{k_v})$$

and $K_{(v)} = K \cap G'_{k_v}$ is a maximal compact subgroup of G'_{k_v}. If $k_v = \mathbf{C}$, then G'_{k_v} is a complex Lie group, viewed as real Lie group, and its rank as such is twice the rank of $K_{(v)}$, whence (a).

The groups G'_{k_v} are the simple factors of $G_{\mathbf{R}}$. If one of them is compact, then $G'_k \cong G_{\mathbf{Q}}$ consists of semi-simple elements, hence $G_{\mathbf{R}}/\Gamma$ is compact [13, 11.6], which proves (b), and G has no proper $\mathbf{Q}$-parabolic subgroup.

3.3. (i) Let X be a bounded symmetric domain, $H(X)$ its group of holomorphic automorphisms and $\mathrm{Is}(X)$ its group of isometries with respect to the underlying riemannian structure. Let $\mathfrak{h}$ be the Lie algebra of $H(X)$. It is known that $\mathrm{Ad}\,\mathfrak{h} = H(X)^0 \subset \mathrm{Is}\,X = \mathrm{Aut}\,\mathfrak{h}$. Thus $H(X)$ is identified with a group of finite index in the group of real points of an algebraic $\mathbf{R}$-group, namely $\mathrm{Aut}\,\mathfrak{h}_{\mathbf{C}}$. Assume that we have put on $\mathrm{Aut}\,\mathfrak{h}_{\mathbf{C}}$ a structure of $\mathbf{Q}$-group subordinated to its natural $\mathbf{R}$-structure. This is equivalent to putting a $\mathbf{Q}$-structure on $\mathfrak{h}$; i.e., fixing a Lie subalgebra $\mathfrak{h}_{\mathbf{Q}}$ over $\mathbf{Q}$ of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{h}_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$. An *arithmetic subgroup* Γ of $H(X)$ is then an arithmetic subgroup of $\mathrm{Aut}\,\mathfrak{h}_{\mathbf{C}}$, viewed as a $\mathbf{Q}$-group. More correctly, one should say that Γ is arithmetically definable, since this definition presupposes the determination of a $\mathbf{Q}$-structure, for which there is usually a wide choice. However, we shall just say arithmetic for the sake of brevity. It is then understood that $\mathrm{Ad}\,\mathfrak{h}_{\mathbf{C}}$ has been identified with a semi-simple $\mathbf{Q}$-group G which has no center; i.e., with $\mathrm{Ad}\,\mathfrak{g}_{\mathbf{C}}$, and $H(X)$ with a subgroup of $\mathrm{Aut}\,\mathfrak{g}_{\mathbf{R}}$. The space X is then the quotient of $\mathrm{Aut}\,\mathfrak{g}_{\mathbf{R}}$, or $H(X)$, or $\mathrm{Ad}\,\mathfrak{g}_{\mathbf{R}}$, by a maximal compact subgroup.

The group G is the direct product of its normal simple $\mathbf{Q}$-groups G_i ($1 \leqq i \leqq m$), and X the product of the symmetric spaces $(K \cap G_{i\mathbf{R}})\backslash G_{i\mathbf{R}}$, which are then also bounded symmetric domains. Let $\Gamma_i = \Gamma \cap G^0_{i\mathbf{R}}$ ($1 \leqq i \leqq m$) and Γ' be the subgroup generated by the Γ_i. It is arithmetic, normal, of finite index, in Γ.

Our problem is the compactification of X/Γ. It turns out that the passage from X/Γ' to X/Γ offers no difficulty (cf. 8.9). Since X/Γ' is the product of the X_i/Γ_i, the essential case to consider is when G is simple over $\mathbf{Q}$, and $\Gamma \subset G^0_{\mathbf{R}}$.

(ii) We introduce some notation pertaining to our main case of interest. We keep the assumption of (i), and assume moreover G to be simple over $\mathbf{Q}$, and $\Gamma \subset G^0_{\mathbf{R}}$. Then $G = R_{k/\mathbf{Q}}G'$, where G' is an absolutely simple k-group, and k a totally real number field. Let Σ be the set of distinct isomorphisms of k into $\mathbf{R}$. There is a 1–1 correspondence between elements of Σ and normalized archimedean valuations of k given by $|a|_v = |\sigma(a)|$ ($a \in k$), and we have $G_{k_v} \cong ({}^{\sigma}G')_{\mathbf{R}}$, [38, Ch. I]. We may then also write

$$X = \textstyle\prod_{\sigma \in \Sigma} X_\sigma, \qquad (X_\sigma = K_{(\sigma)}\backslash {}^{\sigma}G_{\mathbf{R}} = K^0_{(\sigma)}\backslash {}^{\sigma}G^0_{\mathbf{R}})\ ,$$

where X_σ is an irreducible symmetric bounded domain. For simplicity, we shall also write G_σ for ${}^{\sigma}G^0_{\mathbf{R}}$.

We assume further that if Γ is an arithmetic subgroup of G, then X/Γ is not compact. This implies (3.2) that no X_σ is reduced to a point and that G' has a non-trivial maximal k-split torus, say S'. Then ${}^{\sigma}S'$ is a maximal $\sigma(k)$-

split torus of ${}^{\sigma}G'$; there is a canonical isomorphism $\varphi_\sigma\colon S' \to {}^{\sigma}S'$ which induces an isomorphism of ${}_k\Phi$ onto ${}_{\sigma(k)}\Phi({}^{\sigma}G') = {}_k\Phi_\sigma$. Furthermore the maximal $\mathbf{Q}$-split subtorus S of $R_{k/\mathbf{Q}}S'$ is a maximal $\mathbf{Q}$-split torus of G. It is canonically isomorphic to S' and is diagonally embedded in $R_{k/\mathbf{Q}}S'$. This means more precisely that the projection pr_σ of S into ${}^{\sigma}G'$ is the composition of the canonical isomorphisms $\varphi\colon S \to S'$ and $\varphi_\sigma\colon S' \to {}^{\sigma}S'$. The isomorphism also induces an isomorphism of ${}_{\mathbf{Q}}\Phi(G)$ onto ${}_k\Phi(G')$. We shall identify ${}_k\Phi(G')$, ${}_{\sigma(k)}\Phi({}^{\sigma}G')$, and ${}_{\mathbf{Q}}\Phi(G)$ by means of these isomorphisms.

In each group ${}^{\sigma}G'$, we choose a maximal $\mathbf{R}$-split torus $T_\sigma \supset {}^{\sigma}S'$, contained in a maximal torus defined over $\sigma(k)$ (apply 2.6 to $Z({}^{\sigma}S')$). We fix an ordering on $X(S')$, hence, using φ and φ_σ, also an ordering on $X({}^{\sigma}S')$ and $X(S)$. For each σ, choose an ordering on $X(T_\sigma)$ compatible with the given one on $X({}^{\sigma}S')$, and let $r\colon X(T_\sigma) \to X({}^{\sigma}S') \cong X(S)$ be the restriction homomorphism. By 2.9, the canonical numbering on the set ${}_{\mathbf{R}}\Delta_\sigma$ of simple $\mathbf{R}$-roots of G with respect to T_σ is compatible by restriction with the canonical numbering of ${}_{\mathbf{Q}}\Delta$.

Let ${}_k\Delta = \{\beta_1, \cdots, \beta_s\}$. For i between 1 and s, we let $c(i, \sigma)$ be the index of the critical simple $\mathbf{R}$-root of ${}^{\sigma}G$ restricting on β_i. Then, the remark just made shows that $i < j$ implies $c(i, \sigma) < c(j, \sigma)$ for all $\sigma \in \Sigma$.

A sequence of elements indexed by Σ will often be denoted in boldface and used as a multi-index or a multi-exponent. In particular, let b be between 1 and s. Then

$$F_{\mathbf{b}} = \textstyle\prod_{\sigma \in \Sigma} F_{c(b,\sigma)} \;,$$

is the product of the standard boundary components $F_{c(b,\sigma)}$ of X_σ, where *standard* refers to the choice of T_σ and ${}_{\mathbf{R}}\Delta_\sigma$. It is also understood that the Lie algebra of $K_{(\sigma)}$ is orthogonal to that of T_σ. Since $c(j, \sigma)$ is an increasing function of j, for each σ, we have $\bar{F}_{\mathbf{j}} \subset \bar{F}_{\mathbf{i}}$ $(1 \leqq i \leqq j \leqq s)$.

Let $F = \prod_\sigma F_{i(\sigma)}$ be a product of standard boundary components. We let $S_F = \prod S_{i(\sigma)}$ be the product of the unbounded realizations associated to the $F_{i(\sigma)}$, (1.6), J_F be the functional determinant in S_F, and j_F be the functional determinant in the Harish-Chandra bounded realization of F. If $F = F_{\mathbf{b}}$ we also write $S_{\mathbf{b}}$, $J_{\mathbf{b}}$, $j_{\mathbf{b}}$ for S_F, J_F and j_F. In the notation of 1.8, we have therefore

$$J_F(x, g) = \textstyle\prod_\sigma J_{i(\sigma)}(x_\sigma, g_\sigma) \qquad \big(x = (x_\sigma),\, g = (g_\sigma);\, x_\sigma \in X_\sigma,\, g_\sigma \in G_\sigma\big) \;,$$
$$j_F(x, g) = \textstyle\prod_\sigma j_{i(\sigma)}(x_\sigma, g_\sigma) \qquad \big(x = (x_\sigma),\, g = (g_\sigma);\, x_\sigma \in F_{i(\sigma)},\, g_\sigma \in L(F_\sigma)\big) \;.$$

By 1.11, applied to each irreducible factor of X, the functional determinant $J_F(x, g)$, $(g \in N(F))$, is constant along the fibres of the canonical projection σ_F of X onto F, (defined in 1.7, remark).

The natural compactification $\bar{X}$ of X is the product of the natural compactifications $\bar{X}_\sigma$ of the X_σ. We shall also write $o_{\mathfrak{b}}$ for the point with components $o_{c(b,\sigma)} \in \bar{X}_\sigma$, in the notation of 1.5. Thus

$$F_{\mathfrak{b}} = o_{\mathfrak{b}} \cdot N(F_{\mathfrak{b}}) = o_{\mathfrak{b}} \cdot L(F_{\mathfrak{b}}) = o_{\mathfrak{b}} \cdot G(F_{\mathfrak{b}}) , \qquad (1 \leqq b \leqq s) .$$

We recall that $G(F_{\mathfrak{b}}) = N(F_{\mathfrak{b}})/Z(F_{\mathfrak{b}})$. We shall denote by $\varpi_{\mathfrak{b}}$ the natural projection of $N(F_{\mathfrak{b}})$ onto $G(F_{\mathfrak{b}})$. Applying 1.3 and 1.5 to each irreducible factor of X, we see that $G(F_{\mathfrak{b}})$ is connected, that $N(F_{\mathfrak{b}}) = L(F_{\mathfrak{b}}) \cdot Z(F_{\mathfrak{b}})$, and that $Z(F_{\mathfrak{b}})$ is the greatest normal subgroup of $N(F_{\mathfrak{b}})$ with identity component $Z(F_{\mathfrak{b}})^0$.

3.4. It will be sometimes convenient to use the following variation on the notion of arithmetic group.

Let H be a connected *real* Lie group. A subgroup Γ is of *arithmetic type*, or *arithmetically definable*, if there exists a connected $\mathbf{Q}$-group G, a continuous surjective homomorphism $f\colon G^0_{\mathbf{R}} \to H$ with *compact* kernel N and an arithmetic subgroup Γ' of G such that $f(\Gamma') = \Gamma$. Since N is compact, the group Γ is then obviously discrete.

Let H be semi-simple. It is easily seen that, without restricting generality, G may be assumed to be semi-simple, and to be almost simple over $\mathbf{Q}$ if H is simple. If G is simple over $\mathbf{Q}$, and $\dim N > 0$, then H/Γ is compact. In fact, we have in this case $G = R_{k/\mathbf{Q}}G'$, where k is an algebraic number field, and G' an absolutely simple k-group. The group $G_{\mathbf{R}}$ is the product of the groups G'_{k_v} where k_v runs through the archimedean completions of k, and these are the simple normal subgroups of $G_{\mathbf{R}}$. Therefore N contains at least one of them, G'_k consists of semi-simple elements, and $G_{\mathbf{R}}/\Gamma'$ is compact [13, 11.6]. This implies of course the compactness of H/Γ and of $K\backslash H/\Gamma$, where K is a compact subgroup of H.

3.5. *Rational boundary components.* Let G be a connected semi-simple $\mathbf{Q}$-group, whose symmetric space of non-compact type, $X = K\backslash G_{\mathbf{R}}$, where K is a maximal compact subgroup of $G_{\mathbf{R}}$, is a bounded symmetric domain. For a discrete subgroup Γ of $G_{\mathbf{R}}$ and a boundary component F of X, (1.2), we let $\Gamma(F)$ be the image of $\Gamma \cap N(F)$ in $G(F) = N(F)/Z(F)$ by the natural projection. The component F is said to be Γ-rational if

(i) the quotient $U(F)/(U(F) \cap \Gamma)$ is compact,

(ii) the group $\Gamma(F)$ is discrete.

Clearly a Γ-rational component is Γ'-rational for any group Γ' commensurable with Γ. In particular, if F is Γ-rational for one arithmetic group Γ, it is so for all arithmetic groups; in that case, we shall drop the prefix Γ- and speak of *rational boundary components*.

Let now Γ be arithmetic. We remark first that (i) is equivalent to

(i)′ $N(F)_{\mathbf{C}}$ is defined over $\mathbf{Q}$.

The implication (i)′ $\Rightarrow$ (i) follows from the standard fact that, if U is a unipotent $\mathbf{Q}$-group and Γ an arithmetic subgroup of U, then $U_{\mathbf{R}}/\Gamma$ is compact. Assume now (i) to hold. Let V be the smallest algebraic subgroup of $U(F)_{\mathbf{C}}$ containing $U(F) \cap \Gamma$. It is invariant under all automorphisms of $\mathbf{C}$, since $\Gamma \subset G_{\mathbf{Q}}$, hence it is defined over $\mathbf{Q}$. Since $U(F)/\Gamma$ is compact, the quotient $U_{\mathbf{R}}/V_{\mathbf{R}}$ is compact, too. But $U_{\mathbf{R}}$ and $V_{\mathbf{R}}$ are homeomorphic to euclidean spaces, hence $U(F)_{\mathbf{C}} = V$, which shows that $U(F)_{\mathbf{C}}$ is defined over $\mathbf{Q}$. The group $N(F)_{\mathbf{C}}$, being equal to the normalizer of its unipotent radical, is then also defined over $\mathbf{Q}$.

It will turn out that (i)′ $\Rightarrow$ (ii) in our case; however we have preferred to start from a definition which makes sense for any symmetric space and any Satake compactification.

3.6. Clearly, (ii) is implied by

(ii)′ the group $\Gamma(F)$ is of arithmetic type.

Assuming (i), we now prove that (ii)′ is implied by either of the two following equivalent conditions:

(iii) There exists a normal connected $\mathbf{Q}$-subgroup C of $N(F)_{\mathbf{C}}$ containing $U(F)_{\mathbf{C}}$ and $L(F)_{\mathbf{C}}$ and such that $C_{\mathbf{R}}/L(F) \cdot U(F)$ is compact.

(iv) There exists a connected normal $\mathbf{Q}$-subgroup B of $N(F)_{\mathbf{C}}$, contained in $Z(F)^0_{\mathbf{C}}$, containing $U(F)_{\mathbf{C}}$, such that $Z(F)/(B_{\mathbf{R}} \cap G^0_{\mathbf{R}})$ is compact.[2]

We show, to begin with, that (iii) and (iv) are equivalent. First assume (iii). Let H be a maximal connected reductive $\mathbf{Q}$-subgroup of C, and L a maximal connected reductive $\mathbf{Q}$-subgroup of $N(F)_{\mathbf{C}}$ containing H. We may write $L = H \cdot H'$, with H' normal, defined over Q, and $H \cap H'$ finite. Then $D = H' \cdot U(F)_{\mathbf{C}}$ is a normal $\mathbf{Q}$-subgroup such that $Z(F)/D^0_{\mathbf{R}}$ is compact. Moreover, $D_{\mathbf{R}} \cap G^0_{\mathbf{R}} \subset Z(F)$ by (1) of 1.3, whence (iv). The other implication is proved in the same way.

Assume (iv) holds. The projection $N(F) \to G(F)$ is then the composition of the restriction to $N(F)$ of the $\mathbf{Q}$-morphism $N(F)_{\mathbf{C}} \to N(F)_{\mathbf{C}}/B$ with the projection of $N(F)/(B_{\mathbf{R}} \cap G^0_{\mathbf{R}})$ onto $G(F)$, which has a compact kernel, whence (ii)′.

We note finally that if $G = G_1 \times \cdots \times G_m$ is a direct product of normal $\mathbf{Q}$-subgroups, then a boundary component $F = F_1 \times \cdots \times F_m$ of X is rational if and only if F_i is a rational boundary component of $X_i = (K \cap G_i)\backslash G_{i\mathbf{R}}$ for all i. This follows immediately from the two following facts: the parabolic subgroups of G are the products of the parabolic subgroups of the G_i; the

[2] These conditions are of course fulfilled if $Z(F)_{\mathbf{C}}$ is defined over $\mathbf{Q}$. In fact, this is the requirement made in [10]; however, it has turned out to be too restrictive.

group Γ is commensurable with the product of the groups $\Gamma \cap G_i$, which are arithmetic (3.1).

3.7. THEOREM. *We keep the assumption of* 3.3 (i). *A boundary component F of X is rational if and only if $N(F)_{\mathbf{C}}$ is defined over* $\mathbf{Q}$. *If F is rational, $\Gamma(F)$ is of arithmetic type. The map $F \mapsto N(F)_{\mathbf{C}}$ defines a bijection of the set of proper rational boundary components onto the set of proper maximal parabolic $\mathbf{Q}$-subgroups of $G_{\mathbf{C}}$.*

By the last remark of 3.6, we may assume G to be simple over $\mathbf{Q}$. If X/Γ is compact, where Γ is an arithmetic group, then $\mathrm{rk}_{\mathbf{Q}}(G) = 0$ [13, 11.4, 11.6], G has no proper parabolic $\mathbf{Q}$-subgroup [14, 8.3-5], and there is no proper rational boundary component.

Assume now X/Γ to be non-compact. In the notation of 3.3 (ii), we have $G = R_{k/\mathbf{Q}}G'$ with G' absolutely simple and k totally real. Let F be a boundary component of X. If F is rational, then $N(F)_{\mathbf{C}}$ is a $\mathbf{Q}$-subgroup by 3.5 (i)′. Assume conversely that $N(F)_{\mathbf{C}}$ is defined over $\mathbf{Q}$. We have then $N(F)_{\mathbf{C}} = R_{k/\mathbf{Q}}P$, where P is a parabolic k-subgroup of G', [14, 6.19], hence

$$F = \prod\nolimits_{\sigma \in \Sigma} F_\sigma\ , \qquad N(F) = \prod N(F_\sigma)\ ,$$
$$\big(N(F_\sigma) = ({}^\sigma P)_{\mathbf{R}} \cap G_\sigma,\ \sigma \in \Sigma\big)\ ,$$

where F_σ is a boundary component of X_σ. Let V_0' be the center of the unipotent radical $V' = R_u(P)$ of P, and C' the connected centralizer of V_0' in P. The groups V', V_0' and C' are clearly defined over k. It follows immediately from the properties of the functor $R_{k/\mathbf{Q}}$ that $C = R_{k/\mathbf{Q}}C'$ is the centralizer in $N(F)_{\mathbf{C}}$ of the center $V_0 = R_{k/\mathbf{Q}}V_0'$ of the unipotent radical of $N(F)_{\mathbf{C}}$, and that $C^0_{\mathbf{R}} = \prod_\sigma C^0_\sigma$, where C_σ is the connected centralizer in $N(F_\sigma)$ of the center of $U(F_\sigma)$. By 1.20, C_σ contains $L(F_\sigma)\cdot U(F_\sigma)$ and the quotient $C_\sigma/L(F_\sigma)\cdot U(F_\sigma)$ is compact. Therefore condition (iii) of 3.6 is fulfilled; since, together with (i), it implies (ii), (ii)′, by 3.6, our first two assertions are proved.

Let F be rational. Then $N(F)_{\mathbf{C}} = \prod_\sigma N(F_\sigma)_{\mathbf{C}}$ and $N(F_\sigma)_{\mathbf{C}}$ is a proper maximal parabolic $\mathbf{R}$-subgroup of ${}^\sigma G'$, hence $N(F)_{\mathbf{C}}$ is a proper maximal parabolic $\mathbf{Q}$-subgroup of G.

Conversely, let P be a proper maximal parabolic $\mathbf{Q}$-subgroup of G. We have $P = R_{k/\mathbf{Q}}P'$, where P' is a proper maximal parabolic k-subgroup of G', and therefore $P = \prod_\sigma {}^\sigma P'$, and ${}^\sigma P'$ is a proper maximal parabolic $\sigma(k)$-subgroup of ${}^\sigma G'$. By 2.10, ${}^\sigma P'$ is also a proper maximal parabolic $\mathbf{R}$-subgroup of ${}^\sigma G'$; consequently (1.5), ${}^\sigma P'_{\mathbf{R}} \cap G^0_{\mathbf{R}} = N(F_\sigma)$ where F_σ is a boundary component of X, and $P_{\mathbf{R}} \cap G^0_{\mathbf{R}} = N(F)$, $(F = \prod F_\sigma)$. The boundary component F is then rational by the first part of the theorem. Since two boundary components with the same normalizer are identical, the proof of the theorem is complete.

3.8. THEOREM. *We keep the notation and conventions of* 3.3 (ii). *Let*[3] $\theta(b) = \{\beta_{b+1}, \cdots, \beta_s\}$, $L'_b = {}_kL_{\theta(b)}$ (cf. 2.2) *and* $L_b = R_{k/\mathbf{Q}}L'_b$, $(1 \leqq j \leqq s)$. *If* $b \neq s$, *then* $(L_{b,\mathbf{R}})^0 \cong L(F_{\mathrm{b}})$. *In particular* $L(F_{\mathrm{b}})_{\mathbf{C}}$ *is defined over* $\mathbf{Q}$, *almost* $\mathbf{Q}$-*simple, and its* $\mathbf{Q}$-*rank is equal to* $s - b$. *For any arithmetic subgroup* Γ *of* G, *the quotient* $F_{\mathrm{s}}/\Gamma(F_{\mathrm{s}})$ *is compact. Given a rational boundary component* F, *there exist one and only one index* b $(1 \leqq b \leqq s)$ *and an element* $x \in G_{\mathbf{Q}}$ *such that* $F = F_{\mathrm{b}} \cdot x$.

By 2.10 (b), we have $L'_b = L(F_{c(b,\sigma)})_{\mathbf{C}}$ for all $\sigma \in \Sigma$. Since $L_b = \prod_\sigma {}^\sigma L'_b$, this proves the first assertion. The set $\theta(b)$ being connected, L'_b is almost k-simple, (2.2), hence L_b is almost $\mathbf{Q}$-simple [14, 6.21 (ii)].

For each σ, the index $c(s, \sigma)$ is the last critical index, therefore (1.3, 1.5) $Z(F_{\mathrm{s}})$ contains $S'_{\mathbf{R}}$. Let C be the connected centralizer in $N(F_{\mathrm{s}})_{\mathbf{C}}$ of the center of the unipotent radical of $N(F_{\mathrm{s}})_{\mathbf{C}}$. By 1.20, the intersection of ${}^\sigma G' \cap C$ with ${}^\sigma S'$ is finite for every $\sigma \in \Sigma$. In particular, S normalizes C and $S \cap C$ is finite. This implies that the $\mathbf{Q}$-rank of C is zero, for otherwise there would exist a maximal $\mathbf{Q}$-split subtorus T of $N(F)_{\mathbf{C}}$ with $\dim(T \cap C) \neq 0$, and it could not be conjugate to S (since C is normal), contradicting the conjugacy theorem for maximal split tori. It follows then from [14, 8.5] that the unipotent elements of $C_{\mathbf{Q}}$ all belong to the unipotent radical of C, and that $X(C)_{\mathbf{Q}} = 0$. The quotient $C_{\mathbf{R}}/(\Gamma \cap C)$ is then compact [13, 11.8]. Since the projection $N(F_{\mathrm{s}}) \to G(F_{\mathrm{s}})$ maps $C^0_{\mathbf{R}}$ onto $G(F_{\mathrm{s}})$ and $\Gamma \cap C^0_{\mathbf{R}}$ into $\Gamma(F_{\mathrm{s}})$, we see that $G(F_{\mathrm{s}})/\Gamma(F_{\mathrm{s}})$ is compact, whence our second assertion.

Let F be a rational boundary component. Then $N(F)_{\mathbf{C}}$ is a proper maximal parabolic $\mathbf{Q}$-subgroup of G. On the other hand, the group $N(F_{\mathrm{b}})_{\mathbf{C}}$ is, in the notation of 2.2, the standard parabolic group ${}_{\mathbf{Q}}P_{\psi(b)}$, $\big(\psi(b) = {}_{\mathbf{Q}}\Delta - \{\beta_b\}$; $b = 1, \cdots, s\big)$. The groups ${}_{\mathbf{Q}}P_{\psi(j)}$ $(j = 1, \cdots, s)$ are all the proper standard maximal parabolic $\mathbf{Q}$-subgroups (2.2); there exists therefore one and only one b for which we may find $x \in G_{\mathbf{Q}}$ such that $x \cdot N(F)_{\mathbf{C}} \cdot x^{-1} = N(F_{\mathrm{b}})_{\mathbf{C}}$. We have then $F = F_{\mathrm{b}} \cdot x$, which ends the proof.

3.9. COROLLARY. *Let* F, F' *be rational boundary components of* X *such that* $F' \subset \bar{F}$. *Let* b, c *be the integers such that* $F \subset F_{\mathrm{b}} \cdot G_{\mathbf{Q}}$ *and* $F' \subset F_{\mathrm{c}} \cdot G_{\mathbf{Q}}$. *Then there exists* $g \in G_{\mathbf{Q}}$ *such that* $F \cdot g = F_{\mathrm{b}}$, $F' \cdot g = F_{\mathrm{c}}$.

There is nothing to prove unless $b \neq c$; in particular, we may assume $b \neq s$. Therefore (3.8), $L(F_{\mathrm{b}})_{\mathbf{C}}$ is defined over $\mathbf{Q}$, almost $\mathbf{Q}$-simple, and we may apply 3.8 to $X = F_{\mathrm{b}}$. The proof of 3.9 is then the same as that of the

[3] We hope the reader will not be unduly confused by the occasionally similar (or, by chance, even coinciding) notation used for real Lie groups in §1, and for their complex forms in §§2, 3. Also, L'_b has a different meaning here than in §1.3.

similar remark made in 1.5, using 3.8 to insure that u, v may be taken in $G_{\mathbf{Q}}$ and $L(F_{\mathrm{b}})_{\mathbf{Q}}$ respectively.

3.10. REMARKS. (1) The group $G(F_{\mathrm{b}})$ is the quotient of $L(F_{\mathrm{b}})$ by its center, which is finite; it may be identified with the adjoint group of $L(F_{\mathrm{b}})$. Thus, if $b \neq s$, the group $G(F_{\mathrm{b}})_{\mathrm{C}}$ may be viewed in a canonical way as a group defined over $\mathbf{Q}$, in such a way that ϖ_{b} induces a $\mathbf{Q}$-morphism of $N(F_{\mathrm{b}})_{\mathrm{C}}$ onto $G(F_{\mathrm{b}})_{\mathrm{C}}$, and that $\Gamma(F_{\mathrm{b}})$ is arithmetic. The images of $P \cap L(F_{\mathrm{b}})_{\mathrm{C}}$ and $S \cap L(F_{\mathrm{b}})_{\mathrm{C}}$ under ϖ_{b} are a minimal parabolic $\mathbf{Q}$-subgroup and a maximal $\mathbf{Q}$-split torus respectively.

(2) Theorem 3.8 shows, independently of 3.7, that F is rational if $N(F)_{\mathrm{C}}$ is defined over $\mathbf{Q}$ and conjugate to one of the groups $N(F_{\mathrm{b}})_{\mathrm{C}}$ with $b \neq s$. Our original proof of 3.7 consisted of 3.8 and of a separate discussion of the case $b = s$. The proof of 3.7 given here, which is based on 1.20, was suggested by [31].

3.11. PROPOSITION. *We keep the notation and assumptions of* 3.3 (ii). *Let* $\chi_{\mathrm{b}}(g) = \det \mathrm{Ad}_{\mathfrak{u}}\, g$ $(g \in N(F_{\mathrm{b}}))$, *where* $\mathfrak{u} = \mathfrak{u}(F_{\mathrm{b}})$ *is the Lie algebra of* $U(F_{\mathrm{b}})$, *and let* η_{b} *be the restriction to* $Z(F_{\mathrm{b}})$ *of the functional determinant* J_{b}. *Then there exists a rational number* $n_b > 0$ *such that*

$$\eta_{\mathrm{b}}(g) = \chi_{\mathrm{b}}(g)^{-n_b} \qquad (g \in A_b \cdot U(F_{\mathrm{b}}))\ ,$$
$$|\eta_{\mathrm{b}}(g)| = |\chi_{\mathrm{b}}(g)|^{-n_b} \qquad (g \in Z(F_{\mathrm{b}}))\ .$$

Using the notation of 1.16, we may write

$$\text{(1)}\qquad \begin{aligned}\chi_{\mathrm{b}}(g) &= \textstyle\prod_{\sigma \in \Sigma} \det\cdot \mathrm{Ad}_{\mathfrak{u}_\sigma}\, g_\sigma \\ &= \textstyle\prod_{\sigma\in\Sigma} \chi_{c(b,\sigma)}(g_\sigma) \qquad (g = (g_\sigma),\ \mathfrak{u}_\sigma = \mathfrak{u}(F_{c(b,\sigma)}))\ ,\end{aligned}$$

and similarly

$$\text{(2)}\qquad \eta_{\mathrm{b}}(g) = \textstyle\prod_{\sigma\in\Sigma} \eta_{c(b,\sigma)}(g_\sigma)\ .$$

By 1.16, there exists a rational number $n_{c(b,\sigma)} > 0$ such that

$$\text{(3)}\qquad |\eta_{c(b,\sigma)}(g)| = |\chi_{c(b,\sigma)}(g)|^{-n_{c(b,\sigma)}}\ , \qquad (g \in Z(F_{c(b,\sigma)}))$$

$$\text{(4)}\qquad \eta_{c(b,\sigma)}(g) = \chi_{c(b,\sigma)}(g)^{-n_{c(b,\sigma)}} \qquad (g \in A_{c(b,\sigma)})\ .$$

Let $g = (g_\sigma) \in {}_{\mathbf{Q}}A$. We have already remarked that $g_\sigma = \varphi_\sigma \circ \varphi(g)$, in the notation of 3.3; by the properties of the functor $R_{k/\mathbf{Q}}$, this implies

$$\text{(5)}\qquad \det \mathrm{Ad}_{\mathfrak{u}_\sigma}\, g = \chi_{c(b,\sigma)}(g_\sigma) \qquad (\sigma \in \Sigma;\ g \in {}_{\mathbf{Q}}A)\ .$$

The proposition will then be a consequence of (1) to (5) once it is shown that $n_{c(b,\sigma)}$ is independent of σ. By 2.9, ${}_k\Phi$ is of type C_s if and only if ${}_{\mathbf{R}}\Phi({}^\sigma G')$ is of type C_{t_σ} and $c(s, \sigma) = t_\sigma$ for *every* $\sigma \in \Sigma$. Thus if ${}_k\Phi$ is of type C_s and $b = s$, we are in case (i) of 1.15 for every σ, hence $n(b, \sigma) = 1$ $(\sigma \in \Sigma)$. If either

$b \neq s$ or ${}_k\Phi$ is of type BC_s, then, for every $\sigma \in \Sigma$, we are in one of the cases (ii), (iii) of 1.15. Then $n_{c(b,\sigma)}$ is given by an expression which depends only on the multiplicities of the restrictions to $A_{c(b,\sigma)}$ of $\alpha_{\sigma,c(b,\sigma)}$ and $2\alpha_{\sigma,c(b,\sigma)}$. In view of the different identifications made in 3.3, these are also the multiplicities of the restrictions of β_b and $2\cdot\beta_b$ to $A_{\mathbf{b}}$, and they are consequently independent of σ. This proves our assertion; and, in view of 1.16, shows that

$$(6) \qquad n_b = 1 \qquad ({}_k\Phi \text{ of type } C_s \text{ and } b = s)$$

$$(7) \qquad n_b = (p_b + 4q_b)\cdot(2p_b + 4q_b)^{-1} \qquad \text{otherwise,}$$

where p_b and q_b are the multiplicities of the restrictions of β_b and $2\cdot\beta_b$ to the intersection ${}_{\mathbf{Q}}A_{\mathbf{b}}$ of the kernels of the simple $\mathbf{Q}$-roots β_i $(i \neq b)$.

3.12. *The functional determinant on* $F_{\mathbf{b}}$. Let $\mathbf{q}_b = (q_{c(b,\sigma)})_{\sigma\in\Sigma}$, where $q_{c(b,\sigma)}$ is the rational number $\geqq 1$ attached by 1.17 to X_σ and the boundary component $F_{c(b,\sigma)}$. Put

$$|j_F(\sigma_{\mathbf{b}}(x), g)|^{\mathbf{q}_b} = \prod\nolimits_{\sigma\in\Sigma} |j_{c(b,\sigma)}(\sigma_{c(b,\sigma)}(x_\sigma), g_\sigma)|^{q_{c(b,\sigma)}} \qquad (g = (g_\sigma) \in N(F_{\mathbf{b}}))\ .$$

If we apply 1.17 to each component $F_{c(b,\sigma)}$ and use 3.11, we see that

$$(1) \qquad |J_{\mathbf{b}}(x, g)| = |j_F(\sigma_{\mathbf{b}}(x), g)|^{\mathbf{q}_b}\,|\chi_{\mathbf{b}}(g)|^{-n_b}\ , \qquad (x \in S_{\mathbf{b}};\ g \in N(F_{\mathbf{b}}))\ .$$

3.13. Let B be a normal connected $\mathbf{Q}$-subgroup of $N(F_{\mathbf{b}})_{\mathbf{C}}$ satisfying (iv) of 3.6. We may write B as a semi-direct product over $\mathbf{Q}$ of $S_{\mathbf{b}}\cdot U(F_{\mathbf{b}})_{\mathbf{C}}$ by a reductive $\mathbf{Q}$-group H. Write further $H = H'\cdot\mathfrak{D}H$ as an almost direct product of its connected center H' by its derived group $\mathfrak{D}H$. We know that $Z(F_{\mathbf{b}})/U(F_{\mathbf{b}})$ is the almost direct product of $B_{\mathbf{R}}/U(F_{\mathbf{b}})$ by a compact group. Furthermore, $Z(F_{\mathbf{b}})/{}_{\mathbf{Q}}A_b\cdot U(F_{\mathbf{b}})$ modulo its derived group is compact. This follows from 1.3, 1.5, applied to each factor $N(F_{c(b,\sigma)})$. It follows then that $B_{\mathbf{R}}/U(F_{\mathbf{b}})$ modulo its derived group is compact, therefore $H'_{\mathbf{R}}$ is compact. Since $J_{\mathbf{b}}$ and $\chi_{\mathbf{b}}$ are both characters on $Z(F_{\mathbf{b}})$, they are equal to one on $\mathfrak{D}H$, whence the equality

$$(1) \qquad \chi_{\mathbf{b}}(g)^{-n_b} = J_{\mathbf{b}}(x, g) \qquad (g \in (\mathfrak{D}H)\cdot{}_{\mathbf{Q}}A_{\mathbf{b}}\cdot U(F_{\mathbf{b}}))\ .$$

Let now Γ_0 be an arithmetic subgroup of $N(F_{\mathbf{b}})_{\mathbf{C}}$. The group $\Gamma_0 \cap H$ is commensurable with $(\Gamma_0 \cap H')\cdot(\Gamma_0 \cap \mathfrak{D}H)$, where both factors are arithmetic [13; 6.4, 6.11]. Since $H'_{\mathbf{R}}$ is compact, the group $\Gamma_0 \cap H'$ is finite. Of course, $\chi_{\mathbf{b}}$ takes the values ± 1 on Γ_0. It follows then that the image of $\Gamma_0 \cap B$ under $J_{\mathbf{b}}$ is a finite group (of roots of unity). This proves therefore the

3.14. PROPOSITION. *We keep the previous notation. Let* Γ_0 *be an arithmetic subgroup of* $N(F_{\mathbf{b}})_{\mathbf{C}}$. *Then there exists a positive integer* d *such that*

$$J_{\mathbf{b}}(x, g\cdot\gamma)^d = J_{\mathbf{b}}(x, g)^d \qquad (x \in X,\ g \in N(F_{\mathbf{b}}),\ \gamma \in \Gamma_0 \cap B)\ .$$

3.15. PROPOSITION. *Let* G *be as in* 3.3 (i). *Assume that* G *has no normal*

$\mathbf{Q}$-subgroup of dimension 3. *Then every proper rational boundary component has complex codimension* $\geqq 2$.

It suffices to prove this when G is simple over $\mathbf{Q}$. If it is absolutely simple, then our assertion follows from 1.5 (iii). Let now G be not absolutely simple. Then the set Σ of 3.3 (ii) has at least two elements. By 3.7, a rational boundary component F is a product $\prod_\sigma F_\sigma$, where F_σ is a *proper* boundary component of X_σ. We have then $\dim_{\mathbf{C}} X_\sigma - \dim_{\mathbf{C}} F_\sigma \geqq 1$ for each σ; since $\operatorname{card} \Sigma \geqq 2$, we are done.

4. Fundamental sets and compactification

4.1. Let G be a connected semi-simple $\mathbf{Q}$-group, and P a minimal parabolic $\mathbf{Q}$-subgroup. We write $P = M\cdot S\cdot U$ as in 2.2 and let ${}_{\mathbf{Q}}\Delta$ be the set of simple $\mathbf{Q}$-roots for the ordering associated to U. Since M centralizes S, we also have $P = S\cdot V$ where $V = M\cdot U$ is the semi-direct product of M and U.

We shall often write ${}_{\mathbf{Q}}A$ for $S^0_{\mathbf{R}}$. For $t > 0$, let

$$(1) \qquad {}_{\mathbf{Q}}A_t = \{a \in {}_{\mathbf{Q}}A \mid a^\beta \leqq t\,(\beta \in {}_{\mathbf{Q}}\Delta)\}\ .$$

A fundamental property of this subset is given by the:

LEMMA. *Let ω be a compact subset of $(M\cdot U)_{\mathbf{R}}$. Then the union of the sets $a\cdot\omega\cdot a^{-1}$, $(a \in {}_{\mathbf{Q}}A_t)$ is relatively compact.*

Since S centralizes M and normalizes U, it is enough to prove this when $\omega \subset U_{\mathbf{R}}$. But then ${}_{\mathbf{Q}}A_t$ is a bounded set of operators on $\mathfrak{u}_{\mathbf{R}}$ by $a \mapsto \operatorname{Ad} a$, in view of the very definition of ${}_{\mathbf{Q}}A_t$. Since the exponential is a homeomorphism of $\mathfrak{u}_{\mathbf{R}}$ onto $U_{\mathbf{R}}$, our assertion follows (see [13, Prop. 4.2]).

4.2. Let K be a maximal compact subgroup of $G_{\mathbf{R}}$ whose Lie algebra is orthogonal to that of $S_{\mathbf{R}}$, with respect to the Killing form, π the natural projection of $G_{\mathbf{R}}$ onto $X = K\backslash G_{\mathbf{R}}$, and $o = \pi(K)$. A *Siegel domain* $\mathfrak{S}'' = \mathfrak{S}''_{t,\omega}$ in $G_{\mathbf{R}}$ (with respect to K, S, and U) is a set of the form

$$\mathfrak{S}'' = \mathfrak{S}''_{t,\omega} = K\cdot{}_{\mathbf{Q}}A_t\cdot\omega \qquad (\omega \text{ compact in } V_{\mathbf{R}})\ .$$

In this paper, we shall in fact be more concerned with the intersection $\mathfrak{S}' = (K \cap P)\cdot{}_{\mathbf{Q}}A_t\cdot\omega$ of $\mathfrak{S}''$ with $P_{\mathbf{R}}$, to be called a *Siegel domain in* P, and with

$$\mathfrak{S} = \mathfrak{S}_{t,\omega} = o\cdot\mathfrak{S}'' = o\cdot\mathfrak{S}'\ ,$$

to be called a *Siegel domain in* X. If we replace the sign $\leqq$ by $<$ in 4.1 (1), and ω by an open relatively compact subset of $V_{\mathbf{R}}$, then we get *open Siegel domains* in $G_{\mathbf{R}}$, $P_{\mathbf{R}}$ and X. We note that ${}_{\mathbf{Q}}A$ centralizes M and that $K \cap P \subset M$, so that we may also write $\mathfrak{S}' = {}_{\mathbf{Q}}A_t\cdot\omega'$ with $\omega' = (K \cap P)\cdot\omega$.

The $\mathbf{Q}$-roots are, in a natural way, positive-valued functions on $P^0_{\mathbf{R}}$ which are equal to one on $V^0_{\mathbf{R}}$. Thus we also have

$$s^{\beta} \leqq t \qquad (s \in \mathfrak{S}'_t;\ \beta \in {}_{\mathbf{Q}}\Delta)\ .$$

4.3. Let Γ be an arithmetic subgroup of G. It is known that there exists a Siegel domain $\mathfrak{S}$ and a finite subset $C \subset G_{\mathbf{Q}}$ such that $\Omega = \mathfrak{S}\cdot C$ is a *fundamental set* for Γ in X; i.e., verifies

(F1) $\Omega\cdot\Gamma = X$, and

(F2) given $x \in G_{\mathbf{Q}}$, the set of $\gamma \in \Gamma$ for which $\Omega x \cap \Omega\gamma \neq \varnothing$ is finite (the *Siegel property*).

(See [20]. The result is stated there for $\Omega' = \pi^{-1}(\Omega)$ in $G_{\mathbf{R}}$, but this is clearly equivalent.)

If $S = \{e\}$, then X/Γ is compact, and conversely [13, 11.6, 11.4]. Of course any compact set has the Siegel property (for any discontinuous group Γ, and any $x \in G_{\mathbf{R}}$ in fact), so that in this case, any compact set verifying (F1) is a fundamental set. Also, in this case, a(n open) Siegel domain is just a(n open relatively) compact set.

REMARK. In considering later arithmetic subgroups of $H(X)$, we shall implicitly use a slight extension of these results to non-connected semi-simple groups, which is not stated in the literature, but reduces readily to the connected case. Namely, if G' is a $\mathbf{Q}$-group whose connected component is G, we replace P and K by their normalizers in G' and $G'_{\mathbf{R}}$ respectively. Let Γ be an arithmetic subgroup of G'. The only non-obvious point is to see that a Siegel domain has the Siegel property in G'. To see this, one uses the corresponding result in G and the following facts:

(i) $G'_{\mathbf{Q}}$ is generated by $G_{\mathbf{Q}}$ and $N(P)_{\mathbf{Q}}$, which follows from [14, § 4.13];

(ii) if $h \in N(P)_{\mathbf{R}}$, then $\mathfrak{S}\cdot h$ is contained in a Siegel domain (with respect to K, S, P).

4.4. We now assume that G is simple over $\mathbf{Q}$, X is a bounded symmetric domain, $\Gamma \subset G^0_{\mathbf{R}}$ and X/Γ is not compact. We take the notation and conventions of 3.3 (ii). In each group G_σ, we shall use the notation of § 1. Putting

$$(1) \qquad {}_{\mathbf{R}}\Delta_\sigma = \{\alpha_{\sigma,1}, \cdots, \alpha_{\sigma,t_\sigma}\}\ ,$$

we may write,

$$(2) \qquad \alpha_{\sigma,i} = (\gamma_{\sigma,i} - \gamma_{\sigma,i+1})/2\ , \qquad (1 \leqq i \leqq t_\sigma)$$

$$(3) \qquad \begin{aligned} \alpha_{\sigma,t_\sigma} &= \gamma_{t_\sigma}\ , && ({}_{\mathbf{R}}\Phi_\sigma \text{ of type } C_{t_\sigma})\ , \\ \alpha_{\sigma,t_\sigma} &= \gamma_{t_\sigma}/2\ . && ({}_{\mathbf{R}}\Phi_\sigma \text{ of type } BC_{t_\sigma})\ . \end{aligned}$$

In the product of the groups $T_{\sigma\mathbf{R}}$, the torus ${}_{\mathbf{Q}}A$ is contained in the identity component of the intersections of the kernels of the roots $\alpha_{\sigma,i}$ ($\alpha_{\sigma,i} \in {}_{\mathbf{R}}\Delta_\sigma$, $1 \leqq i \leqq t_\sigma$, i not critical). An element $a = (a_\sigma) \in {}_{\mathbf{Q}}A$ satisfies therefore the relations

$$(4) \qquad \gamma_{\sigma,i}(\log a_\sigma) = \gamma_{\sigma,i+1}(\log a_\sigma) \qquad (\sigma \in \Sigma, 1 \leqq i \leqq t_\sigma, i \text{ not critical})$$

with the convention that $\gamma_{\sigma,i} = 0$ if $i > t_\sigma$. Moreover, $S_{\mathbf{R}}$ being *diagonally* imbedded in $G_{\mathbf{R}}$ (see 3.3), we have

$$(5) \qquad a^{\beta_j} = a_\sigma^{\alpha_{\sigma,i}} \qquad \big(i = c(j, \sigma), j = 1, \cdots, s; \sigma \in \Sigma\big) .$$

Therefore, $a \in {}_{\mathbf{Q}}A_t$ if and only if it verifies (4), (5), and

$$(6) \qquad \gamma_{\sigma,i}(\log a_\sigma) - \gamma_{\sigma,i+1}(\log a_\sigma) \leqq 2 \log t \qquad \big(i = c(j, \sigma)\big)$$

if $c(j, \sigma) \neq t_\sigma$, and

$$\gamma_{\sigma,t_\sigma}(\log a_\sigma) \leqq \log t \, \big(\text{resp. } \gamma_{\sigma,t_\sigma}(\log a_\sigma) \leqq 2 \log t\big),$$

if t_σ is critical; i.e., if $t_\sigma = c(s, \sigma)$, and if ${}_{\mathbf{R}}\Phi_\sigma$ is of type C (resp. BC).

The subset ${}_{\mathbf{Q}}A_t \cdot \omega' = \mathfrak{S}' \cap P$ is contained in every standard parabolic subgroup, in particular in the groups $N(F_{\mathrm{b}})$. From 3.8 and the construction of the groups $L(F_{\mathrm{b}})$, we deduce immediately the following assertion: For $b \neq s$, $\mathfrak{S}' \cap L(F_{\mathrm{b}})$ is a Siegel domain of $L(F_{\mathrm{b}})$ with respect to $K \cap L(F_{\mathrm{b}})$, $S \cap L(F_{\mathrm{b}})$, and $U \cap L(F_{\mathrm{b}})$; its image under ϖ_{b} is a Siegel domain of $G(F_{\mathrm{b}})$; and

$$o_{\mathrm{b}} \cdot \varpi_{\mathrm{b}}\big(\mathfrak{S}'' \cap N(F_{\mathrm{b}})\big) = o_{\mathrm{b}} \cdot \varpi_{\mathrm{b}}(\mathfrak{S}')$$

is a Siegel domain of F_{b}.

If $b = s$, then $F_{\mathrm{b}}/\Gamma(F_{\mathrm{b}})$ is compact, and $o_{\mathrm{b}} \cdot \varpi_{\mathrm{b}}(\mathfrak{S}'' \cap N(F_{\mathrm{b}}))$ is compact.

4.5. LEMMA. *We keep the assumptions and notation of* 4.4. *The closure* $\bar{\mathfrak{S}}$ *of a Siegel domain* $\mathfrak{S}$, *in the natural compactification of* X, *is contained in the union of the standard rational boundary components* F_{b} ($0 \leqq b \leqq s$). *The intersection* $F_{\mathrm{b}} \cap \bar{\mathfrak{S}}$ *is equal to* $o_{\mathrm{b}} \cdot \varpi_{\mathrm{b}}(\mathfrak{S}')$ *and is a Siegel domain in* F_{b} ($0 \leqq b \leqq s$); *moreover, any Siegel domain in* F_{b} *is contained in a Siegel domain obtained in this way by taking* $\mathfrak{S}$ *itself sufficiently large.*

Let a_ν and v_ν ($\nu = 1, 2, \cdots$) be sequences of elements in ${}_{\mathbf{Q}}A_t$ and ω respectively. We may assume that $v_\nu \to v$ and that $\lim a_\nu^{\beta_j} = d_j$ exists for every $j \leqq s$. If all the d_j are >0, then $o \cdot a_\nu \cdot v_\nu$ tends to a point of $\mathfrak{S}$ itself. Otherwise let b be the greatest index such that $d_b = 0$. It follows from 4.4 that

$$(1) \qquad \gamma_{\sigma,i}(\log a_{\nu,\sigma}) \to -\infty \qquad \big(i \leqq c(b, \sigma), \sigma \in \Sigma\big)$$

and that $\gamma_{\sigma,i}(\log a_{\nu,\sigma})$ has finite limits if $i > c(b, \sigma)$. We have then

$$\lim o \cdot a_\nu = o_{\mathrm{b}} \cdot a ,$$

where a is the element of $L(F_{\mathrm{b}}) \cap \mathfrak{S}$ such that $a^{\beta_j} = d_j$ ($j > b$) if $b \neq s$, and is the identity if $b = s$. This implies

$$\lim_{\nu\to\infty} o\cdot a_\nu\cdot v_\nu = o_{\mathbf{b}}\cdot a\cdot v \in o_{\mathbf{b}}\cdot \varpi_{\mathbf{b}}({}_{\mathbf{Q}}A_t\cdot\omega) \ .$$

Conversely, given $x \in o_{\mathbf{b}}\cdot\pi({}_{\mathbf{Q}}A_t\cdot\omega)$, it is clear that we can find a sequence $a_\nu\cdot v_\nu$ as above such that $x = \lim o\cdot a_\nu\cdot v_\nu$. Taking into account the next to last paragraph of 4.4, this proves the lemma.

The following lemma is the analogue, in the present context, of Lemma 1 in [33].

4.6. LEMMA. *We keep the assumptions of* 4.4. *Let $\mathfrak{S}$ be a Siegel domain in X, C a finite subset of $G_{\mathbf{Q}}$, and $\Omega = \mathfrak{S}\cdot C$. Then $\bar\Omega$ is contained in the union of finitely many rational boundary components. There exist finitely many elements $\gamma_i \in \Gamma$ $(1 \leqq i \leqq q)$ having the following property: for every $\gamma \in \Gamma$ such that $\bar\Omega\gamma \cap \bar\Omega \neq \varnothing$, there exists γ_j verifying the condition $a\cdot\gamma_j = a\cdot\gamma$ for every $a \in \bar\Omega\gamma^{-1} \cap \bar\Omega$.*

By 4.5, $\bar\Omega$ is contained in the union of the boundary components $F_{\mathbf{b}}\cdot c (c \in C)$ which are all rational. Moreover, if we put $\mathfrak{T}_b = \mathfrak{S} \cap F_{\mathbf{b}}$, then we may write $\bar\Omega = \bigcup_\lambda \mathfrak{T}_\lambda\cdot c_\lambda$ where λ runs through a finite set Λ, $c_\lambda \in C$, and $\mathfrak{T}_\lambda = \mathfrak{T}_{b(\lambda)}$ for a suitable $b(\lambda)$. Thus $\mathfrak{T}_\lambda$ is a Siegel domain of $F_{b(\lambda)}$ and has the Siegel property (4.4, 4.5).

For each pair $\lambda, \mu \in \Lambda$ for which there exists $\gamma \in \Gamma$ verifying

$$(1) \qquad \mathfrak{T}_\lambda\cdot c_\lambda\cdot\gamma \cap \mathfrak{T}_\mu\cdot c_\mu \neq \varnothing \ ,$$

choose one element $\gamma_{\lambda\mu} \in \Gamma$ fulfilling that condition and let $d_{\lambda\mu} = c_\lambda\cdot\gamma_{\lambda\mu}\cdot c_\mu^{-1}$. We have $b(\lambda) = b(\mu)$, $\mathfrak{T}_\lambda = \mathfrak{T}_\mu \subset F_{b(\lambda)}$, and $d_{\lambda\mu} \in N(F_{b(\lambda)})_{\mathbf{Q}}$. Let γ be an element of Γ satisfying (1). We define $e_{\lambda\mu}(\gamma) = c_\mu\cdot\gamma_{\lambda\mu}^{-1}\cdot\gamma\cdot c_\mu^{-1}$. We have $e_{\lambda\mu}(\gamma) \in N(F_{b(\lambda)})_{\mathbf{Q}}$. Condition (1) implies

$$(2) \qquad \mathfrak{T}_\lambda\cdot\varpi_b(d_{\lambda\mu})\cdot\varpi_b\big(e_{\lambda\mu}(\gamma)\big) \cap \mathfrak{T}_\lambda \neq \varnothing \ .$$

If $b = s$, then $\mathfrak{T}_\lambda$ is relatively compact, and has trivially the Siegel property for any discontinuous group. If $b \neq s$, then ϖ_b is canonically a $\mathbf{Q}$-morphism (3.10 (1)). Since $c_\mu\cdot\Gamma\cdot c_\mu^{-1}$ is arithmetic [13, § 6.4] and $\mathfrak{T}_\lambda$ is a Siegel domain (4.4), the Siegel property obtains. In both cases, it shows that there are only finitely many possibilities for $\varpi_b(e_{\lambda\mu}(\gamma))$ when γ varies through the elements of Γ verifying (1). We have thus shown the existence of a finite subset D_λ of $c_\lambda^{-1}(N(F_{\mathbf{b}(\lambda)})_{\mathbf{Q}})c_\mu$ such that (1) implies the existence of $\tau(\gamma) \in D_\lambda$ verifying

$$x\cdot\tau(\gamma) = x\cdot\gamma \ , \qquad (x \in \mathfrak{T}_\lambda\cdot c_\lambda) \ .$$

This means that, if $\bar\Omega\cdot\gamma \cap \bar\Omega \neq \varnothing$, there are only finitely many possibilities for the action of γ on $\bar\Omega\cdot\gamma^{-1} \cap \bar\Omega$, whence the lemma.

4.7. LEMMA. *We keep the assumptions of* 4.4. *Let X^* be the union of the rational boundary components of X. Then there exists a fundamental* *set* $\Omega = \mathfrak{S}\cdot C$ *for* Γ *such that* $X^* = \bar{\Omega}\cdot\Gamma = \bar{\Omega}\cdot G_{\mathbf{Q}}$.

By 4.6, we have $\bar{\Omega} \subset X^*$, and hence $\bar{\Omega}\cdot G_{\mathbf{Q}} \subset X^*$, for every fundamental set of the type considered in 4.2.

By 4.2, 4.3, and 4.5, there exists a fundamental set Ω such that $\bar{\Omega} \cap F_{\mathrm{b}}$ is a fundamental set for $\Gamma(F_{\mathrm{b}})$ $(0 \leqq b \leqq s)$. For this set we have then $F_{\mathrm{b}} \subset \bar{\Omega}(\Gamma \cap N(F_{\mathrm{b}}))$. Let now F be a rational boundary component. There exists $c \in G_{\mathbf{Q}}$ and b such that $F = F_{\mathrm{b}}\cdot c$ (3.8). Let $\gamma \in N(F_{\mathrm{b}}) \cap \Gamma$. By the Siegel property, there exists a finite subset D of Γ (depending on γ), such that

$$\Omega\cdot\gamma\cdot c \subset \Omega\cdot D\ .$$

This implies

$$\bar{\Omega}\cdot\gamma\cdot c \subset \bar{\Omega}\cdot D \subset \bar{\Omega}\cdot\Gamma\ ;$$

therefore,

$$F = F_{\mathrm{b}}\cdot c \subset \bar{\Omega}\cdot\Gamma\ ,$$

and $X^* \subset \bar{\Omega}\cdot\Gamma$, which completes the proof.

4.8. We now turn to the general case and let Γ be an arithmetic subgroup of $H(X)$ as in 3.3 (i). We take the notation of 3.3 (i). The union X^* of the rational boundary components of X is the product of the similarly defined spaces X_i^* corresponding to the different simple **Q**-factors G_i of G. The group $H(X)$ operates continuously on the natural compactification of X (11.2), and hence $H(X)_{\mathbf{Q}}$ operates on X^*. Furthermore, (11.2), the restriction of each element $h \in H(X)$ to each boundary component is holomorphic. By 4.7, applied to X_i^* and $\Gamma_i = \Gamma \cap G_{i\mathbf{R}}^0$, and 4.3, there exists then a fundamental set Ω for Γ in X such that $\bar{\Omega}\cdot\Gamma = X^*$. For $x \in X^*$, we let $\Gamma_x = \{\gamma \in \Gamma, x\cdot\gamma = x\}$. The space X^* will be endowed with the topology $\mathcal{S}(\Omega, \Gamma)$ in which a fundamental set of neighborhoods of $x \in X^*$ is the set of all subsets U of X^* containing x and having the following properties: $U\cdot\gamma = U$ if $\gamma \in \Gamma_x$ and $U\cdot\gamma \cap \bar{\Omega}$ is a neighborhood of $x\cdot\gamma$ in $\bar{\Omega}$, in the natural topology of $\bar{\Omega}$ whenever $x\cdot\gamma \in \bar{\Omega}$. This topology was introduced, in a similar setting, in [33], and will be called the *Satake topology* for X^*.

4.9. THEOREM. *We keep the assumptions of* 4.8. *The Satake topology is the unique topology on* X^* *having the following properties*:

(i) *It induces the natural topology on* X *and on the closure of any fundamental set* Ω *for any arithmetic group* $\Gamma \subset H(X)$.

(ii) *The elements of the group* $H(X)_{\mathbf{Q}}$ *operate continuously on* X^*.

(iii) *If* x *and* x' *are not equivalent with respect to* Γ, *then there exist neighborhoods* U *of* x *and* U' *of* x' *such that* $U\cdot\Gamma \cap U' = \emptyset$.

(iv) *For each $x \in X^*$, there exists a fundamental set of neighborhoods $\{U\}$ of x such that $U\cdot\gamma = U$ if $\gamma \in \Gamma_x$, and $U\cdot\gamma \cap U = \varnothing$ if $\gamma \notin \Gamma_x$.*

PROOF. We wish to appeal to Theorem 1′ of [33]. We first consider a fundamental set as in 4.8, used to define the topology. As was remarked in 4.8, every element of $H(X)_{\mathbf{Q}}$ operates continuously on X^*, viewed as a subspace of $\bar{X}$ in the natural topology, and this implies condition (2) of Theorem 1′, loc. cit. As to the condition (3), it is just Lemma 4.6. Thus, Theorem 1′ applies. Also, by [33, Remark 2, p. 563], the topology induced by $\mathcal{S}(\Omega, \Gamma)$ on X is the natural one. This proves 4.9, with $H(X)_{\mathbf{Q}}$ replaced by Γ in (ii).

Let now $c \in H(X)_{\mathbf{Q}}$, and let Ω be a fundamental set for $c\cdot\Gamma\cdot c^{-1} \cap \Gamma$. It is also one for Γ and $c\cdot\Gamma\cdot c^{-1}$ in view of the Siegel property. It follows directly from the definitions, and from the fact that c acts continuously on $\bar{X}$, that c carries $\mathcal{S}(c\cdot\Gamma\cdot c^{-1}, \Omega)$ onto $\mathcal{S}(\Gamma, \Omega\cdot c)$. However, as pointed out in [33, Remark 3, p. 563], these two topologies of X^* are identical, whence (ii).

In the sequel we shall also write $\mathcal{S}$ for $\mathcal{S}(\Omega, \Gamma)$.

4.10. PROPOSITION. *Let F be a rational boundary component of X, and $x \in F$. Then x has a neighborhood U in X^* verifying* 4.9 (iv), *and such that a rational boundary component F' intersects U if and only if $F \subset \bar{F}'$. The closure of F in X^* is the union of the rational boundary components contained in the closure of F in the natural compactification $\bar{X}$ of X.*

We let $\mathcal{S}$ and $\mathcal{T}$ denote respectively the Satake topology of X^* and the topology of the natural compactification $\bar{X}$. Let $\bar{\Omega}$ be as in 4.8. Then $\mathcal{S}$ and $\mathcal{T}$ coincide on $\bar{\Omega}$. We may (and shall) assume that $x\cdot\Gamma$ contains an interior point of $\bar{\Omega} \cap F$. It follows from 4.6 that we may find finitely many elements $\gamma_1, \cdots, \gamma_m \in \Gamma$ such that $\bar{\Omega} \cap x\cdot\Gamma = \{x\cdot\gamma_1, \cdots, x\cdot\gamma_m\}$. Let U_i ($1 \leqq i \leqq m$) be a subset of X^* such that $U_i\cdot\gamma_i$ is a neighborhood of $x\cdot\gamma_i$ in $\bar{\Omega}$. Then $U = \bigcup_i U_i\cdot\Gamma_x$ is a neighborhood of x, which satisfies 4.9 (iv) if the U_i are sufficiently small, and we get in this way a fundamental system of neighborhoods of x in $\mathcal{S}$ [33, p. 562]. By 4.6, $\bar{\Omega}$ is the union of its intersections with finitely many rational boundary components. We may therefore assume U_i to be such that if the rational boundary component F' intersects $U_i\cdot\gamma_i$, then $x\cdot\gamma_i$ belongs to the closure of $F' \cap \bar{\Omega}$.

Let now $F' \cap U \neq \varnothing$. There exist then $\gamma \in \Gamma_x$ and an index i such that $F' \cap U_i\cdot\gamma \neq \varnothing$, whence $F'\cdot\gamma^{-1}\cdot\gamma_i \cap U_i\cdot\gamma_i \neq \varnothing$; by the condition just imposed on U_i, this implies that $x\cdot\gamma_i$ belongs to the closure of $F'\cdot\gamma^{-1}\cdot\gamma_i \cap \bar{\Omega}$, either in $\mathcal{S}$ or $\mathcal{T}$ since both coincide on $\bar{\Omega}$.

Consequently x belongs to the closure of $F'\cdot\gamma^{-1}$, and hence to that of F', in both $\mathcal{S}$ and $\mathcal{T}$. This proves our first assertion and shows that if x belongs to the closure of F' in $\mathcal{S}$, then it does so in $\mathcal{T}$, too. To prove the converse, we may assume G to be $\mathbf{Q}$-simple, and $F = F_{\mathrm{b}}$ to be standard. Then F_{c} $(c \geqq b)$ belongs to the closure of F_{b} in both the $\mathcal{S}$ and the $\mathcal{T}$ topologies.

Let F' be a rational boundary component contained in the $\mathcal{T}$-closure of F. There exist $c \geqq b$ and $g \in N(F)_{\mathbf{Q}}$ such that $F' \cdot g = F_{\mathrm{c}}$ (3.8, 3.9). Since g defines a homeomorphism in the $\mathcal{S}$-topology (4.9 (ii)), it follows that F' is also in the $\mathcal{S}$-closure of F.

4.11. COROLLARY. *The quotient $V^* = X^*/\Gamma$, endowed with the quotient topology, is a compact Hausdorff space. $V = X/\Gamma$ is an open everywhere dense subset. V^* is the finite union of subspaces $V_i = F_i/\Gamma(F_i)$, where F_i runs through a set of representatives of equivalence classes modulo Γ of rational boundary components. The closure of V_i is the union of V_i and of subspaces V_j of strictly smaller dimension.*

The first three assertions are obvious consequences of 4.9 and 4.10, once it is shown that, when G is $\mathbf{Q}$-simple, $F_{\mathrm{b}} \cdot H(X)_{\mathbf{Q}}/\Gamma$ is covered by the images of finitely many rational boundary components. Let $\Omega = \mathfrak{S} \cdot C$ be the fundamental set of 4.7, where $\mathfrak{S}$ is a Siegel domain and C a finite subset of $G_{\mathbf{Q}}$. Let F be a rational boundary component of type F_{b}. There exist $c \in C$ and $\gamma \in \Gamma$ such that $\bar{\mathfrak{S}} \cdot c \cdot \gamma \cap F \neq \emptyset$. But the intersection of $\bar{\mathfrak{S}}$ with the orbit of F_{b} under $G^0_{\mathbf{R}}$ is contained in F_{b} by 4.5; therefore, $F_{\mathrm{b}} \cdot c \cdot \gamma \cap F \neq \emptyset$, and $F_{\mathrm{b}} \cdot c \cdot \gamma = F$, so that $F_{\mathrm{b}} \cdot H(X)_{\mathbf{Q}}/\Gamma$ is a quotient of $F_{\mathrm{b}} \cdot C$.

The last assertion of the corollary follows from 4.10, because in $\bar{X}$, the closure of a boundary component F is the union of F and of boundary components of strictly smaller dimension (1.5).

REMARK. The proper rational boundary components correspond to the proper maximal parabolic $\mathbf{Q}$-subgroups (3.7). Since parabolic $\mathbf{Q}$-subgroups are conjugate if and only if they are conjugate by an element of $G_{\mathbf{Q}}$ (2.2), the V_i's of the corollary are in one-to-one correspondence with the orbits of Γ in $G_{\mathbf{Q}}/P_{\mathbf{Q}}$, where P runs through the standard maximal parabolic $\mathbf{Q}$-subgroups.

Let G be the quotient of the symplectic group $\mathbf{Sp}(2n, \mathbf{C})$ by its center, and $\Gamma = \mathbf{Sp}(2n, \mathbf{Z})$ be Siegel's modular group. The proper maximal parabolic $\mathbf{Q}$-subgroups are the stability groups of the rational isotropic subspaces. It is elementary that two rational isotropic subspaces of the same dimension q are transforms of each other by an element of Γ, so that we have $G_{\mathbf{Q}} = \Gamma \cdot P_{\mathbf{Q}}$ for every maximal parabolic $\mathbf{Q}$-subgroup. The boundary component corresponding to such a subspace is isomorphic to Siegel's upper half-plane of degree q, and thus $V^* = \bigcap_{0 \leqq q \leqq n} V_q$, where V_q is the quotient of the Siegel upper half plane of degree q by the corresponding modular group. We get therefore again the compactification introduced by Satake and later considered in [35].

4.12. *Truncated Siegel domains.* In order to give a more precise description of a fundamental set of neighborhoods of a point in V^*, we need to introduce certain subsets of a Siegel domain. To define them, we assume again, for convenience, that G is simple over $\mathbf{Q}$.

Let $\mathfrak{S}' = {}_{\mathbf{Q}}A_t \cdot \omega$ be a Siegel domain in $P_{\mathbf{R}}$ and $\mathfrak{S} = o \cdot \mathfrak{S}'$ the corresponding Siegel domain in X (4.2). Fix an index $b \leqq s$. For a positive number u and a subset $E \subset F_{\mathbf{b}}$, we let

$$\mathfrak{S}'_b(u, E) = \{s \in \mathfrak{S}' \mid s^{\beta_b} \leqq u,\ o_{\mathbf{b}} \cdot s \in E\}\ , \tag{1}$$

and

$$\mathfrak{S}_b(u, E) = o \cdot \mathfrak{S}'_b(u, E)\ . \tag{2}$$

The sets described by (1), (2) will be called $F_{\mathbf{b}}$*-adapted truncated Siegel domains.* The subscript b will sometimes be omitted, if it is clear from the context, or replaced by F.

An element $s \in \mathfrak{S}'$ can be written uniquely in the form

$$s = a_1 \cdot h_b \cdot a_2 \cdot w \qquad \big(a_1 \in \mathfrak{D}Z(F_{\mathbf{b}}) \cap {}_{\mathbf{Q}}A,\ a_2 \in L(F_{\mathbf{b}}) \cap {}_{\mathbf{Q}}A,\ h_b \in {}_{\mathbf{Q}}A_b,\ w \in \omega\big)\ , \tag{3}$$

where ${}_{\mathbf{Q}}A_b$ is the kernel of all simple $\mathbf{Q}$-roots β_i $(i \neq b)$. Of course, all elements on the right hand side depend (continuously) on s. However, for simplicity, we shall not make this explicit in the notation as long as no confusion arises. We collect a few remarks about this decomposition.

(i) The element h_b annihilates all simple $\mathbf{Q}$-roots except β_b, and a_1 (resp. a_2) annihilates β_i for $i \geqq b + 1$ (resp. $i < b$); hence,

$$s^{\beta_i} = a_1^{\beta_i}\ , \qquad (i \leqq b - 1) \tag{4}$$

$$s^{\beta_i} = a_2^{\beta_i}\ , \qquad (i \geqq b + 1)\ . \tag{5}$$

If the roots β_i $(i \geqq b + 1)$ are multiplicatively bounded on a_2, then a_2 is bounded; hence, there exists t' such that the set of products $h \cdot a_1$ is contained in ${}_{\mathbf{Q}}A_{t'}$. In view of 4.5, it is clear that the β_i's $(i \geqq b + 1)$ are multiplicatively bounded if E is relatively compact, as will be the case unless the contrary is stated. If so, the discussion of 4.5 implies

$$\textstyle\bigcap_{u>0} \bar{\mathfrak{S}}'_b(u, E) = o_{\mathbf{b}} \cdot \mathfrak{S}'_b(u_0, E) \qquad (\text{some } u_0 > 0)\ . \tag{6}$$

(ii) The group ${}_{\mathbf{Q}}A_b$ centralizes $K' = K \cap P_{\mathbf{R}}$, and is contained in the isotropy group of $o_{\mathbf{b}}$. Since $\mathfrak{S}' = K' \cdot {}_{\mathbf{Q}}A_t \cdot \omega$, $(\omega \subset V_{\mathbf{R}})$, we have then

$$h \cdot \mathfrak{S}'_b(u, E) \subset \mathfrak{S}'_b(v, E)\ , \qquad (v = u \cdot h^{\beta_b},\ h \in {}_{\mathbf{Q}}A_b)\ , \tag{7}$$

and hence also

$$o_{\mathbf{b}}\cdot\mathfrak{S}' = o_{\mathbf{b}}\cdot\mathfrak{S}'_b(u, F_{\mathbf{b}})\ , \qquad (u > 0)\ ,$$
$$o_{\mathbf{b}}\cdot\mathfrak{S}'(u, E) = o_{\mathbf{b}}\cdot\mathfrak{S}'(v, E) \qquad (u, v > 0)\ .$$

(iii) Let (X_i) be the basis of ${}_{\mathbf{Q}}\mathfrak{a}$ with respect to which the simple $\mathbf{Q}$-roots are written as in 1.2. Then the Lie algebra of ${}_{\mathbf{Q}}A \cap \mathfrak{D}Z(F_{\mathbf{b}})$ (resp. ${}_{\mathbf{Q}}A \cap L(F_{\mathbf{b}})$, resp. ${}_{\mathbf{Q}}A_b$) is generated by $X_i - X_{i+1}$ $(i = 1, \cdots, b-1)$ (resp. X_i $(i = b+1, \cdots, s)$, resp. $X_1 + \cdots + X_b$).

4.13. LEMMA. *We keep the assumptions of* 4.4. *Let* $x \in F_{\mathbf{b}}$, *and let* x^* *be the image of* x *in* V^*. *Let* $\Omega = \mathfrak{S}\cdot C$ *be a fundamental set verifying* 4.7, *where* $\mathfrak{S}$ *is an open Siegel domain whose closure contains* x. *Then there exist finitely many elements* $e_i \in N(F_{\mathbf{b}})_{\mathbf{Q}}$ *with the following property: the image* U *in* V^* *of* $U' = \bigcup_i \bar{\mathfrak{S}}(u, E_i)\cdot e_i$, *where* E_i *is a relatively compact neighborhood of* $x\cdot e_i^{-1}$ *in* $F_{\mathbf{b}}$, *is a neighborhood of* x^*; *the set* U *(resp.* $U'\cdot\Gamma_x$*) describes a fundamental set of neighborhoods of* x^* *in* V^* *(resp.* $x \in X^*$*) if* $u \to 0$ *and, independently,* E_i *runs through a fundamental set of neighborhoods of* $x\cdot e_i^{-1}$ *in* $F_{\mathbf{b}}$. *Furthermore, we get an equivalent set of neighborhoods of* x^* *if we replace* $\mathfrak{S}$ *by any open Siegel domain* $\mathfrak{S}' \supset \mathfrak{S}$.

It is clear from 4.5 that $\bar{\mathfrak{S}}_b(u, E)$ contains an open neighborhood of x in $\bar{\mathfrak{S}}_b$, and 4.12 shows that we get in this way a fundamental set of neighborhoods of x in $\bar{\mathfrak{S}}$. We let D be the finite set of elements $c \in C$ such that $x \in \bar{\mathfrak{S}}\cdot c$. If $d \in D$, then we get similarly a fundamental set of neighborhoods of x in $\bar{\mathfrak{S}}\cdot d$ by taking subsets $\mathfrak{S}_b(u, E)\cdot d$, where E is a neighborhood of $x\cdot d^{-1}$ in $F_{\mathbf{b}}$. Since $\bar{\Omega}$ is the finite union of the closed subsets $\bar{\mathfrak{S}}\cdot c$, $(c \in C)$, it follows that x has a fundamental set of neighborhoods in $\bar{\Omega}$ of the form $\bigcup_d \bar{\mathfrak{S}}_b(u, E_d)\cdot d$, where d runs over D and E_d is a neighborhood of $x\cdot d^{-1}$.

By 4.6, $x\cdot\Gamma \cap \bar{\Omega}$ is finite. Let γ_j $(1 \leqq j \leqq q)$ be elements of Γ such that $x\cdot\Gamma \cap \bar{\Omega} = \{x\cdot\gamma_1, \cdots, x\cdot\gamma_q\}$, and put $x\cdot\gamma_j = x_j$. Let Γ_x be the isotropy group of x in Γ. From 4.10, it follows that if $U_j\cdot\gamma_j$ is a neighborhood of x_j in $\bar{\Omega}$, then $\bigcup_j U_j\cdot\Gamma_x$ is a neighborhood of x in X^*, and that we get in this way a fundamental set of neighborhoods. For U_j, we may by the above take a finite union of sets $\bar{\mathfrak{S}}_b(u, E)\cdot c$, where $c \in C$ is such that $x_j \in \bar{\mathfrak{S}}\cdot c$, and E is a neighborhood of $x_j\cdot c^{-1}$ in $F_{\mathbf{b}}$. From this, our first assertion follows, with e_i running over the set of products $c\cdot\gamma_j^{-1}$ for which $x\cdot\gamma_j\cdot c^{-1} \in \bar{\mathfrak{S}}$. Such elements are rational over $\mathbf{Q}$, and they belong to $N(F_{\mathbf{b}})$, since they transform x into a point of $F_{\mathbf{b}}$.

As for the second assertion, the images of the sets $\bigcup_j U'_j$, which are constructed simply by replacing $\mathfrak{S}$ by $\mathfrak{S}'$ while allowing c and γ_j to run over the same sets of objects as in the definition of U_j, are relatively compact neighborhoods of x^*, and for $u \to 0$ and $E_{j,c}$ decreasing through a basis of compact neighborhoods of $x_j \cdot c^{-1}$ for each j and c, we obtain a decreasing family of *compact* neighborhoods of x^*; it follows from 4.12 (6) that the intersection of all these is just x^*. Since V^* is compact and Hausdorff, and since all the spaces we consider are second countable, it follows that these neighborhoods also give a basis of neighborhoods of x^* in V^*, which proves the last assertion.

REMARK. Let E_n $(n = 1, \cdots)$ be a decreasing sequence of neighborhoods of x in F_{b}, whose intersection is x, and let u_n be a sequence of real numbers tending to 0. Then the images in V^* of the sets $U_n = \bigcup_i \mathfrak{S}(u_n, E_n \cdot e_i^{-1}) \cdot e_i$ form a fundamental set of neighborhoods of x^*.

4.14. LEMMA. *Let b $(1 \leqq b \leqq s)$ and a Siegel domain $\mathfrak{S}'_0$ in $P_{\mathbf{R}}$ be given. Then there exists a Siegel domain $\mathfrak{S}' \supset \mathfrak{S}'_0$ in $P_{\mathbf{R}}$ such that $\mathfrak{S}_b(u, E) = o \cdot \mathfrak{S}'(u, E)$ is connected for every $u > 0$ and every connected open subset E of $o_{\mathrm{b}} \cdot \mathfrak{S}'$.*

We choose an Iwasawa decomposition $P_{\mathbf{R}} = K' \cdot A \cdot N$ of $P_{\mathbf{R}}$ where

$$K' = K \cap P\,, \qquad A = (M \cap A) \times {}_{\mathbf{Q}}A\,, \qquad N = (M \cap N) \cdot U_{\mathbf{R}}\,.$$

We may assume as before that ${}_{\mathbf{Q}}T \subset {}_{\mathbf{R}}T \subset T$, and choose compatible orderings on the root systems. The groups $Z(F_{\mathrm{b}}) \cap A \cdot N$ and $L(F_{\mathrm{b}}) \cap A \cdot N$ are connected (for otherwise they would have infinitely many components, which is impossible since $Z(F_{\mathrm{b}})_{\mathbf{C}}$, $(A \cdot N)_{\mathbf{C}}$, and $L(F_{\mathrm{b}})_{\mathbf{C}}$ are algebraic); they are also contractible, and the product mapping yields a homeomorphism of

$$\big(L(F_{\mathrm{b}}) \cap A \cdot N\big) \times \big(Z(F_{\mathrm{b}}) \cap A \cdot N\big)$$

onto $A \cdot N$. Furthermore $(L(F_{\mathrm{b}}) \cap A)$ and $(L(F_{\mathrm{b}}) \cap N)$ are the A and N part of an Iwasawa decomposition of $L(F_{\mathrm{b}})$. It follows that $(a, n) \mapsto o \cdot a \cdot n$ and $(a, n) \mapsto o_{\mathrm{b}} \cdot a \cdot n$ yield homeomorphisms of $A \times N$ onto X and of

$$\big(L(F_{\mathrm{b}}) \cap A\big) \times \big(L(F_{\mathrm{b}}) \cap N\big)$$

onto F_{b}. We also have ($\approx$ meaning homeomorphism)

$$L(F_{\mathrm{b}}) \cap A \cdot N \approx \big({}_{\mathbf{Q}}A \cap L(F_{\mathrm{b}})\big) \times \big(L(F_{\mathrm{b}}) \cap V_{\mathbf{R}} \cap A \cdot N\big)\,,$$
$$Z(F_{\mathrm{b}}) \cap A \cdot N \approx \big({}_{\mathbf{Q}}A \cap Z(F_{\mathrm{b}})\big) \times \big(Z(F_{\mathrm{b}}) \cap V_{\mathbf{R}} \cap A \cdot N\big)\,,$$

where the homeomorphisms are given by the product mapping.

Let us write $\mathfrak{S}'_0 = K' \cdot {}_{\mathbf{Q}}A_{t_0} \cdot \omega_0$, $(\omega_0 \subset V_{\mathbf{R}})$. We take $t > t_0$, and choose ω_2, ω_1 open, relatively compact, and connected in $L(F_{\mathrm{b}}) \cap V_{\mathbf{R}} \cap A \cdot N$ and $Z(F_{\mathrm{b}}) \cap V_{\mathbf{R}} \cap A \cdot N$ such that $K' \cdot \omega_1 \cdot \omega_2 \supset K' \cdot \omega_0$, (note that K' meets every connected component of $P_{\mathbf{R}}$). We claim that $\mathfrak{S}' = \mathfrak{S}'_{t,\omega}$, $(\omega = K' \cdot \omega_1 \cdot \omega_2)$, verifies our conditions. Since K' is contained in the isotropy group of o_{b} in $P_{\mathbf{R}}$, and is the isotropy group of o in $P_{\mathbf{R}}$, an elementary argument shows that it suffices to prove that $K' \backslash S'_{\mathrm{b}}(u, x)$ is connected for every $u > 0$ and every

$x \in o_{\mathbf{b}} \cdot \mathfrak{S}'$. We may write

$$x = o_{\mathbf{b}} \cdot a' \cdot w' \qquad (a' \in L(F_{\mathbf{b}}) \cap {}_{\mathbf{Q}}A,\ w' \in \omega_2)\ .$$

An arbitrary element $s \in \mathfrak{S}'$ can be written as a product

$$s = k' \cdot a_1 \cdot h_b \cdot a_2 \cdot w_1 \cdot w_2\ ,$$

where a_1, h_b, a_2 are as in 4.12, and $k' \in K'$, $w_i \in \omega_i$ $(i = 1, 2)$. The remarks made above imply that $o_{\mathbf{b}} \cdot s = x$ if and only if $a_2 = a'$, $w_2 = w'$. Thus w_1 runs through ω_1, which is connected by assumption, k' through K', and $a_1 \cdot h_b$ runs through the elements $y \in Z(F_{\mathbf{b}}) \cap {}_{\mathbf{Q}}A$ satisfying the conditions

$$y^{\beta_i} \leqq t\ (i < b)\ , \qquad y^{\beta_b} \leqq u \cdot t\ ,$$

which define clearly a connected set; whence our contention.

4.15. PROPOSITION. *We keep the notation of* 4.11. *Every point $v^* \in V^*$ has a fundamental set of open neighborhoods U such that $U \cap V$ is connected.*

Let Γ' be a normal subgroup of finite index of Γ. Then $V^* = V^*(\Gamma) = X^*/\Gamma$ is the quotient of X^*/Γ' by Γ/Γ'. Let G_i $(1 \leqq i \leqq m)$ be the simple $\mathbf{Q}$-factors of G and, as in 3.3 (i), let Γ' be the group generated by the $\Gamma'_i = G^0_{i\mathbf{R}} \cap \Gamma$. Then $X^*/\Gamma' = \prod_i X^*_i/\Gamma_i$. It suffices therefore to consider the case where G is simple over $\mathbf{Q}$ and $\Gamma \subset G^0_{\mathbf{R}}$. We may furthermore assume $x \in F_{\mathbf{b}}$. We use the notation of 4.13, 4.14, take $\mathfrak{S}'_0$ with $\mathfrak{S}_0$ verifying 4.7, and $\bar{\mathfrak{S}}_0$ containing x. We choose the elements $e_i \in N(F_{\mathbf{b}})_{\mathbf{Q}}$ as in the proof of 4.13; in particular, $x \cdot e_i^{-1} \in o_{\mathbf{b}} \cdot \mathfrak{S}'_0$. Moreover, we may so arrange things that each $x \cdot e_i^{-1}$ is contained in the (relative) interior of $F_{\mathbf{b}} \cap \bar{\mathfrak{S}}_0$; this is an easy consequence of the two following facts: the fixed set C of 4.13 is finite, and the number of elements in the intersection of any orbit of Γ in X^* with the closure of any Siegel domain (constructed with respect to the given torus, ordering, etc.) is finite.

Let E be a connected open neighborhood of x in F, small enough so that $E \cdot e_i^{-1} \subset o_{\mathbf{b}} \cdot \mathfrak{S}'_0$ for all i. Given $u > 0$, there exists $g_0 \in \mathfrak{S}'_{0,b}(u, E)$ such that $x = o_{\mathbf{b}} \cdot g_0$ (see 4.12 (ii)). By construction, $o_{\mathbf{b}} \cdot g_0 \cdot e_i^{-1} \in o_{\mathbf{b}} \cdot \mathfrak{S}'_0$; hence (loc. cit.), $o_{\mathbf{b}} \cdot g_0 \cdot e_i^{-1} \in o_{\mathbf{b}} \cdot \mathfrak{S}'_{0,b}(u, E \cdot e_i^{-1})$; there exists therefore u_i in the isotropy subgroup of $o_{\mathbf{b}}$ in $N(F_{\mathbf{b}})$ such that $u_i \cdot g_0 \cdot e_i^{-1} \in \mathfrak{S}'_{0,b}(u, E \cdot e_i^{-1})$.

The groups $K \cap L(F_{\mathbf{b}})$ and $K \cap Z(F_{\mathbf{b}})$ are maximal compact in $L(F_{\mathbf{b}})$ and $Z(F_{\mathbf{b}})$, and the first one is the isotropy group of $o_{\mathbf{b}}$ in $L(F_{\mathbf{b}})$. Moreover,

$$Z(F_{\mathbf{b}}) = (K \cap Z(F_{\mathbf{b}})) \cdot (P \cap Z(F_{\mathbf{b}}))\ .$$

Since $N(F_{\mathbf{b}}) = L(F_{\mathbf{b}}) \cdot Z(F_{\mathbf{b}})$ by 3.3, we may write (not uniquely):

$$u_i = k'_i \cdot k''_i \cdot s_i \qquad (k'_i \in K \cap L(F_{\mathbf{b}});\ k''_i \in K \cap Z(F_{\mathbf{b}});\ s_i \in Z(F_{\mathbf{b}}) \cap P)\ .$$

Let $\mathfrak{S}'$ be an open Siegel domain in $P_{\mathbf{R}}$ satisfying 4.14 which contains $\mathfrak{S}'_0$ and the elements $s_i \cdot g_0$. The images in V^* of the sets $U = \bigcup_i \bar{\mathfrak{S}}_b(u, E' \cdot e_i^{-1}) \cdot e_i$, as u runs over all strictly positive numbers, and E' over a basis of connected open neighborhoods of x contained in E, form a fundamental set of neighborhoods of v^*, and their interiors form a fundamental set of open neighborhoods. It is then enough to show that the interior of $U \cap X$ is connected. This latter set is the union over i, of the open sets $o \cdot \mathfrak{S}'_b(u, E \cdot e_i^{-1}) \cdot e_i$. We are therefore reduced (4.14) to proving that

$$(1) \qquad o \cdot \mathfrak{S}'_b(u, E' \cdot e_i^{-1}) \cdot e_i \cap o \cdot \mathfrak{S}'_b(u, E') \neq \varnothing ,$$

for all $u > 0$, all i, and all E' subject to our previous conditions.

We let h be an element of ${}_{\mathbf{Q}}A_b$ such that $h^{\beta_b} \leqq t^{-1} \cdot u$.

By construction, $o_{\mathbf{b}} \cdot s_i \cdot g_0 = x$ and $s_i \cdot g_0 \in \mathfrak{S}'$; hence, $s_i \cdot g_0 \in \mathfrak{S}'_b(t, E')$, and therefore (4.12 (ii)):

$$(2) \qquad h \cdot s_i \cdot g_0 \in \mathfrak{S}'_b(u, E') .$$

By 4.12 (ii), we have

$$o \cdot h \cdot u_i \cdot g_0 \cdot e_i^{-1} \in o \cdot \mathfrak{S}'_b(u, E' \cdot e_i^{-1}) .$$

But ${}_{\mathbf{Q}}A_b$ centralizes $L(F_{\mathbf{b}})$ and $K \cap Z(F_{\mathbf{b}})$; therefore,

$$o \cdot h \cdot u_i \cdot g_0 \cdot e_i^{-1} = o \cdot h \cdot k'_i \cdot k''_i \cdot s_i \cdot g_0 \cdot e_i^{-1} = o \cdot h \cdot s_i \cdot g_0 \cdot e_i^{-1} ,$$

whence

$$(3) \qquad o \cdot h \cdot s_i \cdot g_0 \in o \cdot \mathfrak{S}'_b(u, E' \cdot e_i^{-1}) \cdot e_i ,$$

which, together with (2), proves (1).

4.16. PROPOSITION. *We keep the previous notation. Let $J(x, g)$ be the functional determinant function in the unbounded realization associated to the rational boundary component $F = F_{\mathbf{b}}$ (3.3) and $g \in N(F)_{\mathbf{Q}}$. Then $J(x, g)$ is multiplicatively bounded on $\mathfrak{S}_F$, where $\mathfrak{S}_F = \mathfrak{S}_b(u, E)$ (E compact in F) is an F-adapted truncated Siegel domain.*

We have $\mathfrak{S}_F = o \cdot \mathfrak{S}'_F$, and

$$(1) \qquad J(x, g) = J(o \cdot s, g) = J(o, s)^{-1} \cdot J(o, s \cdot g) \qquad (s \in \mathfrak{S}'_F)$$

We want to estimate both factors on the right hand side. We write as in 4.12,

$$(2) \qquad s = a_1 \cdot h \cdot a_2 \cdot q \qquad \left(a_1 \in \mathfrak{D}Z(F) \cap {}_{\mathbf{Q}}A,\ a_2 \in L(F) \cap {}_{\mathbf{Q}}A,\ h \in {}_{\mathbf{Q}}A_b,\ q \in \omega\right) .$$

As pointed out in 4.12 (i), since $s \in \mathfrak{S}'_F$, the element a_2 is bounded on $\mathfrak{S}'_F$, and there exists $t' \geqq t$ such that

$$(3) \qquad h \cdot a_1 \in {}_{\mathbf{Q}}A_{t'} .$$

From (2), we get

$$s = a_2 \cdot h \cdot a_1 \cdot q \cdot (h \cdot a_1)^{-1} \cdot h \cdot a_1 = c \cdot h \cdot a_1 \qquad (c = a_2 \cdot h \cdot a_1 \cdot q \cdot (h \cdot a_1)^{-1}) ;$$

hence, by 1.11,

$$J(o, s) = J(o, c) \cdot J(o, h) . \tag{4}$$

By 4.1, and the above remark on a_2, the element c varies in a compact set when $s \in \mathfrak{S}'_F$; hence,

$$J(o, s) \asymp J(o, h) \qquad (s \in \mathfrak{S}'_F) . \tag{5}$$

The Bruhat decomposition gives

$$g = u \cdot t \cdot w \cdot v , \qquad (u, v \in U_{\mathbf{Q}}, t \in Z(S)_{\mathbf{Q}}, w \in N_{N(F)}(S)_{\mathbf{Q}})$$

whence

$$s \cdot g = d \cdot f \cdot v \qquad (d = c \cdot h \cdot a_1 \cdot u \cdot (h \cdot a_1)^{-1} \cdot t \cdot w; f = w^{-1} \cdot h \cdot a_1 \cdot w) .$$

Since c varies in a compact set, 4.1 and (2) show that so does d. We may write

$$v = v' \cdot v'' \qquad (v' \in L(F) \cap U, v'' \in Z(F) \cap U) .$$

The element v' commutes with the subtorus $Z(F) \cap S = S_\theta$ ($\theta = \{\beta_{b+1}, \cdots, \beta_s\}$). Since θ is a connected component of the set ${}_{\mathbf{Q}}\Delta - \{\beta_b\}$ of simple $\mathbf{Q}$-roots of $N(F)$, the Weyl group relative to $\mathbf{Q}$ of $N(F)$ also leaves S_θ stable. Therefore v' commutes with f, we have $s \cdot g = d \cdot v' \cdot f \cdot v''$, and, by (1.11),

$$J(o, s \cdot g) = J(o, d \cdot v') \cdot J(o, f) \cdot J(o, v'') . \tag{6}$$

But v' and v'' are fixed and d varies in a compact set. Consequently

$$J(o, s \cdot g) \asymp J(o, f) \qquad (s \in \mathfrak{S}'_F) . \tag{7}$$

The restriction of $J(o, g)$ to $Z(F)$ is a character (1.8, 1.11), which is of course equal to one on the derived group. Furthermore, the relative Weyl group of $N(F)$ is generated by the symmetries with respect to the hyperplanes annihilating the simple $\mathbf{Q}$-roots $\beta_i \neq \beta_b$, and hence it acts trivially on ${}_{\mathbf{Q}}A_b$. We have therefore

$$J(o, f) = J(o, h) , \tag{8}$$

and the proposition follows now from (1), (5), (7), and (8).

II. Automorphic Forms

5. Poincaré series

5.1. In this section, G is a complex connected reductive group defined over $\mathbf{R}$, whose group of real points has a compact center, H, a subgroup of finite index of $G_{\mathbf{R}}$, K, a maximal compact subgroup of H, V, a finite dimensional complex Hilbert space, and ρ: $K \to \mathbf{GL}(V)$ a unitary representation.

A function $f\colon H \to V$ is of *type* ρ on the left (resp. right) if $f(k\cdot g) = \rho(k)\cdot f(g)$ (resp. $f(g\cdot k) = \rho(k^{-1})\cdot f(g)$) ($g \in H$, $k \in K$). It is of *finite type*, with respect to K, or K-finite, on the right (resp. left) if the set of right translates $r_k f$ (resp. left translates $l_k f$) of f by elements of K spans a finite dimensional vector space over $\mathbf{C}$ of V-valued functions. This is in particular the case if f is of type ρ, and the general case can be subsumed into that one, at the cost of changing V, as is easily seen.

5.2. The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ is identified in the customary manner with the algebra of right invariant differential operators on H. In particular, if $X \in \mathfrak{g}$ and f is a differentiable V-valued function, then

$$Xf(g) = \left(\frac{d}{dt} f(e^{t\cdot X}\cdot g)\right)\Big|_{t=0}\,.$$

The center $\mathfrak{Z}(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$ is then identified with the algebra of left and right invariant differential operators on H. A smooth function (or a distribution) is called $\mathfrak{Z}(\mathfrak{g})$-finite if it is annihilated by an ideal $\mathfrak{U}$ of finite codimension of $\mathfrak{Z}(\mathfrak{g})$. If $\mathfrak{U}$ has codimension one, such a function is an eigenfunction of $\mathfrak{Z}(\mathfrak{g})$, which is our main case of interest. It is known that if f is $\mathfrak{Z}(\mathfrak{g})$-finite, and is K-finite on the right (resp. left), then it is annihilated by an elliptic right (resp. left) invariant, hence analytic, differential operator, and is consequently necessarily analytic. However, since analyticity is obvious for the functions to be considered later, we omit the proof. The convergence proofs for the Poincaré series could be based on that fact (and indeed are in [35, Exp. 10]). Here, we shall use instead the following result (in which, in fact, the assumptions on H may be slightly relaxed) of Harish-Chandra [23, Th. 1].

5.3. LEMMA. *Let U be a neighborhood of the identity in H. Let $f\colon H \to V$ be a C^∞-function which is of finite type on the right (resp. left) with respect to K, and is $\mathfrak{Z}(\mathfrak{g})$-finite. Then there exists $\alpha \in C_c^\infty(U)$ such that $\alpha(k\cdot g\cdot k^{-1}) = \alpha(g)$ ($g \in H$, $k \in K$) and that $f = f*\alpha$ (resp. $f = \alpha * f$).*

As is usual, C_c^∞ refers to C^∞-functions with compact support, and $*$ stands for the convolution. Thus in particular

$$f*\alpha(g) = \int_H f(g\cdot h^{-1})\cdot\alpha(h)dh = \int_H f(h)\cdot\alpha(h^{-1}\cdot g)dh\ ,$$

where dh is a Haar measure on H, chosen once and for all.

5.4. THEOREM. *Let Γ be a discrete subgroup of H. Let $f\colon H \to V$ be a function which belongs to $L^1(H)\otimes V$, is $\mathfrak{Z}(\mathfrak{g})$-finite, and is of finite type on the right, with respect to K. Then the series*

$$p_f(g) = \sum\nolimits_{\gamma\in\Gamma} f(g\cdot\gamma)\ , \qquad p_{\|f\|}(g) = \sum\nolimits_{\gamma\in\Gamma} \|f(g\cdot\gamma)\|\ ,$$

are absolutely and uniformly convergent on compact subsets, and are bounded on H.

It is enough to prove this for $p_{||f||}$. Since Γ is discrete, we may find a symmetric compact neighborhood U of e such that $U^2 \cap \Gamma = \{e\}$. By 5.3, there exists $\alpha \in C_c^\infty(U)$ such that

$$f(g\cdot\gamma) = \int_H f(g\cdot\gamma\cdot h^{-1})\cdot\alpha(h)dh\ ; \qquad (g \in H, \gamma \in \Gamma)$$

this can also be written

$$f(g\cdot\gamma) = \int_H f(g\cdot h^{-1})\alpha(h\cdot\gamma)dh\ .$$

Let M be the maximum of $|\alpha|$ on H. Since α has its support in $U = U^{-1}$, we have then

$$(1) \qquad ||f(g\cdot\gamma)|| \leqq M\cdot\int_{U\cdot\gamma^{-1}} ||f(g\cdot h^{-1})||\cdot dh\ .$$

But $U\cdot\gamma \cap U\cdot\gamma' = \varnothing$ if $\gamma \neq \gamma'$, hence

$$\sum\nolimits_{\gamma\in\Sigma} ||f(g\cdot\gamma)|| \leqq M\int_H ||f(g\cdot h^{-1})||\, dh \leqq M\cdot||f||_{L^1}\ ,$$

which proves that $p_{||f||}$ is bounded on H.

Let now C be a compact subset of H. Given $\varepsilon > 0$, there exists a compact subset D of H such that

$$(2) \qquad \int_{H-D} ||f(h)||\, dh \leqq \varepsilon\ .$$

The inequality (1) can also be written

$$(3) \qquad ||f(g\cdot\gamma)|| \leqq M\int_{g\cdot\gamma\cdot U} ||f(h)||\, dh\ .$$

Let ψ be the set of elements $\gamma \in \Gamma$ for which $C\cdot\gamma\cdot U \cap D \neq \varnothing$; it is finite. Given $g \in C$, the translates $g\cdot\gamma\cdot U(\gamma \in \Gamma, \gamma \notin \psi)$ are pairwise disjoint subsets of $H - D$, hence

$$\sum\nolimits_{\gamma\in\Gamma-\psi} ||f(g\cdot\gamma)|| \leqq M\cdot\int_{H-D} ||f(h)||\, dh \leqq M\cdot\varepsilon \qquad (g \in C)\ ,$$

from which the uniform convergence of $p_{||f||}$ on C follows.

5.5. REMARKS. (1) The above proof is due to Harish-Chandra. Our original argument was longer, and was a variation on one of Godement's [35, Exp. 10].

(2) If f is of finite type on the *left*, and satisfies the other assumptions of the theorem, then a similar argument, or the one of Godement, shows

readily that p_f is absolutely and uniformly convergent on compact sets. However, it does not seem necessarily true then that p_f is bounded.

5.6. The series p_f, where f satisfies the assumptions of 5.4, will be called a *Poincaré series*. Our next aim is to show that the usual Poincaré series on bounded symmetric domains are associated in a simple way to Poincaré series in the above sense.

Up to the end of this section, we assume that $X = K\backslash H$ is a bounded symmetric domain, let $H = G^0_{\mathbf{R}}$, and use the notation of § 1. Let, further,

$$\mu(x, g) = (e^{tX}\cdot g)_0 \in K_{\mathbf{C}} \ , \qquad (g \in H,\ x \in D)$$

be the canonical automorphy factor of the bounded realization D of X (1.8), and let

$$(1) \qquad \mu_\rho(x, g) = \rho\big(\mu(x, g)\big) \ ,$$

where ρ also denotes the natural extension to $K_{\mathbf{C}}$ of ρ. Since $\mu(x, k) = k$ $(k \in K)$, we have in particular,

$$(2) \qquad \mu_\rho(o, k) = \rho(k) \qquad (k \in K) \ .$$

To a function $F\colon D \longrightarrow V$, we associate $f\colon H \longrightarrow V$ by

$$(3) \qquad f(g) = \mu_\rho(o, g)\cdot F\big(\zeta(g)\big) \qquad (g \in H) \ .$$

From (2), and the cocycle identity (1.8), it follows that

$$(4) \qquad f(k\cdot g) = \rho(k)\cdot f(g) \qquad (k \in K,\ g \in H) \ .$$

It is easily seen that $F \mapsto f$ is in fact a bijection of the set of V-valued functions on D onto the set of V-valued functions on H which satisfy (4).

5.7. LEMMA. *We keep the notation of* 5.6. *Then*

(a) *the function F on D is holomorphic if and only if $Y\cdot f = 0$ for all $Y \in \mathfrak{p}^-$;*

(b) *Let $f\colon H \longrightarrow V$ be a function which satisfies* 5.6 (4) *and* $Y\cdot f = 0$, $(Y \in \mathfrak{p}^-)$. *Then f is $\mathfrak{Z}(\mathfrak{g})$-finite.*

The second assertion is proved in [35, Exp. 10, pp. 6–8]. As a matter of fact, it is only explicitly stated there that f is an eigenfunction of $\mathfrak{Z}(\mathfrak{g})$ if ρ is irreducible, but the proof also yields (b).

Part (a) is also known, and mentioned in [35, Exp. 10, p. 6]. For the sake of completeness, we indicate a proof. If we view $\mathfrak{g}$ as a Lie algebra of differential operators on D, *via* the action of $G^0_{\mathbf{R}}$, then the elements of $\mathfrak{p}^-$ are the linear combinations with constant coefficients of the partial derivatives $\partial/\partial\bar{z}_i$, where the z_i are coordinates in $\mathfrak{p}^+$. Therefore it is enough to show

$$(1) \qquad Yf(g) = \mu_\rho(o, g)\cdot YF\big(\zeta(g)\big) \qquad (Y \in \mathfrak{p}^-;\ g \in H) \ .$$

Let first $Y \in \mathfrak{g}_{\mathbf{R}}$. We may write

$$(2) \qquad Y = Y_- + Y_0 + Y_+ \qquad (Y_\pm \in \mathfrak{p}_\pm;\ Y_0 \in \mathfrak{k}_{\mathbf{C}})$$

where, obviously

$$(3) \qquad Y_0 = \left(\frac{d}{dt}(e^{tY})_0\right)_{t=0}.$$

We have

$$Yf(e) = \frac{d}{dt}\left(\mu_\rho(o, e^{tY})\cdot F(o\cdot e^{tY})\right)\Big|_{t=0},$$

hence, using (3), and denoting by $d\rho\colon \mathfrak{g}_{\mathbf{C}} \to \mathfrak{gl}(V)$ the differential of ρ,

$$(4) \qquad Yf(e) = d\rho(Y_0)\cdot F(o) + Y\cdot F(o).$$

By linearity this formula extends to all $Y \in \mathfrak{g}$. If $Y \in \mathfrak{p}^-$, then $Y_0 = 0$, and (4) yields (1) for $g = e$. Now the correspondence 5.6 (3) associates to the right translate $f' = r_g\cdot f (g \in H)$ the function F' given by $F'(x) = \mu_\rho(x, g)\cdot F(x\cdot g)$. Since μ_ρ is holomorphic in x, we have $YF'(o) = \mu_\rho(o, g)\cdot YF(o\cdot g)$; on the other hand $Y(r_g f)(e) = Yf(g)$, hence (4), applied to f' and F', yields (1).

5.8. LEMMA. *Let $J(x, g)$ be the functional determinant function in the bounded realization of X. Then $g \mapsto J(o, g)^a$ is in $L^1(H)$ for every integer $a \geqq 2$.*

We know that $J(x, k) = \det \mathrm{Ad}_{\mathfrak{p}^+}\, k^{-1}$ is a scalar independent of x, of modulus one $(k \in K)$. By the cocycle formula, $g \mapsto |J(o, g)|^a$ is therefore left and right invariant under K, and in particular may be viewed as a function on D. We have

$$\int_{G_{\mathbf{R}}} |J(o, g)|^a\, dg = \int_D |J(o, x)|^a\, dx,$$

where dx is a suitable invariant volume element. Up to a positive factor,

$$dx = |J(o, x)|^{-2}\,\omega,$$

where ω is the euclidean volume element in $\mathfrak{p}^+$. The domain of integration being bounded, it is then enough to show that $|J(o, x)|$ is bounded on D. Since $G_{\mathbf{R}} = K\cdot A\cdot K$, it suffices to check this on $o\cdot A$, where it follows from 1.12.

5.9. Let $X = X_1 \times \cdots \times X_q$ be the decomposition of X into irreducible bounded symmetric domains. The X_i's correspond canonically to the almost simple, non-compact, almost direct factors of the derived group of $G_{\mathbf{R}}$, and are therefore stable under $G_{\mathbf{R}}$. Each $g \in G^0_{\mathbf{R}}$ induces a complex analytic homeomorphism g_i of X_i such that

$$(x_1, \cdots, x_q)\cdot g = (x_1\cdot g_1, \cdots, x_q\cdot g_q) \qquad (x_i \in X_i).$$

Letting J_i be the functional determinant function in the canonical bounded realization D_i of X_i, we have

$$J(x, g) = \prod_i J_i(x_i, g_i) .$$

If $\mathbf{a} = (a_1, \cdots, a_q)$ is a sequence of integers, define $J^{\mathbf{a}}$ by

$$J(x, g)^{\mathbf{a}} = \prod_i J_i(x_i, g_i)^{a_i} .$$

Let Γ be a discrete subgroup of H. Let $\varphi: D \to V$ be a polynomial mapping. Put

$$(1) \qquad P_\varphi(x) = P(x) = \sum_{\gamma\in\Gamma} J(x, \gamma)^{\mathbf{a}}\cdot\varphi(x\cdot\gamma) .$$

Up to the fact that (for later use) we allow a multi-exponent **a**, this is just a Poincaré series in the usual sense. If it converges, it represents an automorphic form of weight **a**, i.e., it verifies

$$(2) \qquad P(x) = J(x, \gamma)^{\mathbf{a}}\cdot P(x\cdot\gamma) \qquad (x \in X; \gamma \in \Gamma) ,$$

as follows from the cocycle formula. To φ we associate, as in 5.6, the function $f: H \to V$ defined by

$$f(g) = J(o, g)^{\mathbf{a}}\cdot\varphi\big(\zeta(g)\big) \qquad (g \in H) .$$

Then by the cocycle formula, we have formally

$$(3) \qquad p_f(g) = \sum_{\gamma\in\Gamma} f(g\cdot\gamma) = J(o, g)^{\mathbf{a}}\cdot P\big(\zeta(g)\big) .$$

As before, let

$$(4) \qquad p_{||f||}(g) = \sum_{\gamma\in\Gamma} ||f(g\cdot\gamma)|| .$$

5.10. Theorem. *Assume* $a_i \geqq 2$ $(i = 1, \cdots, q)$. *Then the series* p_f *is a Poincaré series in the sense of* 5.6. *Consequently* P_φ, p_f *and* $p_{||f||}$ *converge absolutely and uniformly on compact sets, and* $p_{||f||}$ *is bounded on* H.

By 5.6 (4), f is of finite type on the *left*. Together with 5.7, this shows that f is $\mathfrak{Z}(\mathfrak{g})$-finite. Since φ is a polynomial mapping on D, it is bounded; hence, f is in $L^1(H) \otimes V$ by 5.8. We have

$$J(o, g\cdot k) = J(o, g)\cdot\prod_i\cdot(\det \operatorname{Ad} k_i^{-1})^{a_i} \qquad \big(k = (k_1, \cdots, k_q), k_i \in K_i\big) .$$

On the other hand, since k acts on D by means of a linear transformation, namely $\operatorname{Ad}_{\mathfrak{p}+} k^{-1}$, it transforms φ into a polynomial mapping of the same degree, hence the set of transforms of φ under K is contained in a finite dimensional vector space. It follows then that f is K-finite *on the right*, too. It satisfies therefore all the assumptions of 5.4.

5.11. We conclude this paragraph with some remarks to be used in § 10, in the application of our main embedding theorem. Let ν be the one-dimensional representation $k \mapsto \det \mathrm{Ad}_{\mathfrak{p}^+} k^{-1}$ of $K_{\mathbf{C}}$. Then, in the notation of 5.6, the automorphy factor μ_ν is just the functional determinant J in the bounded realization of X. Letting $\rho(m)$ ($m \in \mathbf{Z}$) stand for the tensor product $\rho \otimes \nu^m$, we have

$$(1) \qquad \mu_{\rho(m)}(x, g) = J(x, g)^m \cdot \mu_\rho(x, g) , \qquad (x \in X, g \in H) .$$

For a linear transformation A of a finite dimensional Hilbert space we let $\| A \|^2 = \mathrm{Tr}\,(A^* \cdot A)$, where A^* is the adjoint of A. We claim that, given ρ, there exists m_0 such that the function β: $g \mapsto \| \mu_{\rho(m)}(0, g) \|$ belongs to $L^1(H)$ for all $m \geqq m_0$.

PROOF. We have

$$(2) \qquad \mu(o, k \cdot g \cdot k') = k \cdot \mu(o, g) \cdot k' \qquad (g \in H, k, k' \in K \cap H) ,$$

so that β is right and left invariant under K. Clearly, $\mu_{\rho(n)} = J^{n-m} \cdot \mu_{\rho(m)}$, ($n, m \in \mathbf{Z}$); therefore, as in 5.8, it is enough to show that β is bounded on A for m big enough. For $a \in A$, we have

$$(3) \qquad \begin{aligned} \mu(o, a) &= a_0 \\ &= \textstyle\prod_i \exp(\log\cosh \gamma_i \cdot H_i) , \qquad (a = \exp(\gamma_1 X_1 + \cdots + \gamma_t X_t)) , \end{aligned}$$

as was recalled in 1.12. The transformations $\rho(m)(a_0)$ are simultaneously diagonalizable, and their eigenvalues are of the form a_0^δ, where δ runs through the weights of $\rho(m)$. These are the sums $m \cdot \nu + \lambda$, where λ runs through the weights of ρ. From 1.10, applied to each irreducible factor of X, we see that $\nu(H_i) < 0$ for all i. Consequently, there exists m_0 such that $\delta(H_i) < 0$ for all i and all weights δ of $\rho(m)$ if $m \geqq m_0$ By (3), β is then bounded on A.

As in 5.10, it follows that if φ: $X \to V$ is a polynomial mapping, the series

$$\textstyle\sum_{\gamma \in \Gamma} J(x, \gamma)^m \cdot \mu_\rho(x, g) \cdot \varphi(x \cdot \gamma) ,$$

is a Poincaré series for $m \geqq m_0$, and any discrete subgroup Γ of H, to which the conclusion of 5.10 applies.

6. Poincaré-Eisenstein series

6.1. In this section, G is a connected semi-simple $\mathbf{Q}$-group, H a subgroup of finite index of $G_{\mathbf{R}}$, K a maximal compact subgroup of H, V a finite dimensional complex Hilbert space, P a parabolic $\mathbf{Q}$-subgroup of G, and χ_0: $p \mapsto \det \mathrm{Ad}_{\mathfrak{u}}\, p$ ($p \in P$) the determinant of $\mathrm{Ad}\, p$ in the Lie algebra of the unipotent radical $U = R_u(P)$ of P.

Furthermore we assume P to be in the standard form (2.2). Thus $P = {}_{\mathbf{Q}}P_\theta$ where $\theta \subset {}_{\mathbf{Q}}\Delta$. Let $\theta' = {}_{\mathbf{Q}}\Delta - \theta$. We have then (2.3):

$$\chi_0 = \textstyle\sum_{\alpha \in \theta'} e_\alpha d_\alpha , \qquad (e_\alpha \in \mathbf{Q}, e_\alpha > 0) ,$$

where the d_α are fundamental highest weights for P. For every set $s =$

$(s_\alpha)_{\alpha\in\theta'}$ of complex numbers, we let $\Delta(p, s)$ be the complex valued function on $P_{\mathbf{R}}$ defined by

$$\Delta(p, s) = \prod\nolimits_{\alpha\in\theta'} |\, d_\alpha(p)\,|^{-s_\alpha} \,. \tag{1}$$

Let $f\colon H \longrightarrow V$ be a continuous function which satisfies

$$f(g\cdot p) = f(g)\cdot\Delta(p, s) \qquad (g \in H,\, p \in P \cap H)\,. \tag{2}$$

Let Γ be an arithmetic subgroup of G, contained in H, and Γ_∞ a subgroup of finite index of $\Gamma \cap P$. The series

$$E_f(g) = \sum\nolimits_{\gamma\in\Gamma/\Gamma_\infty} f(g\cdot\gamma) \qquad (g \in H) \tag{3}$$

is called an *Eisenstein series*. It follows from a theorem of Godement (unpublished; for a sketch of the proof, see [12]) that this series converges absolutely and uniformly on compact sets if

$$\mathcal{R}s_\alpha > e_\alpha \qquad (\alpha \in \theta')\,. \tag{4}$$

We note that since a rational character, defined over $\mathbf{Q}$, takes only the values ± 1 on an arithmetic subgroup, (2) implies that f is right invariant under Γ_∞, so that the summation in (3) makes sense, and E_f is right invariant under Γ. We shall need the following generalization of this result.

6.2. THEOREM. *We keep the notation of* 6.1. *Let $f'\colon H \longrightarrow V$ be a continuous function which is right invariant under Γ_∞ and such that*

$$m(g) = \sup\nolimits_p \|\, f'(g\cdot p)\,\| \cdot |\,\Delta(p, s)\,|^{-1} \qquad (p \in P \cap H)$$

is finite for every $g \in H$, and is bounded on compact sets. If s verifies 6.1(4), *then*

$$E_{f'} = \sum\nolimits_{\gamma\in\Gamma/\Gamma_\infty} f'(x\cdot\gamma)\,, \tag{1}$$

converges absolutely and uniformly on compact sets.

We only show how this reduces to the Godement theorem. Replacing f' by $\|\, f'\,\|$ we may assume the s_α to be real, and f' to be a real-valued positive function. Let f'' be a strictly positive continuous function on H such that

$$f''(g\cdot p) = f''(g)\cdot\Delta(p, s) \qquad (g \in H,\, p \in P_{\mathbf{R}} \cap H)\,.$$

Such functions obviously exist; we may for instance simply put $f''(k\cdot p) = \Delta(p, s)$. Since Δ is trivial on $P \cap K$, this is legitimate, and defines the required function on $H = K\cdot(P \cap H)$. Writing $g = k\cdot p$ $(k \in K,\, p \in P \cap H)$, we have

$$f'(g)\cdot\big(f''(g)\big)^{-1} = f'(k\cdot p)\cdot\Delta(p, s)^{-1}\cdot f''(k)^{-1}\,.$$

f'' has a strictly positive minimum on K, and $f'(k\cdot p)\Delta(p, s)^{-1}$ remains bounded when k runs through K and p through $P \cap H$ by assumption. There exists therefore a strictly positive constant c such that

$$f'(g) \leqq c \cdot f''(g) \qquad (g \in H) ,$$

whence the reduction to the case of 6.1.

We now introduce a generalization of Eisenstein series and Poincaré series, to be called Poincaré-Eisenstein series.

6.3. Let B be a normal connected $\mathbf{Q}$-subgroup of P which contains the split radical $S \cdot U$ of P, and let $C = P/B$. This is a reductive connected $\mathbf{Q}$-group which has no non-trivial rational character defined over $\mathbf{Q}$. The natural projection maps $H \cap P$ and $P_{\mathbf{R}}$ onto subgroups of finite index of $C_{\mathbf{R}}$. Furthermore, if $f\colon H \to V$ verifies

$$(1) \qquad \| f(h \cdot c) \| = \| f(h) \cdot \Delta(c, s) \| , \qquad (h \in H,\ c \in B_{\mathbf{R}} \cap H) ,$$

then the restriction of $\| f \cdot \Delta(\ , s)^{-1} \|$ to $H \cap P$ is right invariant under $B \cap H$, and may be identified with a function on the open subgroup $(H \cap P)/(H \cap B) = C_1$ of $C_{\mathbf{R}}$.

Let now Γ and Γ_∞ be as in 6.1, and $\Gamma_0 = \Gamma_\infty \cap B$. We define the Poincaré-Eisenstein series E_f by:

$$(2) \qquad E_f(h) = \sum\nolimits_{\gamma \in \Gamma/\Gamma_0} f(h \cdot \gamma) \qquad (h \in H) .$$

6.4. THEOREM. *We keep the notation of* 6.3, *and assume that* $C^0_{\mathbf{R}}$ *has a compact center. Let* $f\colon H \to V$ *and* $f'\colon C_1 \to V$ *be continuous functions which verify*

(i) $\| f(k \cdot h \cdot b) \| = \| f(h) \| \cdot \Delta(b, s)$ $(k \in K, h \in H, b \in B \cap H)$, *where* $s = (s_\alpha)$ *is real, and*

(ii) *the function* f' *belongs to* $L^1(C_1) \otimes V$, *is* $\mathfrak{Z}(\mathfrak{c}_1)$*-finite* (cf. 5.1), *is of finite type on the right with respect to some maximal compact subgroup, and is equal in norm to* $f \cdot \Delta(\ , s)^{-1}$.

Then, if s *satisfies* 6.1 (4), *the series* 6.3 (2) *converges absolutely and uniformly on compact sets.*

PROOF. Note first that

$$(1) \qquad \| f(h \cdot b \cdot p) \| = \| f(h \cdot p) \| \cdot \Delta(b, s) \qquad (h \in H, b \in B \cap H, p \in P \cap H) .$$

In fact, we have $b \cdot p = p \cdot b'$ $(b' = p^{-1} \cdot b \cdot p)$, hence

$$\| f(h \cdot b \cdot p) \| = \| f(h \cdot p \cdot b') \| = \| f(h \cdot p) \| \cdot \Delta(b', s) .$$

But $\Delta(b', s) = \Delta(b, s)$ since P acts trivially by inner automorphisms on its character group, whence (1). The function Δ, being equal to one on Γ_0, and Γ_0 being normal in Γ_∞, this implies in particular

$$(2) \qquad f(h \cdot \tau \cdot \sigma) = f(h \cdot \sigma) \qquad (h \in H, \tau \in \Gamma_0, \sigma \in \Gamma_\infty) .$$

Let

$$(3)\qquad \begin{aligned} p_f(h) &= \textstyle\sum_{\sigma\in\Gamma_\infty/\Gamma_0} f(h\cdot\sigma)\ , \\ p_{\|f\|}(h) &= \textstyle\sum_{\sigma\in\Gamma_\infty/\Gamma_0} \|f(h\cdot\sigma)\| \end{aligned} \qquad (h\in H)\ .$$

Write $h = k_h\cdot p_h$, where $k_h\in K$ and $p_h\in P$ are determined up to an element of $K\cap P$. In view of (i), (ii), we have, π denoting the projection $H\cap P\to C_1$,

$$(4)\qquad p_{\|f\|}(h\cdot p)\cdot\Delta(p, s)^{-1} = \Delta(p_h, s)\cdot p_{\|f'\|}(\pi(p_h\cdot p))\ ,$$

where

$$(5)\qquad p_{\|f'\|}(\pi(q)) = \textstyle\sum_{\sigma\in\Gamma_\infty/\Gamma_0} \|f'(\pi(q\cdot\sigma))\| \qquad (q\in P\cap H)\ .$$

By (ii) and 5.4, the series in (5) is uniformly bounded; therefore, the left hand side of (4) is bounded when p varies in $P\cap H$ and p_h runs over a compact set. Moreover, (2) and (3) show that p_f is right invariant under Γ_∞. We may write

$$(6)\qquad E_f = \textstyle\sum_{\gamma\in\Gamma/\Gamma_\infty} p_f(h\cdot\gamma)\ ,$$

$$(7)\qquad E_{\|f\|} = \textstyle\sum_{\gamma\in\Gamma/\Gamma_\infty} p_{\|f\|}(h\cdot\gamma)\ ,$$

so that the theorem now follows from 6.2.

7. Poincaré-Eisenstein series on bounded symmetric domains

In this section, G is a simple, connected $\mathbf{Q}$-group satisfying the assumptions of 3.3 (ii); *the notation of* 3.3 *is used.*

7.1. Let $F = F_{\mathrm{b}}$ $(1\leqq b\leqq s)$ be a standard rational boundary component, $\sigma_{\mathrm{b}}\colon X\to F$ the canonical projection. The group $N(F)_{\mathrm{C}}$ is defined over $\mathbf{Q}$, and has a connected normal $\mathbf{Q}$-subgroup B, containing the split radical $S_{\mathrm{b}}\cdot U_{\mathrm{b}}$ of $N(F)_{\mathrm{C}}$, such that $B^0_{\mathrm{R}}\subset Z(F)$ and $Z(F)/B^0_{\mathrm{R}}$ is compact (3.6, 3.7).

Let J_F or J_{b}, or simply J, be the functional determinant in the unbounded realization S_{b} associated to F, and φ a polynomial on F, in the coordinates of the canonical bounded realization of F. Let Γ be an arithmetic subgroup of G, contained in G^0_{R}, $\Gamma_\infty = N(F)\cap\Gamma$, and $\Gamma_0 = B^0_{\mathrm{R}}\cap\Gamma$; let l be a positive integer. We shall consider the series

$$(1)\qquad E(x) = E_{\varphi,l,\Gamma}(x) = \textstyle\sum_{\gamma\in\Gamma/\Gamma_0} \varphi(\sigma_{\mathrm{b}}(x\cdot\gamma))\cdot J_F(x, \gamma)^l \qquad (x\in X)\ .$$

For this to make good sense, each term of the right hand side should be right invariant under Γ_0. More generally, we wish to know that

$$(2)\qquad \varphi(\sigma_{\mathrm{b}}(x\cdot g\cdot\lambda))\cdot J_F(x, g\cdot\lambda)^l = \varphi(\sigma_{\mathrm{b}}(x\cdot g))\cdot J_F(x, g)^l \quad (g\in G^0_{\mathrm{R}},\ x\in X,\ \lambda\in\Gamma_0)\ .$$

Since $B^0_{\mathrm{R}}\subset Z(F)$ acts trivially on F, the invariance of the first factor is clear. For the second one, it is enough that l be a multiple of the integer d of 3.14, as we shall assume.

In what follows, E may be supplied with a subscript consisting of any subset of $\{\varphi, l, \Gamma\}$ sufficient to characterize it in a given context. It will be called a Poincaré-Eisenstein series (P-E series for short) *adapted to* F. More generally, for any $g \in G_{\mathbf{Q}}$, we shall also consider the transform $E \circ g$ of E by g, defined by

$$(3) \qquad E \circ g(x) = J_F(x, g^{-1})^l \cdot E(x \cdot g^{-1}) \qquad (x \in X) .$$

By the cocycle identity:

$$(4) \qquad E \circ g(x) = \sum\nolimits_{\gamma \in \Gamma/\Gamma_0} \varphi\big(\sigma_{\mathrm{b}}(x \cdot g^{-1} \cdot \gamma)\big) \cdot J_F(x, g^{-1} \cdot \gamma)^l .$$

We shall see shortly that this series converges absolutely for suitable l. It represents then an automorphic form of weight l for the group $\Gamma^g = g^{-1} \cdot \Gamma \cdot g$.

7.2. THEOREM. *There exists a positive integer l_0 such that, if l is a positive multiple of l_0, then the series $E_{\varphi, l, \Gamma} \circ g$ converges absolutely and uniformly on compact sets.*

It suffices to prove this for E.

The group $N(F)_{\mathbf{C}}$ is the standard parabolic group ${}_{\mathbf{Q}}P_\theta$ where $\theta = {}_{\mathbf{Q}}\Delta - \{\beta_b\}$, in the notation of 2.2. Therefore the set $\theta' = {}_{\mathbf{Q}}\Delta - \theta$ reduces to $\{\beta_b\}$, and $\chi_{\mathrm{b}}(p) = \det \mathrm{Ad}_{\mathfrak{u}}\, p (p \in N(F)_{\mathbf{C}})$ may be taken as the fundamental highest weight relative to $\mathbf{Q}$ (2.3). By 3.12 (1), we have then

$$(1) \qquad |J_F(x, g)| = |j_F(\sigma_{\mathrm{b}}(x), g)|^{q_{\mathrm{b}}} \cdot \Delta(g, n_b) , \qquad \big(g \in N(F)\big)$$

where $\Delta(g, n_b) = \chi_{\mathrm{b}}(g)^{-n_b}$.

We claim that 7.2 holds if we take for l_0 the smallest positive integer l verifying 7.1 (2) and:

$$(2) \qquad l \cdot n_b \in \mathbf{Z} , \qquad l \cdot n_b > 1 ; \qquad l \cdot q_{c(b,\sigma)} \geqq 2 \qquad (\sigma \in \Sigma) .$$

Let f be the complex valued function on $G^0_{\mathbf{R}}$ defined by

$$(3) \qquad f(g) = \varphi\big(\sigma_{\mathrm{b}}(\zeta(g))\big) \cdot J_F(o, g)^l .$$

Since the maximal compact subgroup K leaves o fixed, and $|J_F(o, k)| = 1$, if $k \in K$, we have

$$(4) \qquad |f(k \cdot g)| = |f(g)| \qquad (k \in K \cap G^0_{\mathbf{R}}, g \in G^0_{\mathbf{R}}) .$$

The cocycle formula and 3.11 imply

$$(5) \qquad |f(g \cdot b)| = |f(g)| \cdot \Delta(b, l \cdot n_b) , \qquad (g \in G^0_{\mathbf{R}}, b \in B_{\mathbf{R}} \cap G^0_{\mathbf{R}}) .$$

From (1), we get

$$(6) \qquad |f(g)| \cdot \Delta(g, l \cdot n_b)^{-1} = |\varphi\big(\sigma_{\mathrm{b}} \cdot \zeta(g)\big)| \cdot |j_F(o_{\mathrm{b}}, g)|^{l \cdot q_{\mathrm{b}}} \qquad \big(g \in N(F) \cap G^0_{\mathbf{R}}\big) .$$

Taking (4), (5) and 5.7, 5.8 into account, we see that all conditions of 6.4 are fulfilled by $|f|$, if l is a positive multiple of l_0; therefore

$$(7) \qquad E_f(g) = \sum\nolimits_{\gamma \in \Gamma/\Gamma_0} f(g\cdot\gamma) , \qquad (g \in G^0_{\mathbf{R}})$$

converges absolutely and uniformly on compact subsets. Since

$$(8) \qquad E_{\varphi,l}(o\cdot g) = J(o, g)^{-l}\cdot E_f(g) , \qquad (g \in G^0_{\mathbf{R}})$$

by the cocycle identity, the theorem is proved.

As in 6.4, we may write E in the form

$$(9) \qquad E(o\cdot g) = J(o, g)^{-l}\cdot\sum\nolimits_{\gamma\in\Gamma/\Gamma_\infty} p_f(g\cdot\gamma) ,$$

where

$$(10) \qquad p_f = \sum\nolimits_{\lambda\in\Gamma_\infty/\Gamma_0} f(g\cdot\lambda) = \sum\nolimits_{\lambda\in\Gamma_\infty/\Gamma_0} \varphi\big(\sigma_{\mathrm{b}}\circ\zeta(g\cdot\lambda)\big)\cdot J_F(o, g\cdot\lambda)^l .$$

The argument used in proving 6.4 shows that

$$|f(g\cdot b\cdot p)| = |f(g\cdot p)|\,\Delta(b, l\cdot n_b) \qquad \big(g \in G^0_{\mathbf{R}},\, b \in B\cap G^0_{\mathbf{R}},\, p\in N(F)\big) ;$$

therefore, if we put again

$$(11) \qquad p_{\|f\|} = \sum\nolimits_{\lambda\in\Gamma_\infty/\Gamma_0} |f(g\cdot\lambda)| ,$$

we have

$$(12) \qquad p_{\|f\|}(k\cdot g\cdot b) = p_{\|f\|}(g)\cdot\Delta(b, l\cdot n_b) \qquad (g\in G^0_{\mathbf{R}},\, b\in G^0_{\mathbf{R}}\cap B,\, k\in K) .$$

As was proved in 6.4, $p_{\|f\|}(g\cdot p)\Delta(p, l\cdot n_b)^{-1}$ is bounded when g varies in a compact set of $G^0_{\mathbf{R}}$ and p in $P\cap G^0_{\mathbf{R}}$.

We henceforth assume l to be divisible by the integer l_0 defined above.

7.3. From the geometric point of view, an automorphic form ω on X of weight l for Γ is a holomorphic cross section, invariant under Γ, of the complex line bundle $(\Lambda^n\tau)^{-l}$ ($n = \dim_{\mathbf{C}} X$), where $\Lambda^n\tau$ is the n^{th} exterior power of the tangent bundle τ to X. If X is realized as a domain in $\mathbf{C}^n$, then τ is canonically trivialized, and ω is represented by a holomorphic function ω' on X which verifies

$$(1) \qquad \omega'(x) = j(x, \gamma)^l\cdot\omega'(x\cdot\gamma) , \qquad (x\in X, \gamma\in\Gamma)$$

where j is the functional determinant in the coordinates of the ambient vector space.

By a slight abuse of language, we shall say that ω *is* a P-E series adapted to F_{b}, if it is represented by such a series in the unbounded realization S_{b} associated to $F = F_{\mathrm{b}}$. Let $F^* = F_{\mathrm{b}^*}$ be another standard rational boundary component. We let $\nu = \nu_{F^*F} = \nu_{\mathrm{b}^*}\circ\nu_{\mathrm{b}}^{-1}$, where $\nu_{\mathrm{c}}\colon D\to S_{\mathrm{c}}$ $(1\leqq c\leqq s)$ is the canonical isomorphism (cf. §1). The P-E series E adapted to F is then represented on S_{b^*} by the function E^* given by

$$(2) \qquad E^*\big(\nu(x)\big) = j(x, \nu)^{-l}\cdot E(x) \qquad (x\in S_{\mathrm{b}}) ,$$

where $j(x, \nu)$ is the functional determinant of ν at x. In studying E^*, and more generally $E^*\circ g$ $(g\in G_{\mathbf{Q}})$, we shall use the following notation, where J, J_* stand for J_F, J_{F^*}:

$$(3)\qquad \alpha(s) = J(o, s)^l ,$$

$$(4)\qquad \alpha^*(s) = J_*(o, s)^l ,$$

$$(5)\qquad \beta(s) = \textstyle\sum_{\lambda\in\Gamma_\infty/\Gamma_0}\varphi(\sigma_{\mathbf{b}}\cdot\zeta(s\cdot\lambda))\alpha(s\cdot\lambda) , \qquad (s\in G^0_{\mathbf{R}}) .$$

Thus, β is the function p_f of 7.2 (10), and we have, by 7.1 (4) and 7.2 (9):

$$(6)\qquad (E\circ g)(o\cdot s) = \alpha(s)^{-1}\cdot\textstyle\sum_{\gamma\in\Gamma/\Gamma_\infty}\beta(s\cdot g^{-1}\cdot\gamma) \qquad (s, g\in G^0_{\mathbf{R}}) .$$

More generally

$$(7)\qquad \begin{aligned}&(E^*\circ g)(o, s)\\ &\quad = d^{-1}\cdot\alpha^*(s)^{-1}\cdot\textstyle\sum_{\gamma\in\Gamma/\Gamma_\infty}\beta(s\cdot g^{-1}\cdot\gamma) \qquad (s, g\in G^0_{\mathbf{R}};\ d = j(o, \nu)^l) .\end{aligned}$$

In fact, the obvious equality

$$(8)\qquad j(x, \nu)\cdot J_*(\nu(x), s) = J(x, s)\cdot j(x\cdot s, \nu) , \qquad (s\in G^0_{\mathbf{R}},\ x\in S_{\mathbf{b}})$$

shows that

$$(9)\qquad d\cdot\alpha^*(s) = \alpha(s)\cdot j(o\cdot s, \nu)^l .$$

On the other hand

$$\begin{aligned}(E^*\circ g)(o\cdot s) &= J_*(o\cdot s, g^{-1})^l\cdot E^*(o\cdot s\cdot g^{-1})\\ &= J_*(o\cdot s, g^{-1})^l\cdot j(o\cdot s\cdot g^{-1}, \nu)^{-l}\cdot E(o\cdot s\cdot g^{-1})\\ &= J_*(o, s)^{-l}\cdot J_*(o, s\cdot g^{-1})^l\cdot j(o\cdot s\cdot g^{-1}, \nu)^{-l}\cdot E(o\cdot s\cdot g^{-1}) .\end{aligned}$$

Together with (9), this gives

$$(E^*\circ g)(o\cdot s) = \alpha^*(s)^{-1}\cdot d^{-1}\cdot\alpha(s\cdot g^{-1})\cdot E(o\cdot s\cdot g^{-1}) ,$$

so that (7) follows from (6), and from the definition 7.1 (3) of $E\circ g$.

7.4. LEMMA[4]. *We keep the notation of* 7.3, *fix* b, b^*, l, *put* $\Lambda = ln_b\cdot\chi_b$ *if* $b\geqq 1$, $\Lambda_* = ln_{b^*}\cdot\chi_{b^*}$ *if* $b^*\geqq 1$, *and* $\Lambda = 0$, $\Lambda_* = 0$ *otherwise. Let* $\mathfrak{S}'_{b^*} = \mathfrak{S}^*$ *be an* F_{b^*}*-adapted truncated Siegel domain* (4.12) *in* $P^0_{\mathbf{R}}$. *For* $s\in\mathfrak{S}^*$, *we write* $s = a(s)\cdot v(s)$ $(a(s)\in {}_{\mathbf{Q}}A,\ v(s)\in V_{\mathbf{R}})$. *Then*

(i) $\alpha^*(s)\asymp\alpha^*(a(s))\asymp a(s)^{-\Lambda_*}$, $(s\in\mathfrak{S}^*)$

(ii) $\alpha^*(a(s))\cdot a(s)^\Lambda\asymp 1$, $(b^*\leqq b;\ s\in\mathfrak{S}^*)$

(iii) $\alpha^*(a(s))\cdot a(s)^\Lambda\asymp\prod_{b<i\leqq b^*}a(s)^{-l_i\beta_i}$ $(b<b^*;\ l_i\in\mathbf{Z},\ l_i>0,\ (b<i\leqq b^*))$.

By 3.12 (1), we have

$$\begin{aligned}|\alpha^*(s)| &= |j_{F^*}(\sigma_{\mathbf{b}^*}(o_*), \varpi_{\mathbf{b}^*}(s))|^{l\cdot q_{\mathbf{b}}}\cdot|\chi_{\mathbf{b}^*}(s)|^{-n_{b^*}\cdot l}\\ |\alpha^*(a(s))| &= |j_{F^*}(\sigma_{\mathbf{b}^*}(o_*), \varpi_{\mathbf{b}^*}(a(s)))|^{l\cdot q_{\mathbf{b}}}\cdot|\chi_{\mathbf{b}^*}(a(s))|^{-n_{b^*}\cdot l} .\end{aligned}$$

[4] In this proof, s occurs in two capacities: as the $\mathbf{Q}$-rank of G and as an element of a truncated Siegel domain. We trust this will not cause any confusion.

By the definition of a truncated Siegel domain, $\varpi_{b^*}(\mathfrak{S}^*)$ is a set of elements in $G(F^*)$ which bring the origin into a compact set, and so is relatively compact. Since $v(s)$ varies in a compact set, the set of elements $\varpi_{b^*}(v(s))$ is then also relatively compact. Thus the factors j_{F^*} in the two above equalities are multiplicatively bounded on $\mathfrak{S}^*$. Since $\chi_{b^*}(s) = \chi_{b^*}(a(s))$ for any $s \in N(F^*)$, this proves the first assertion.

If $b^* = 0$, then $\mathfrak{S}^*$ is compact, and the remaining assertions are obvious. So we assume $b^* \geqq 1$.

As in 4.12, we may write, with reference to the index b, an element $a \in {}_{\mathbf{Q}}A$ as $a = a_1 \cdot h \cdot a_2$; the set of all a_1 (resp. h, resp. a_2) which occur in this way form a subgroup A_1 (resp. H, resp. A_2) of ${}_{\mathbf{Q}}A$, we have

$$(1) \qquad \begin{aligned} H &= {}_{\mathbf{Q}}A_{\mathrm{b}}\ , & A_1 \cdot H &= Z(F_{\mathrm{b}}) \cap {}_{\mathbf{Q}}A\ , \\ A_2 &= L(F_{\mathrm{b}}) \cap {}_{\mathbf{Q}}A\ , & A_1 &= \mathfrak{D}Z(F_{\mathrm{b}}) \cap {}_{\mathbf{Q}}A\ , \end{aligned}$$

and the rational character Λ is trivial on $A_1 \cdot A_2$. We have $\beta_i = (\gamma_i - \gamma_{i+1})/2$, $(1 \leqq i < s)$, and $\beta_s = \gamma_s$ (resp. $\beta_s = \gamma_s/2$) in case C_s (resp. BC_s). It is clear that any character χ of ${}_{\mathbf{Q}}A$ trivial on $A_1 \cdot A_2$ is of the form $\chi = m(\gamma_1 + \cdots + \gamma_b)$ with some $m \in \mathbf{Q}$. In particular $\Lambda = m_b(\gamma_1 + \cdots + \gamma_b)$, and $m_b > 0$, if $b \geqq 1$, because the simple $\mathbf{Q}$-roots appear with positive coefficients in $\det \mathrm{Ad}_{\mathfrak{u}} = \chi_b$, where $\mathfrak{u}$ is the Lie algebra of $U(F)$. We want to prove that m_b *is independent of* b for $b \geqq 1$. We have

$$(2) \qquad \Lambda = 2m_b\big(\beta_1 + 2\cdot\beta_2 + \cdots + b(\beta_b + \cdots + \beta_{s-1}) + \nu\cdot b\cdot\beta_s\big)\ ,$$

where $\nu = 1/2$ (resp. 1) in case C_s (resp. BC_s). There is an analogous formula for the restriction of Λ to each irreducible factor ${}^{\sigma}G'$ of G (notation of 3.3 (ii)). Moreover (3.3 (ii)) the first simple $\sigma(k)$-root of ${}^{\sigma}G'$ is the restriction of only one simple $\mathbf{R}$-root, with index $c(1, \sigma)$. Using the fact that ${}_{\mathbf{Q}}A$ is diagonally embedded in $(R_{k/\mathbf{Q}}S')_{\mathbf{R}}$ and applying 1.12 to $G_\sigma = ({}^{\sigma}G')^0_{\mathbf{R}}$ for each σ, we see that

$$m_b = l\cdot\textstyle\sum_\sigma m_{0,\sigma}\cdot c(1, \sigma)\ ,$$

where $m_{0,\sigma}$ is the positive integer associated to G_σ by 1.10, and denoted by m_0 there. This expression is indeed independent of $b \geqq 1$. From now on, we write m for m_b. From (1), we get

$$(3) \qquad a^\Lambda = a^{2m\nu b\beta_s}\cdot\textstyle\prod_{1\leqq i\leqq b} a^{2mi\beta_i}\cdot\prod_{b<i<s} a^{2mb\beta_i} \qquad (b \geqq 1, m > 0, a \in {}_{\mathbf{Q}}A)\ .$$

The roots β_i $(i > b^*)$ are multiplicatively bounded on $a(s)$ $(s \in \mathfrak{S}^*)$. Therefore if $\chi = c_1\beta_1 + \cdots + c_s\beta_s$ $(c_i \in \mathbf{Q})$ is a character on ${}_{\mathbf{Q}}A$, we have

$$(4) \qquad a(s)^\chi \asymp \textstyle\prod_{1\leqq i\leqq b^*} a(s)^{c_i\cdot\beta_i}\ , \qquad (s \in \mathfrak{S}^*)\ .$$

In particular, taking (i) into account:

$$(5)\qquad \alpha^*(a(s)) \asymp \prod_{1\leq i\leq b^*} a(s)^{-2m\cdot i\cdot\beta_i} \qquad (1 \leqq b^* < s)\ ,$$

$$(6)\qquad \alpha^*(a(s)) \asymp a^{-2ms\nu\beta_s}\cdot\prod_{1\leq i<s} a^{-2mi\beta_i}\ , \qquad (s\in\mathfrak{S}^*;\ b^* = s)\ .$$

These relations and (3) yield (ii) and

$$(7)\qquad \alpha^*(a(s))\cdot a(s)^\Lambda \asymp \prod_{b<i\leq b^*} a(s)^{-2m(i-b)\beta_i}\ , \qquad (s\in\mathfrak{S}^*;\ b < b^* < s)\ .$$

$$(8)\qquad \alpha^*(a(s))\cdot a(s)^\Lambda \asymp a^{-2m(s-b)\nu\beta_s}\cdot\prod_{b<i\leq s} a(s)^{-2m(i-b)\beta_i} \qquad (b < b^* = s)\ ,$$

which proves (iii).

7.5. Our main aim in this section is to study the behavior of P-E series near boundary components in X^*. For this purpose, we need to construct a function that will help us to majorize these series in a certain way and to study them termwise.

Let $\rho\colon G \longrightarrow \mathbf{GL}(V)$ be an irreducible representation defined over $\mathbf{Q}$, with highest weight $\Lambda = l\cdot n_b\cdot\chi_b$, where n_b is as in 3.11, such that $V_{\mathbf{Q}}$ contains an element $e_0 \neq 0$ which spans a line stable under $N(F_{\mathrm{b}})_{\mathbf{C}}$, and on which the latter group acts *via* its one dimensional representation Λ. This always exists (2.3). We endow V with a hermitian structure such that K and S are represented by unitary and self-adjoint operators respectively, and such that e_0 has norm one. The function we shall use is defined by

$$(1)\qquad c(g) = \|\rho(g)\cdot e_0\|^{-1} \qquad (g\in G_{\mathbf{R}})\ .$$

Obviously

$$(2)\qquad c(k\cdot g\cdot p) = c(g)\cdot|p^{-\Lambda}| \qquad (p\in N(F))\ .$$

It is also clear that if h varies in a compact set C and $g\in G_{\mathbf{R}}$, then $c(g)\asymp c(h\cdot g)$. In particular, if $\mathfrak{S}'$ is a Siegel domain in the minimal standard parabolic $\mathbf{Q}$-group P, then, using 4.1, we see that

$$(3)\qquad c(a(s)\cdot g) \asymp c(s\cdot g)\ , \qquad (s\in\mathfrak{S}',\ g\in G_{\mathbf{R}})$$

where, as in 7.4, $a(s)$ denotes the component in ${}_{\mathbf{Q}}A$ of s. The main properties of interest to us of the function c are given by the following lemma:

7.6. LEMMA. *We keep the notation of* 7.4, 7.5, *choose* $t > 0$, *and let* a_0 *be the element of* ${}_{\mathbf{Q}}A$ *on which all simple* $\mathbf{Q}$*-roots* β_i *take the value* t. *Then,*

(i) $a^\Lambda\cdot c(a\cdot h) \leqq a_0^\Lambda\cdot c(a_0\cdot h)$ $(h\in G_{\mathbf{R}};\ a\in{}_{\mathbf{Q}}A_t)$.

(ii) $\lim_{a^{\beta_{b^*}}\to 0} a^\Lambda\cdot c(a\cdot g) = 0$ *if* $b^* \geqq b$ *and* $g\notin N(F_{\mathrm{b}^*})\cdot N(F_{\mathrm{b}})$, $(g\in G_{\mathbf{Q}}, a\in{}_{\mathbf{Q}}A_t)$.

We refer to the situation in 7.5. Since ${}_{\mathbf{Q}}A$ is represented by self-adjoint operators in V, the space V is the direct sum of the mutually orthogonal subspaces

$$V_\mu = \{v\in V,\ \rho(a)\cdot v = a^\mu\cdot v,\ (a\in{}_{\mathbf{Q}}A)\}\ ,$$

corresponding to the different $\mathbf{Q}$-weights μ of ρ. The space V_Λ being one-dimensional, spanned by the unit vector e_0, we may write

$$\rho(h)\cdot e_0 = d(h)\cdot e_0 + \sum_\mu f_\mu(h) , \qquad \big(f_\mu(h) \in V_\mu;\ d(h) \in \mathbf{R};\ h \in G_{\mathbf{R}}\big) ,$$

whence

$$c(h)^{-2} = d(h)^2 + \sum_\mu \| f_\mu(h) \|^2 ,$$

and

$$\rho(a\cdot h)\cdot e_0 = a^\Lambda \cdot d(h)\cdot e_0 + \sum_\mu a^\mu \cdot f_\mu(h) , \qquad (a \in {}_{\mathbf{Q}}A,\ h \in G_{\mathbf{R}}) .$$

It is known [14, § 12.14] that every $\mathbf{Q}$-weight is of the form

$$\mu = \Lambda - \sum_{1\leqq i \leqq s} m_i(\mu)\cdot \beta_i \qquad \big(m_i(\mu) \in \mathbf{Z},\ m_i(\mu) \geqq 0\big) ;$$

hence

(1) $$a^{-\Lambda}\cdot \rho(a\cdot h)\cdot e_0 = d(h)\cdot e_0 + \sum_\mu \big(\textstyle\prod_i a^{-m_i(\mu)\cdot\beta_i}\big)\cdot f_\mu(h) \qquad (a \in {}_{\mathbf{Q}}A, h \in G_{\mathbf{R}}) .$$

Using (1) and the definition (7.5) of c, we have

(2) $$| c(a\cdot h)\cdot a^\Lambda |^{-2} = d(h)^2 + \sum_\mu \big(\textstyle\prod_i a^{-2m_i(\mu)\beta_i}\big) \| f_\mu(h) \|^2 .$$

Since the $m_i(\mu)$ are $\geqq 0$, we have

$$a^{-2m_i(\mu)\beta_i} \geqq a_0^{-2m_i(\mu_i)\beta_i} = t^{-2m_i(\mu)} ,$$

for all μ and all i, whence (i).

Let $P = M\cdot S\cdot U$ be the minimal standard parabolic $\mathbf{Q}$-group (2.2) which underlies the definition of the standard boundary components. As recalled in 2.2, the element $g \in G_{\mathbf{Q}}$ may be written as

(3) $$g = u\cdot n_g \cdot u' \qquad \big(u, u' \in U_{\mathbf{Q}};\ n_g \in N(S)_{\mathbf{Q}}\big) ,$$

where n_g is uniquely determined by g. Let w_g be the image of n_g in the relative Weyl group ${}_{\mathbf{Q}}W(G) = N(S)/Z(S)$.

We have $a\cdot g = a\cdot u\cdot a^{-1}\cdot n_g\cdot n_g^{-1}\cdot a\cdot n_g\cdot u'$, and consequently

$$c(a\cdot g) = c(a\cdot u\cdot a^{-1}\cdot n_g)\cdot (n_g^{-1}\cdot a\cdot n_g)^{-\Lambda} .$$

But (4.1), $a\cdot u\cdot a^{-1}$ remains in a relatively compact set, since $a \in {}_{\mathbf{Q}}A_t$, and so the first factor is multiplicatively bounded. Therefore, we are solely concerned with the behavior of

$$a^\Lambda\cdot (n_g^{-1}\cdot a\cdot n_g)^{-\Lambda} = a^\Lambda \cdot a^{-w_g(\Lambda)} .$$

The transform $\nu = w_g(\Lambda)$ of Λ by w_g is a weight of ρ, and therefore has the form

$$\nu = \Lambda - \sum_i m_i(\nu)\beta_i , \qquad \big(m_i(\nu) \geqq 0,\ m_i(\nu) \in \mathbf{Z}\big) .$$

Thus we are reduced to studying the product

$$\prod_i a^{m_i(\nu)\cdot\beta_i} .$$

Since $a \in {}_{\mathbf{Q}}A_t$, each factor is bounded above. Let $\theta(\nu) = \{\beta_i \mid m_i(\nu) > 0\}$. It follows from [14, §12.16] that $\theta(\nu) \cup \Lambda$ is connected; by construction, Λ is orthogonal to all simple $\mathbf{Q}$-roots except β_b. Now, if $g \notin N(F_{\mathbf{b}})$, then $\nu \neq \Lambda$; therefore $\theta(\nu)$ is non-empty, and we must have $m_b(\nu) \neq 0$. It follows that

$$\lim_{a^{\beta_b} \to 0} \prod_i a^{m_i(\nu)\cdot\beta_i} = 0 \qquad (a \in {}_{\mathbf{Q}}A_t)\ ,$$

which proves (ii) if $b^* = b$. Let now $b^* > b$, and assume that the limit is not zero. Then $m_{b^*}(\nu) = 0$, and $\theta(\nu)$ is contained in the set ${}_{\mathbf{Q}}\Delta - \{\beta_{b^*}\}$ of $\mathbf{Q}$-simple roots of $N(F_{\mathbf{b}^*})_{\mathbf{C}}/U(F_{\mathbf{b}^*})_{\mathbf{C}}$. Then, by [14, 12.17], there exist

$$n_1 \in N(S)_{\mathbf{Q}} \cap N(F_{\mathbf{b}^*})\ , \qquad n_2 \in N(S)_{\mathbf{Q}} \cap N(F_{\mathbf{b}})\ ,$$

such that $n_g = n_1 \cdot n_2$. Consequently

$$g = g_1 \cdot g_2\ , \qquad g_1 = u \cdot n_1 \in N(F_{\mathbf{b}^*})_{\mathbf{Q}}\ , \qquad g_2 = n_2 \cdot u' \in N(F_{\mathbf{b}})_{\mathbf{Q}}\ ,$$

which completes the proof of (ii).

7.7. LEMMA. *We keep the notation of* 7.4, 7.5. *Let* $g \in G_{\mathbf{Q}}$.

(i) *There exists a convergent series of positive constant terms which majorizes termwise, in the truncated Siegel domain* $\mathfrak{S}^*$, *the series* (*see* 7.3 (7)):

$$(E^* \circ g)(o \cdot s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} \alpha^*(s)^{-1} \cdot \beta(s \cdot g^{-1} \cdot \gamma)\ ;$$

(ii) *if* $b^* \leqq b$ *and* $g \notin N(F_{\mathbf{b}^*}) \cdot N(F_{\mathbf{b}})$, *then* $\lim_{s^{\beta_{b^*}} \to 0} \alpha^*(s)^{-1} \cdot \beta(s \cdot g) = 0$, $(s \in \mathfrak{S}^*)$;

(iii) *if* $b < b^*$, *then* $\lim_{s^{\beta_{b^*}} \to 0} \alpha^*(s)^{-1} \cdot \beta(s \cdot g) = 0$, $(s \in \mathfrak{S}^*)$.

The function β is equal to p_f, in the notation of 7.2(10), and is majorized by $p_{\|f\|}$. We have already remarked (7.2(12)) that

$$p_{\|f\|}(h \cdot p) \cdot \Delta(p, n_b \cdot l)^{-1} = p_{\|f\|}(h \cdot p) \cdot | p^{-\Lambda} |$$

is bounded when h varies in a compact set of $G_{\mathbf{R}}$ and $p \in P \cap G^0_{\mathbf{R}}$. Since $c(h \cdot p) = c(h) \cdot | p^{-\Lambda} |$ by 7.5 (2), it follows then, as in 6.2, that $p_{\|f\|} \cdot c^{-1}$ is bounded on $G^0_{\mathbf{R}}$. There exists then a constant $\delta > 0$ such that

$$(1) \qquad | \beta(h) | \leqq \delta \cdot c(h)\ , \qquad (h \in G^0_{\mathbf{R}})\ ,$$

and therefore such that

$$(2) \qquad \delta \cdot \sum_{\gamma \in \Gamma/\Gamma_\infty} \alpha^*(s)^{-1} \cdot c(s \cdot h \cdot \gamma)\ ,$$

is a normal majorant of $(E^* \circ h)(s)$ for all $s, h \in G^0_{\mathbf{R}}$, which converges in view of 6.1.

We have $c(s \cdot h) \asymp c(a(s) \cdot h)$ for s in a Siegel domain and $h \in G^0_{\mathbf{R}}$ by 7.5 (3), and $\alpha^*(a(s)) \asymp \alpha^*(s)$ for $s \in \mathfrak{S}^*$, $(s = a(s) \cdot v,\ a(s) \in {}_{\mathbf{Q}}A,\ v \in V_{\mathbf{R}})$ by 7.4. Together with (1), this yields

(3) $$|\alpha^*(s)^{-1}\cdot\beta(s\cdot h)| \prec \cdot\alpha^*(a(s))^{-1}\cdot c(a(s)\cdot h) \qquad (s\in\mathfrak{S}^*,\ h\in G^0_{\mathbf{R}}) .$$

Using 7.4, we get

(4) $$|\alpha^*(s)^{-1}\cdot\beta(s\cdot h)| \prec \left(\prod_{i=b+1}^{b^*} a(s)^{l_i\beta_i}\right)\cdot\left(c(a(s)\cdot h)\cdot a(s)^{\Lambda}\right) ,$$

where the first factor on the right hand side stands for the constant one if $b^* \leqq b$, and the l_i are strictly positive integers if $b^* > b$. Let, as in 7.6, a_0 be the element of ${}_{\mathbf{Q}}A$ on which the simple $\mathbf{Q}$-roots take the value t. Since the first factor on the right hand side of (4) is bounded on $\mathfrak{S}^*$, Lemma 7.6 (i) and (1) imply the existence of a constant $\delta' > 0$ such that

$$c'(h) = \delta'\cdot c(a_0\cdot h)\cdot a_0^{\Lambda} \geqq |\alpha^*(s)^{-1}\cdot\beta(s\cdot h)| \qquad (s\in\mathfrak{S}^*;\ h\in G^0_{\mathbf{R}}) .$$

As a consequence, $E^*\circ g$ is majorized in $\mathfrak{S}^*$ by the series

$$\sum c'(g^{-1}\cdot\gamma) ,$$

which converges, as was noted above (6.1 and 7.5 (2)), whence (i).

In view of (1), (2) it suffices to prove the statements (ii), (iii) with $\alpha^*(s)^{-1}\cdot\beta(s\cdot g)$ replaced by

$$\left(\prod_{i=b+1}^{b^*} a^{l_i\beta_i}\right)\cdot c(a\cdot g)\cdot a^{\Lambda} \qquad (a\in{}_{\mathbf{Q}}A_t) .$$

Let $b^* \leqq b$ and $g \notin N(F_{\mathbf{b}^*})\cdot N(F_{\mathbf{b}})$. Then the first factor is one, and the second one tends to zero as $a^{\beta_{b^*}} \to 0$ by 7.6 (ii), which proves (ii).

Let $b^* > b$. Then $l_{b^*} > 0$ by 7.4; hence, the first factor tends to zero. The second one remains bounded by 7.6 (i), whence (iii).

7.8. THEOREM. *Let E and E^* be as in* 7.1, 7.4. *Let $\mathfrak{S}'_{F^*}$ be a truncated Siegel domain adapted to F^*; let $\varepsilon > 0$ and $g\in G_{\mathbf{Q}}$ be given.*

(i) *Assume $b < b^*$. Then there exists $u_0 > 0$ such that*

$$|E^*\circ g(o\cdot s)| < \varepsilon \qquad \left(s\in\mathfrak{S}'_{F^*},\ a(s)^{\beta_{b^*}} < u_0\right) .$$

(ii) *Assume $b \geqq b^*$. Then there exists $u_0 > 0$ such that*

$$\Big|E^*\circ g(o\cdot s) - j(o,\nu)^l \sum_{\gamma\in g\cdot N(F^*)N(F)\cap\Gamma/\Gamma_\infty} \alpha^*(s)^{-1}\cdot\beta(s\cdot g^{-1}\cdot\gamma)\Big| < \varepsilon$$

for all $s\in\mathfrak{S}'_{F^*}$ satisfying $a(s)^{\beta_{b^*}} < u_0$.

By 7.7 (i) the series $E^*\circ g$ has a constant majorant series in $\mathfrak{S}^*$, so that we may investigate its behavior termwise; 7.7 (ii) and 7.7 (iii) allow us to do this, with the theorem as an immediate consequence.

7.9. PROPOSITION. *We keep the previous notation and assume $b^* \leqq b$. Then the series*

(1) $$E^*\circ g(o\cdot s) = \alpha^*(s)^{-1}\cdot\sum_{\gamma\in(gN(F^*)N(F)\cap\Gamma)/\Gamma_\infty} \beta(s\cdot g^{-1}\cdot\gamma) \qquad (g\in G_{\mathbf{Q}})$$

is an automorphic form on F^*, for the group

$$\Gamma^g(F^*) = \left(Z(F^*)\cap g^{-1}\Gamma g\right)\backslash\left(N(F^*)\cap g^{-}\Gamma g\right) ,$$

lifted to X by means of the canonical projection $\sigma_{\mathfrak{b}^*}\colon X \to F^*$, of type $J_*(x, \ )^l$.

Since $\mathbf{b}^* \leqq \mathbf{b}$ we have a canonical factorization $\sigma = \sigma_{\mathfrak{b}} = \tau \circ \sigma_*$ where $\tau\colon F^* \to F$ is a holomorphic map. In fact τ may be identified with the canonical projection of F^* onto F, the latter being viewed as a standard rational boundary component of F^*.

We show first that $E^* \circ g$ is holomorphic, constant along the fibres of σ_*. By definition

$$\begin{aligned} &E^* \circ g(o \cdot s) \\ &\quad = \alpha^*(s)^{-1} \sum\nolimits_{\gamma \in \Gamma/\Gamma_\infty} \sum\nolimits_{\lambda \in \Gamma_\infty/\Gamma_0} \varphi\big(\sigma(o \cdot s \cdot g^{-1}\gamma \cdot \lambda)\big) \cdot J(o, s \cdot g^{-1} \cdot \gamma \cdot \lambda)^l , \end{aligned}$$

the range of γ being as in (1). For fixed γ, the product of the series on the right hand side by $\alpha(s \cdot g^{-1} \cdot \gamma)^{-1}$ is a Poincaré series for Γ_∞/Γ_0 on F, lifted to X by σ. It is therefore *a fortiori* lifted from a holomorphic function on F^*. In order to finish the proof of our assertion, it suffices to show that

$$\alpha^*(s)^{-1} \cdot \alpha(s \cdot u \cdot v) , \qquad \big(u \in N(F^*),\ v \in N(F)\big)$$

is holomorphic, as a function of $o \cdot s \in X$, and constant along fibres of σ_*. The equality 7.3 (8) yields

$$\begin{aligned} \alpha^*(s)^{-1} \cdot \alpha(s \cdot u \cdot v) &= J_*(o, s)^{-l} \cdot J(o, s \cdot u \cdot v)^l \\ &= j(o, \nu)^l \cdot j(o \cdot s, \nu)^{-1} \cdot J(o, s)^{-l} \cdot J(o, s \cdot u \cdot v)^l \\ &= d \cdot j(o \cdot s, \nu)^{-l} \cdot J(o \cdot s, u \cdot v)^l . \end{aligned}$$

The factors on the right hand side are of course holomorphic in $o \cdot s$; by 3.3 (ii), $j(o \cdot s, \nu)^l$ is constant along the fibres of σ_*. By 1.7, $Z(F^*)^0$ is transitive along the fibers of σ_*. In view of the cocycle identity, it suffices to prove

$$J(x, u) = J(xz, u) \qquad \big(z \in Z(F^*),\ u \in N(F^*)\big) , \tag{2}$$

$$J(xu, v) = J(xzu, v) . \tag{3}$$

We have $zu = uz'$ ($z' \in Z(F^*)$). Since $b^* \leqq b$, $F \subset \bar{F}$, whence $Z(F^*) \subset Z(F)$, so that (2) and (3) follow from 3.3 (ii).

Let now $\gamma_0 \in N(F^*) \cap g^{-1} \cdot \Gamma \cdot g$. Thus $\gamma_0 = g^{-1} \cdot \gamma' \cdot g$ ($\gamma' \in \Gamma$), and it is clear that the series on the right hand side of (1) remains unaltered if s is replaced by $s \cdot \gamma_0$. Moreover

$$\alpha^*(s \cdot \gamma_0)^{-1} = J_*(o, s \cdot \gamma_0)^{-l} = J_*(o, s)^{-l} \cdot J_*(o \cdot s, \gamma_0)^{-l} ;$$

hence,

$$E^* \circ g(o \cdot s) = J_*(o \cdot s, \gamma_0)^l \cdot E^* \circ g(o \cdot s \cdot \gamma_0) ,$$

which ends the proof of the proposition.

8. The operator Φ

8.1. Up to 8.7, we keep the notation and assumptions of 3.3 (ii). As in 4.8, X^* denotes the union of the rational boundary components of X, endowed with the Satake topology. A *good neighborhood* of $x \in X^*$ is one which verifies 4.9 (iv) and 4.10. We let F denote a rational boundary component, and ϖ_F be the canonical epimorphism of $N(F)$ onto $G(F)$.

An automorphic form ω of weight l, for the arithmetic group Γ, will be said to be a P-E series adapted to F if its transform $\omega \circ g$ under some element $g \in G_{\mathbf{Q}}$ which carries F onto the rational boundary component F_{b} is a P-E series adapted to F_{b} for Γ^g, (7.6).

8.2. The functional determinant $J_{\mathrm{b}}(x, g)$ $(g \in N(F_{\mathrm{b}}), x \in X)$ is by 3.3 (ii) constant along the fibres of the projection $\sigma_{\mathrm{b}}\colon X \to F_{\mathrm{b}}$. It defines therefore an automorphy factor on F_{b} for $N(F_{\mathrm{b}})$, hence an action of $N(F_{\mathrm{b}})$ on the trivial line bundle $F_{\mathrm{b}} \times \mathbf{C}$ given by

$$(x, c)\cdot n = \big(x\cdot n,\ J_{\mathrm{b}}(x, n)\cdot c\big) \qquad \big(x \in F_{\mathrm{b}},\ c \in \mathbf{C},\ n \in N(F_{\mathrm{b}})\big)\ .$$

The $N(F_{\mathrm{b}})$-bundle thus defined will be denoted by ξ_{b}. If $F = F_{\mathrm{b}}\cdot g^{-1}$ is as above, the translation by g^{-1} carries ξ_{b} over onto an $N(F)$-bundle denoted by ξ_F. The isomorphism class of ξ_F, viewed as an $N(F)$-bundle, depends only on F.

Let Λ be a discrete subgroup of $N(F)$ whose image Λ' under ϖ_F is discrete, and l an integer. An automorphic form ω for Λ', of type ξ_F^l, is a Λ'-invariant holomorphic cross section of ξ_F^l. The transform $\omega \circ g$ is then an automorphic form of type ξ_{b}^l for $\varpi_{\mathrm{b}}(\Lambda^g)$. It is therefore represented in the canonical bounded realization of F_{b} by a function f which satisfies

$$(1)\qquad f(x\cdot\lambda) = J_{\mathrm{b}}(x, \lambda)^{-l}\cdot f(x)\ , \qquad (x \in F_{\mathrm{b}},\ \lambda \in \Lambda^g)$$

where, by abuse of notation, we write $J_{\mathrm{b}}(x, g)$ for $J_{\mathrm{b}}(x', g)$ if $x \in F_{\mathrm{b}}$, $x' \in \sigma_{\mathrm{b}}^{-1}(x)$, $g \in N(F_{\mathrm{b}})$.

Let $c \leqq b$ and $\nu_{\mathrm{b,c}}\colon S_{\mathrm{c}} \to S_{\mathrm{b}}$ be the canonical isomorphism. We have

$$(2)\qquad j(x, \nu_{\mathrm{b,c}})\cdot J_{\mathrm{b}}\big(\nu_{\mathrm{b,c}}(x), g\big) = J_{\mathrm{c}}(x, g)\cdot j(x\cdot g, \nu_{\mathrm{b,c}}) \qquad (x \in S_{\mathrm{c}},\ g \in G^0_{\mathbf{R}})\ .$$

Let $g \in N(F_{\mathrm{c}})$. Then, using 1.11, we see that $J_{\mathrm{b}}(x, g)$ is constant along the fibres of the projection $\sigma_{\mathrm{c}}\colon S_{\mathrm{c}} \to F_{\mathrm{c}}$, and hence defines an automorphy factor on F_{c}, and an action of $N(F_{\mathrm{c}})$ on the trivial line bundle. However, (2) shows that the automorphy factors given by J_{b} and J_{c} are equivalent, and hence the $N(F_{\mathrm{c}})$-bundle just defined is isomorphic, as an $N(F_{\mathrm{c}})$-bundle, to ξ_{c}.

8.3. Let U be an open subset of X^* which intersects F, and Λ a discrete subgroup of $N(F)$ leaving U invariant. Let ω be an automorphic form of weight l for Λ in U. Let $g \in G_{\mathbf{Q}}$ be such that $F \cdot g = F_{\mathbf{b}}$ for some $\mathbf{b}$. We say that ω extends to $F \cap U$ if $\omega \circ g$ is represented, in the unbounded realization $S_{\mathbf{b}}$, by a function f which extends by continuity to a holomorphic function f' on $(U \cap F) \cdot g$. The extension f' clearly represents an automorphic form σ on $(U \cap F) \cdot g$ of type $\xi^l_{\mathbf{b}}$, for Λ^g, or rather for the image $\varpi_{\mathbf{b}}(\Lambda^g)$ of Λ^g in $G(F_{\mathbf{b}})$. Its transform ω' under g^{-1} is then an automorphic form of type ξ^l_F for $\varpi_F(\Lambda)$, to be called sometimes the extension of ω. This form depends only on ω and F. In fact if $F \cdot g' = F_{\mathbf{b}} (g' \in G_{\mathbf{Q}})$, then $g' = g \cdot n$ $(n \in N(F_{\mathbf{b}})_{\mathbf{Q}})$; hence, $\omega \circ g'$ is represented by $f^*(x) = f(x, n^{-1}) \cdot J(x, n^{-1})^l$. The function f^* extends by continuity to $f^{*\prime} = x \mapsto f'(x, n^{-1}) J_{\mathbf{b}}(x, n^{-1})^l$, which represents $\sigma \circ n$. Its transform under g'^{-1} is then again ω' (cf. Remark of 1.7).

Let $d \geqq b$, and f_d be the function which represents $\omega \circ g$ in the unbounded realisation $S_{\mathbf{d}}$ associated to $F_{\mathbf{d}}$. Then ω extends to $F \cap U$ (where still $F \cdot g = F_{\mathbf{b}}$) if and only if f_d extends by continuity to a holomorphic function f'_d on $(U \cap F) \cdot g$. This follows from the equality $f_d = f \cdot J(\ ,\nu_{\mathbf{d},\mathbf{b}})^l$ and the constancy of $J(\ ,\nu_{\mathbf{d},\mathbf{b}})$ along the fibres of $\sigma_{\mathbf{b}}$. Thus, in order to check that ω extends to $U \cap F$, we may use any unbounded realization $S_{\mathbf{d}}$ $(\mathbf{d} \geqq \mathbf{b})$. Furthermore, the last remark of 8.2 implies easily that $\omega' \circ g$ is represented by $f'_{\mathbf{d}}$ in the trivialization of $\xi^l_{\mathbf{d}}$ which is given by J_d.

8.4. *Local integral automorphic forms.* Let Γ be an arithmetic subgroup of G, and $x \in X^*$. We suppose $x \in F$. An automorphic form ω of weight l for Γ_x on $X \cap N(x)$, where $N(x)$ is a good neighborhood of x, is *integral* if it extends to an automorphic form for $\varpi_F(N(F') \cap \Gamma)$ on $F' \cap N(x)$ for every rational boundary component F' which meets $N(x)$.

If $y \in N(x)$, then $\Gamma_y \subset \Gamma_x$ by 4.9; therefore, the restriction to a good neighborhood $N(y) \subset N(x)$ of y of an integral automorphic form on $N(x)$ is an integral automorphic form, whose extension to $N(y) \cap F'$ is the restriction of the extension of ω to $F' \cap N(x)$.

The form ω is integral on $N(x)$ if and only if for every $g \in G_{\mathbf{Q}}$ such that $F \cdot g = F_{\mathbf{b}}$, the transform $\omega \circ g$ is represented in $S_{\mathbf{b}}$ by a function which extends by continuity to a holomorphic function on $F_{\mathbf{c}} \cap N(x)$ for all $c \leqq b$. In fact, as before, this condition is insensitive to a change of g; moreover, by 3.9, if $F' \cap N(x) \neq \emptyset$, there exists $g \in G_{\mathbf{Q}}$ such that $F' \cdot g = F_{\mathbf{c}}$, $F \cdot g = F_{\mathbf{b}}$ $(c \leqq b)$.

From 3.9 we also deduce that ω is integral if and only if for one $g \in G_{\mathbf{Q}}$ which maps F onto $F_{\mathbf{b}}$, the transform $\omega \circ g$ extends by continuity to $N(x) \cdot g$, and is holomorphic on $(F' \cap N(x)) \cdot g$ for every rational boundary component F'.

8.5. *Integral automorphic forms. The operator* Φ. An automorphic form ω on X of weight l, for Γ, is *integral* if its restriction to $X \cap N(x)$ is integral for every $x \in X^*$ and every good neighborhood $N(x)$ of x. This is the case if and only if for every F and $g \in G_{\mathbf{Q}}$ such that $F \cdot g = F_{\mathbf{b}}$ is standard, the transform $\omega \circ g$ is represented on $S_{\mathbf{b}}$ by a function which extends by continuity to a holomorphic function on $F_{\mathbf{b}}$. Then this function also extends by continuity to a holomorphic function on $F_{\mathbf{c}}$ for every $\mathbf{c} \leqq \mathbf{b}$.

Let ω be integral. It then has an extension to any rational boundary component F, which is an automorphic form for $\Gamma(F)$ of type ξ^l_F, and which will often be denoted by $\Phi_F\omega$. The operator Φ is, by definition, the operator which associates to ω the collection of automorphic forms $\Phi_F\omega$. The definition of the extension of ω given in 8.3 implies that for any rational boundary component F and $g \in G_{\mathbf{Q}}$:

$$(\Phi_{Fg^{-1}}\omega) \circ g = \Phi_F(\omega \circ g) \ . \tag{1}$$

8.6. THEOREM. *Let E be a* P-E *series adapted to the rational boundary component F, for the arithmetic group Γ, of weight l. Then E is an integral automorphic form. Let F^* be a rational boundary component. Then $\Phi_{F^*}E = 0$ if $\dim F^* \leqq \dim F$ and $F^* \not\subset F \cdot \Gamma$. The operator Φ_F maps the module of* P-E *series adapted to F, of weight l, onto the module of Poincaré series for $\Gamma(F)$, of type ξ^l_F.*

If the statement is true for E, F, Γ, then it is also true for $E \circ g, F \cdot g$, and Γ^g $(g \in G_{\mathbf{Q}})$. We may therefore, without loss of generality, assume that $F = F_{\mathbf{b}}$ is a standard rational boundary component.

In order to prove the first assertion, it is enough to show that, for any $x \in X^*$ and $g \in G_{\mathbf{Q}}$ such that $x \cdot g \in F_{\mathbf{c}}$ $(1 \leqq c \leqq s)$, the transform $E \circ g$ is represented on $S_{\mathbf{c}}$ by a function which extends by continuity to a holomorphic function around $x \cdot g$ on $F_{\mathbf{c}}$.

Let E^* be the function which represents E on $S_{\mathbf{c}}$. Then $E \circ g$ is represented by the function defined by $(E^* \circ g)(x) = J_{\mathbf{c}}(x, g^{-1})^l \cdot E^*(x \cdot g^{-1})$. Let $\mathfrak{S}'$ be a Siegel domain in P such that $F_{\mathbf{c}} \cap \bar{\mathfrak{S}}$, where $\mathfrak{S} = o \cdot \mathfrak{S}'$, contains $x \cdot g$ in its interior. This is possible by 4.5. There exists a finite subset C of $N(F_{\mathbf{c}})_{\mathbf{Q}}$, containing the identity, such that $\bar{\Omega} \cdot \Gamma_{x \cdot g}$, where $\Omega = \bigcup_{a \in C} \mathfrak{S}(u, V_a) \cdot a$, runs through a fundamental set of neighborhoods of x in X^* when $u \to 0$ and V_a runs through a fundamental set of neighborhoods of $x \cdot g \cdot a^{-1}$ (see 4.13). We have

$$(E^* \circ g)(s \cdot h) = J_{\mathbf{c}}(s, h)^{-l} \cdot (E^* \circ gh^{-1})(s) \qquad \big(s \in \mathfrak{S}(u, V_a),\ h \in a\Gamma_{x \cdot g},\ a \in C\big) \ .$$

The function $J_{\mathbf{c}}(\ , h)$ is constant along the fibres of $\sigma_{\mathbf{c}}$: $S_{\mathbf{c}} \to F_{\mathbf{c}}$ (3.3 (ii)), and is multiplicatively bounded on $\mathfrak{S}(u, V_a)$ by 4.16. Consequently, we are reduced to showing that $E^* \circ g \cdot h^{-1}$ extends by continuity to a holomorphic function on the interior of $\bar{\mathfrak{S}}(u, V_a) \cap F_c$. Since this series has a normal majorant in $\mathfrak{S}(u, V_a)$ by 7.7, this follows from 7.8, 7.9.

In order to study the limit of $E^* \circ g$ around $x \cdot g$ on F_c, it is enough to consider its behavior on $\mathfrak{S}(u, V)$, where V is a neighborhood of $x \cdot g$. If $c > b$, i.e., if $\dim F^* < \dim F$, this limit is zero by 7.8 (i). Let now $c = b$. Theorem 7.8 (ii) implies that the limit is zero unless $g \cdot N(F_b) \cap \Gamma \neq \emptyset$, i.e., unless $F_b \cdot g^{-1} \subset F_b \cdot \Gamma$. Since

$$\Phi_{F_b}(E \circ g) = (\Phi_{F_b \cdot g^{-1}} E) \circ g \ ,$$

this ends the proof of the second assertion.

Let now $g = e$. Then 7.8 shows that the limit of E on F_b is the limit of the "constant" term $a(s)^{-1} \cdot b(s)$, which is by 7.9 the Poincaré series of type ξ_b^l:

$$\sum\nolimits_{\lambda \in \Gamma_\infty / \Gamma_0} \varphi(\sigma_b(x \cdot \lambda)) J_b(x, \lambda)^l \ ,$$

(see 8.2). Furthermore, it is clear that every such Poincaré series can be obtained in this way by suitable choice of φ in the definition of E.

8.7. The automorphy factor for $N(F_b)$ defined on F_b by J_b is equal in absolute value to $| j_b(\sigma_b(x), g) |^{q_b}$ (cf. 3.12). Therefore, (5.10), it satisfies the condition imposed by H. Cartan [35 Exp. 10 bis; 19 p. 170] so that we may apply Theorems 2, 3 of loc. cit. to the Poincaré series formed by means of J_b on F_b. Since every rational boundary component is the transform of some F_b by an element of G_Q, this implies:

(1) Let F be a rational boundary component of X, $a_1, \cdots, a_q$, q points of F, no two of which are equivalent under $\Gamma(F)$, and t a positive integer. Then there is a positive integer l_0 with the property that, for any multiple l of l_0, there exist a Poincaré series P of type ξ_F^l (see 8.2) which has pre-assigned admissible (i.e., locally invariant, cf. [19, pp. 170-171]) Taylor developments of order t (in suitable local coordinates), at $a_1, \cdots, a_q$.

This means in particular that, for suitable l, we may find P which is not zero at a_1 and zero at a_i ($i \geqq 2$).

8.8. PROPOSITION. *Let F, F' be two rational boundary components such that* $\dim F \geqq \dim F'$, *and that either $F = F'$ or $F \not\subset F' \cdot \Gamma$. Let $x \in F$, $y \in F'$ be not equivalent under Γ. There is an integer l_0, such that if l is a multiple of l_0, there exists an integral automorphic form E which verifies $\Phi_F E(x) \neq 0$,* $\Phi_{F'} E(y) = 0$.

This follows from 8.6 and the result of Cartan mentioned in 8.7 (1).

8.9. We now want to extend 8.8 to the case where G is not necessarily

$\mathbf{Q}$-simple, and where Γ is an arithmetic group of holomorphic automorphisms of X. The notation of 3.3 (i) is used. In particular, X is the product of the spaces $X_i = (K \cap G_{i\mathbf{R}})\backslash G_{i\mathbf{R}}$, the space X^* is the product of the X_i^*, where G_i runs through the $\mathbf{Q}$-simple factors of G, and Γ' is the product of the groups $\Gamma_i = \Gamma \cap G_{i\mathbf{R}}^0$. We do not exclude the possibility that X_i/Γ_i is compact for some i, in which case $X_i = X_i^*$. Let $\mathrm{pr}_i\colon X^* \to X_i^*$ be the natural projection. The notion of an integral automorphic form on X for Γ is defined as in 8.5. If E_i is an integral automorphic form on X_i for Γ_i, of weight l, then the product of the forms $E_i \circ \mathrm{pr}_i$ is an integral automorphic form of weight l on X for Γ'. This shows first that 8.6, 8.8 are valid for Γ'.

Let F, F' be rational boundary components of X, such that $F' \not\subset F\cdot\Gamma$, or $F = F'$, $\dim F' \leqq \dim F$, and let $x \in F$, $y \in F'$, $y \notin x\cdot\Gamma$. Let $\Gamma(F)_x$ be the isotropy group of x in $\Gamma(F) = (N(F) \cap \Gamma)/(Z(F) \cap \Gamma)$, where $N(F)$ and $Z(F)$ are respectively the normalizer and the centralizer of F in $H(X)$. The orbit of x under $N(F) \cap \Gamma$ is the union of finitely many orbits of $N(F) \cap \Gamma'$. By 8.6, 8.8 for Γ', there exists l_0' such that for any multiple l of l_0' we may find a P-E series E' adapted to F of weight l for Γ' verifying

$$(1)\qquad \begin{array}{lll} \Phi_F E'(x) \neq 0\ , & \Phi_F E'(x') = 0 & (x' \in x\cdot\Gamma \cap F;\ x' \notin x\cdot\Gamma' \cap F)\ , \\ \Phi_{F'} E'(y') = 0\ . & & (y' \in y\cdot\Gamma \cap F')\ . \end{array}$$

Assume now that l is also a multiple of the order of $\Gamma(F)_x$, and put

$$E = \sum\nolimits_{\gamma \in \Gamma'\backslash\Gamma} E' \circ \gamma\ .$$

Each summand on the right-hand side is an integral automorphic form of weight l for Γ', hence E is an integral automorphic form of weight l for Γ. We claim that

$$(2)\qquad \Phi_F E(x) \neq 0\ , \qquad \Phi_{F'} E(y) = 0\ ,$$

which will prove our contention, if we take for l_0 some multiple of l_0' and of the order of $\Gamma(F)_x$. We have, for any rational boundary component F^*

$$(3)\qquad (\Phi_{F^*\cdot\gamma^{-1}} E') \circ \gamma = \Phi_{F^*}(E' \circ \gamma)\ ,$$

(see 8.5). Let $F' \not\subset F\cdot\Gamma$ and $F^* = F'$. Then $F^*\cdot\gamma^{-1} \not\subset F\cdot\Gamma'$, the left-hand side of (3) is zero by 8.6, whence the second part of (2) if $F \neq F'$. Let now $F^* = F = F'$. If $\gamma \notin (N(F) \cap \Gamma)\cdot\Gamma'$, then $F\cdot\gamma^{-1} \not\subset F\cdot\Gamma'$, the left-hand side of (3) is zero by 8.6, whence

$$\Phi_F E(z) = \sum\nolimits_{\Gamma'\backslash(N(F)\cap\Gamma)\cdot\Gamma'} \big((\Phi_F E') \circ \gamma\big)(z) \qquad (z \in F)$$

which implies $\Phi_F E(y) = 0$ and, because of the conditions imposed in (1).

$$\Phi_F E(x) = \sum\nolimits_{(N(F)\cap\Gamma')\backslash(N(F)\cap\Gamma)} j_F(x, \gamma)^{-1}\cdot\Phi_F E'(x\cdot\gamma)\ .$$

If $x\cdot\gamma \notin x\cdot\Gamma'$, then the corresponding summand is zero by construction of E'. There remains to consider those terms for which $x\cdot\gamma \in x\cdot\Gamma'$. In that case $\gamma \in (N(F) \cap \Gamma')\cdot\Gamma(F)_x$. Since we sum modulo $N(F) \cap \Gamma'$, we may assume that $\gamma \in \Gamma(F)_x$. Then $j_F(x, \gamma)^l = 1$, since l is a multiple of the order of $\Gamma(F)_x$, and the corresponding summand is equal to $\Phi_F E'(x)$. As a result, $\Phi_F E(x)$ is a non-zero multiple of $\Phi_F E'(x)$, which ends the proof.

8.10. Assume again for convenience, G to be **Q**-simple. An automorphic form of weight l is a *cusp form* if it belongs to the kernel of Φ.

It is known [35; Exp. 10, § 4] that every cusp form whose weight is a multiple of some suitable fixed integer l_0 is a linear combination of Poincaré series; in particular, if X/Γ is compact, every automorphic form of weight ml_0 is a linear combination of Poincaré series. It follows therefore from 8.6, by an obvious induction procedure on $\dim F$, that there exists an integer l_0 with the following property: every automorphic form for Γ, of weight l divisible by l_0, is a linear combination of transforms under elements of $G_{\mathbf{Q}}$ of P-E series for conjugates of Γ under $G_{\mathbf{Q}}$.

III. THE COMPACTIFICATION AS AN ANALYTIC SPACE

9. An analyticity criterion

9.1. In this section, V is a locally compact Hausdorff space, satisfying the second axiom of countability, which is the union of a locally finite countable family of disjoint subspaces $V_0, V_1, \cdots$, each of which is provided with the structure of an irreducible normal analytic space.

An $\mathcal{A}$-function on an open subset U of V is a complex valued continuous function on U whose restriction to $U \cap V_i$ is analytic ($0 \leq i \leq m$). If we associate to U the C-module of $\mathcal{A}$-functions defined on U, we get a presheaf which is easily seen to be a sheaf, the *sheaf $\mathcal{A}$ of germs of $\mathcal{A}$-functions*. The continuous sections of $\mathcal{A}$ over an open subset U of V are the $\mathcal{A}$-functions defined on U. We let $\mathcal{A}_v$ be the stalk of $\mathcal{A}$ at $v \in V$.

9.2. THEOREM. *We keep the notation of* 9.1 *and make the following assumptions*:

(i) *For each positive integer d, the union $V_{(d)}$ of the V_i's whose dimension is $\leq d$ is closed.* $\dim V_0 = \dim V$, $\dim V_i < \dim V_0$ *if* $i \neq 0$ *and* V_0 *is dense in* V.

(ii) *Each point $v \in V$ has a fundamental set of open neighborhoods (U_α) such that $U_\alpha \cap V_0$ is connected for every α.*

(iii) *The restrictions to V_i of local $\mathcal{A}$-functions define the structural sheaf of V_i.*

(iv) *Each point* $v \in V$ *has a neighborhood* U_v *whose points are separated by the* $\mathfrak{A}$*-functions defined on* U.

Then $(V, \mathfrak{A})$ *is an irreducible normal analytic space and for each* $d \leqq \dim V_0$, $V_{(d)}$ *is an analytic subspace of* $(V, \mathfrak{A})$ *with dimension equal to* $\max_{\dim V_i \leqq d} (\dim V_i)$.

The proof of 9.2, will be broken up into several lemmas, and will be concluded at the end of 9.7. We note first that, in view of (i), the subspace V_i is locally closed in $V_{(d)}$ ($d = \dim V_i$), hence is locally closed in V.

We shall use the following remark on normal analytic spaces.

9.3. LEMMA. *Let* Y *be a normal analytic space. Then the ring of analytic functions on* Y *is integrally closed in the ring of complex-valued continuous functions on* Y.

Being normal, Y is the disjoint union of its irreducible components, which are open in Y. We may therefore assume Y to be irreducible.

Let h be a continuous, complex-valued function on Y which satisfies a relation

$$h^n(y) + \sum_{0<i\leqq n} a_i(y)\cdot h^{n-i}(y) = 0 \qquad (y \in Y)\,, \tag{1}$$

where the a_i are analytic functions on Y.

Let $a \in Y$, $\mathcal{O}_a$ be the local ring of Y at a, and K_a be the field of quotients of $\mathcal{O}_a$. Let $P = P(T) = T^n + a_1\cdot T^{n-1} + \cdots + a_n \in \mathcal{O}_a[T]$, where a_i also denotes the germ defined at a by a_i. We assume a to be a regular point. Using the Gauss factorization lemma, and (1), we can find a factor

$$Q = T^m + b_1\cdot T^{m-1} + \cdots + b^m \in \mathcal{O}_a[T]$$

of P which is irreducible in $K_a[T]$, and such that

$$Q_y(h(y)) = h^m(y) + b_1(y)\cdot h^{m-1}(y) + \cdots + b_m(y) = 0 \qquad (y \in U)\,, \tag{2}$$

where U is a sufficiently small neighborhood of a. Here $Q_y \in \mathbf{C}[T]$ denotes the polynomial obtained from Q by replacing b_i by $b_i(y)$, ($y \in U$). By consideration of the resultant of Q and dQ/dT, we see that the set of points $y \in U$, for which Q_y and dQ_y/dT have a common root, is a proper analytic subset Z of U. If $y \notin Z$, then the implicit function theorem and (2) show that h is analytic around y. Hence, h is analytic at a set of points defined locally as the complement of a proper, local analytic subset of U. It is then analytic in U by Riemann's extension theorem [1, 44.42, p. 420]. Since the set of singular points of Y is a proper analytic subset, a further application of the Riemann extension theorem shows that h is analytic on Y.

9.4. LEMMA. *We keep the notation of* 9.1 *and assume* (i), (ii) *of* 9.2. *Then* $\mathfrak{A}_v$ *is integrally closed for every* $v \in V$.

Except for the use of 9.3, the proof is the same as that of the corresponding assertion in [35, Exp. 11, p. 7], and we describe it briefly.

By (ii), v has a fundamental system of neighborhoods U such that $U \cap V_0$ is an irreducible analytic space. Therefore, if f, g are $\mathfrak{A}$-functions on U whose product is identically zero, then one of them must be identically zero on $U \cap V_0$, hence on U by continuity. This shows that $\mathfrak{A}_v$ is integral.

Let now $f, g \in \mathfrak{A}_v$ with g not identically zero and f/g in the integral closure of $\mathfrak{A}_v$. There exists then a relation of the form

$$(1) \qquad (f/g)^n + \sum_{0<i\leqq n} a_i\cdot(f/g)^{n-i} = 0 \qquad (a_i \in \mathfrak{A}_v;\ i = 0, \cdots, n-1)\ .$$

If U is a sufficiently small neighborhood of v, then f, g may be viewed as $\mathfrak{A}$-functions on U, and g is not identically zero on $U \cap V_0$. Since $U \cap V_0$ is normal, there exists then an analytic function h on $U \cap V_0$ such that

$$h(x)\cdot g(x) = f(x) \qquad (x \in U \cap V_0)\ .$$

As in [35, loc. cit.] it follows from (1) and (i) that h extends by continuity to a continuous function on U, which will then verify

$$(2) \qquad h^n(x) + \sum_{0<i\leqq n} a_i(x)\cdot h^{n-i}(x) = 0\ ,$$

for all $x \in U$. By 9.3, the restriction of h to $V_i \cap U$ is then analytic; hence, h is an $\mathfrak{A}$-function on U, and $f/g \in \mathfrak{A}_v$.

9.5. LEMMA. *We keep the notation of* 9.1 *and the assumptions* (i), (iv) *of* 9.2. *Let $v \in V$, U' be an open relatively compact neighborhood of v whose points are separated by $\mathfrak{A}$-functions, and U be a neighborhood of v whose closure is contained in U'. Then there exist finitely many $\mathfrak{A}$-functions on U which separate the points of U.*

Let $f_1, \cdots, f_s$ be a finite set of $\mathfrak{A}$-functions on U'. Define a holomorphic map $f: U' \to \mathbf{C}^s$ by $f(u) = (f_1(u), \cdots, f_s(u))$, and let

$$\varphi = f \times f\colon U' \times U' \to \mathbf{C}^s \times \mathbf{C}^s\ .$$

Let Δ and D be the diagonals of $U' \times U'$ and $\mathbf{C}^s \times \mathbf{C}^s$ respectively. Clearly $\varphi^{-1}(D) \supset \Delta$, and we have $\varphi^{-1}(D) = \Delta$ if and only if f is injective.

$U' \times U'$ is the disjoint union of the locally closed analytic spaces $(U' \cap V_i) \times (U' \cap V_j)$. Similarly $(U' \times U') - \Delta$ is a disjoint union of locally closed subspaces, each endowed with the structure of a separable normal analytic space, namely the complements of the diagonal in the subspaces $(U' \cap V_i) \times (U' \cap V_j)$. Therefore $U' \times U' - \Delta$ may be written as disjoint union of countably many subspaces M_j, each of which is an irreducible analytic space, locally analytically embedded in some V_i. The restriction of φ to M_j is analytic, hence $U \cap \varphi^{-1}(D) \cap M_j$ is an analytic subspace. Let M_{jk}

be its irreducible components, and let

$$\delta_U(f_1, \cdots, f_s) = \max_{j,k} \dim M_{jk}$$

Put $\delta_U(f_1, \cdots, f_s) = -1$ if all the M_{jk} are empty, i.e., if f is injective on U. It is clearly enough to show that if, $\delta_U(f_1, \cdots, f_s) \geqq 0$, then there exists an open neighborhood U'' of U, and finitely many $\mathfrak{A}$-functions $f'_1, \cdots, f'_t$ on U'' such that $\delta_U(f'_1, \cdots, f'_t) < \delta_U(f_1, \cdots, f_s)$.

Let us enumerate the M_{jk} as $Y_1, Y_2, \cdots$, and let $y_i = (u_i, v_i) \in Y_i$ ($i = 1, 2, \cdots$). Then $u_i \neq v_i$, so there exists an $\mathfrak{A}$-function g_i on U' such that $g_i(u_i) \neq g_i(v_i)$ ($i = 1, \cdots$). Define g^*_i on $U' \times U'$ by $g^*_i(x, y) = g_i(x) - g_i(y)$. We may, and shall, assume that $|g_i| \leqq 1/2$ on U', hence that $|g^*_i| \leqq 1$ on $U' \times U'$. Let U'' be an open neighborhood of $\bar{U}$ whose closure is contained in U'. We claim that we may choose constants c_i such that the sequence $g^*_{(m)} = \sum_{1 \leqq i \leqq m} c_i g^*_i$ converges uniformly on $\bar{U}'' \times \bar{U}''$ to a function g^* such that $g^*(u_i, v_i) \neq 0$ ($i = 1, 2, \cdots$). In fact, supposing $c_1, \cdots, c_{m-1}$ chosen in such a way that $g^*_{(i)}(u_i, v_i) \neq 0$ for $i = 1, \cdots, m-1$, we select c_m verifying the following conditions

$$\begin{aligned} &|c_m| \leqq 4^{-m}\,; \qquad g^*_{(m)}(u_m, v_m) \neq 0\,, \\ &|c_m \cdot g^*_m(u_i, v_i)| \leqq 4^{-m} \cdot \mathrm{Min}_{1 \leqq j < m} |g^*_{(j)}(u_i, v_i)| \qquad (1 \leqq i < m)\,. \end{aligned}$$

Then the constants c_i are easily proven to satisfy our condition. In this case, $g = \sum c_i \cdot g_i$ converges uniformly on U'' and is an $\mathfrak{A}$-function on U'' such that $g^*(x, y) = g(x) - g(y)$, $(x, y \in U'')$. This implies that $g(u_i) - g(v_i) = g^*(u_i, v_i) \neq 0$ ($i = 1, \cdots$), hence that

$$\delta_U(f_1, \cdots, f_s, g) < \delta_U(f_1, \cdots, f_s)\,.$$

9.6. LEMMA. *We keep the notation of* 9.1 *and the assumptions* 9.2 (i), 9.2 (ii). *Let U be a relatively compact open neighborhood of $v \in V$, $f_1, \cdots, f_s$ a finite set of $\mathfrak{A}$-functions on U which separate the points of U, and $f\colon u \mapsto (f_1(u), \cdots, f_s(u))$ the associated mapping of U into $\mathbf{C}^s$. Then there exists a relatively compact neighborhood U' of v in U such that f induces a homeomorphism of U' (resp. $U' \cap V_i$, $i = 0, 1, \cdots$) onto an analytic (resp. locally analytic) set in some open domain N of $\mathbf{C}^s$, and that $f(U')$ is locally analytically irreducible at each of its points.*

Let U_1 be an open neighborhood of v such that $\bar{U}_1$ is contained in U. Since f is injective on U, there is an open neighborhood N of $f(v)$ such that $f(\bar{U}_1 - U_1) \cap N$ is empty. Put $U' = f^{-1}(N) \cap U_1$. Let C be compact in N, and $C' = f^{-1}(C) \cap U'$. The set C' is contained in $f^{-1}(C) \cap \bar{U}_1$, which is compact. Let b belong to the closure of C' in $\bar{U}_1$. Then $f(b) \in C \subset N$, so $b \in U_1 \cap f^{-1}(C) \subset U'$; thus $b \in f^{-1}(C) \cap U' = C'$, so C' is compact. Consequently. f is proper on U', and therefore is a homeomorphism of U' onto $f(U') \subset N$. Now, let $V_{i_1}, \cdots, V_{i_r}$ be those V_i of smallest dimension d_0 which meet U'. By 9.2(i), the intersection of each with U' is closed in U' and since f is proper on U', it follows that $f((V_{i_1} \cap \cdots \cap V_{i_r}) \cap U')$ is a closed analytic subset of dimension d_0 of N. Assume now that for some integer $d \geqq d_0$ we have proved that $S = f((V_{(d)}) \cap U')$ is a closed analytic set in N of dimension $\leqq d$. Let V_j be of dimension $d+1$. By [21, Ch. V, C5, p. 162] $f(V_j \cap U')$ is analytic of dimension $d+1$ in $N - S$. Then, by a theorem of Remmert-Stein [21, Ch. V, D5, p. 169] the closure of $f(V_j \cap U)$ in N is an analytic set in N. The fact that $f(U')$ is an analytic set now follows by induction on d. Since f is bijective on U' and since each of its coordinates is an $\mathfrak{A}$-function, it follows that for each $x \in U'$, f induces an injection of the local ring of $f(U')$ at $f(x)$ into $\mathfrak{A}_x$; the latter being an integral domain, we see that $f(U')$ is irreducible at every point.

9.7. LEMMA. *We keep the notation of* 9.1 *and the assumptions of* 9.2. *Let U' be as in* 9.6, *and put $Y = f(U')$. Let $\tilde{Y}$ be the normalization of Y. Then f induces an isomorphism of ringed spaces of $(U', \mathfrak{A}|_{U'})$ onto $\tilde{Y}$.*

Since Y is analytically irreducible (of dimension $d = \dim V_0$) at each point, the canonical projection of $\tilde{Y}$ onto Y is a homeomorphism, and we may identify $\tilde{Y}$ with Y, endowed with the structural sheaf $\tilde{\mathfrak{O}}$ whose stalk at y is the integral closure $\tilde{\mathfrak{O}}_y$ of the local ring $\mathfrak{O}_y$ of Y at y. We have to prove that f induces an isomorphism of $\mathfrak{A}_u$ onto $\tilde{\mathfrak{O}}_{f(u)}$ for every $u \in U'$.

Let first $g \in \tilde{\mathfrak{O}}_{f(u)}$. There is a neighborhood of $f(u)$ in which g defines a continuous function which satisfies an integral dependence relation

$$g^n(x) + \sum\nolimits_{0<i\leqq n} b_i(x)\cdot g^{n-i}(x) = 0 \,, \tag{1}$$

where the b_i are analytic on Y around $f(u)$. The function $g \circ f$ is then continuous around u, and satisfies there a relation similar to (1), with b_i replaced by $a_i = b_i \circ f$. The a_i's are continuous around u. By 9.6, they are $\mathfrak{A}$-functions; hence (9.3), the restriction of $g \circ f$ to V_i around u is analytic. Therefore $g \circ f$ is an $\mathfrak{A}$-function.

It is well-known [8, p. 179] that an analytic homeomorphism of one complex manifold onto another is an isomorphism. Let a be an $\mathfrak{A}$-function at $u \in U'$, i.e., $a \in \mathfrak{A}_u$, and let N be a neighborhood of $f(u)$. If N is chosen small enough, then $a \circ f^{-1}$ is continuous in N and analytic on $f(V_0) \cap N$ (viewed as a subset of $\tilde{Y}$), except possibly at the image points of the singularities of V_0. Hence [1, 44.42, p. 420], $a \circ f^{-1}$ is analytic on all of $f(V_0) \cap N$. By 9.6, $(f(U') - f(U' \cap V_0)) \cap N$ is a proper analytic subset of $\tilde{Y} \cap N$. Hence, $a \circ f^{-1}$ is analytic on $\tilde{Y} \cap N$, so that, finally, f^* is an isomorphism of $\tilde{\mathfrak{O}}_{f(u)}$ onto $\mathfrak{A}_u$.

By 9.5, each point $v \in V$ has a neighborhood U' as in 9.6. Therefore, the first assertion of the theorem follows from 9.7. The assertion about $V_{(d)}$ follows at once from the induction procedure indicated in 9.6.

9.8. COROLLARY. *Let U be an open subset of V, and f a continuous function on U which is analytic on $V_0 \cap U$. Then f is an $\mathcal{A}$-function.*

This follows from the theorem and the Riemann extension theorem.

10. Analytic structure and projective embeddings of the compactification

10.1. We now revert to the set up of 3.3 (i). In particular, G is a connected semi-simple $\mathbf{Q}$-group, with center reduced to $\{e\}$, whose symmetric space $X = K\backslash G_{\mathbf{R}}$ of non-compact type is a bounded domain and $H(X)$ is the group of all holomorphic automorphisms of X, in which $G^0_{\mathbf{R}}$ is of finite index. Moreover, X^* is the union of the rational boundary components of X, endowed with the Satake topology (4.8), Γ an arithmetic group of automorphisms of X, $V^* = X^*/\Gamma$ the compactification of $V = X/\Gamma$ introduced in § 4, and $\pi: X^* \to V^*$ the canonical projection. There are finitely many rational boundary components F_i $(0 \leqq i \leqq m, F_0 = X)$ such that V^* is the disjoint union of the quotients $V_i = F_i/\Gamma(F_i)$. Since $V_i \neq V_j$ if $i \neq j$, we have $F_i \not\subset F_j \cdot \Gamma (i \neq j)$.

10.2. The group $\Gamma(F_i)$ acts in a properly discontinuous fashion on F_i; hence, V_i is canonically endowed with the structure of an irreducible normal analytic space [17]. We are thus in the situation of 9.1 and introduce the sheaf $\mathcal{A}$ of germs of $\mathcal{A}$-functions on V^*. An $\mathcal{A}$-function on an open subset U of V^* is a continuous complex-valued function whose restriction to $V_i \cap U$ is analytic $(0 \leqq i \leqq m)$.

Let $x \in X^*$ and $v = \pi(x)$. Let U be a good neighborhood of x in X^* (8.1). Then $U' = \pi(U)$ may be identified with U/Γ_x, hence $V_i \cap U'$ with $(F_i \cap U)/(\Gamma_x \cap N(F_i))$. The definitions of the analytic structure on V_i and of $\mathcal{A}$ imply that $f: U' \to \mathbf{C}$ is an $\mathcal{A}$-function if and only if $f \circ \pi$ is a continuous function on U, which is invariant under Γ_x, and whose restriction to $F \cap U$ is analytic for every rational boundary component F. In particular, the quotient ω/ω' of two integral automorphic forms ω, ω' for Γ_x on U, of the same weight, where ω' does not take the value zero in U, may be identified with an $\mathcal{A}$-function of U'.

10.3. Let i be the index such that $v \in V_i$. The canonical projection $\sigma_i: X \to F_i$ (1.7, remark) induces an analytic map σ'_i of $(X \cap U)/\Gamma_x = V \cap U'$ onto a neighborhood of v in $V_i \cap U' = (U \cap F_i)/\Gamma_x$. Let j be such that $v \in \bar{V}_j$, and let $w \in V_j \cap U'$. There exists $\gamma \in \Gamma$ such that $F_j \cdot \gamma \cap U \neq \emptyset$ and $\bar{F}_j \cdot \gamma \ni x$.

The canonical projection $\sigma_{F_i, F_j\cdot\gamma}$ induces a holomorphic map of

$$\pi(F_j\cdot\gamma \cap U) = (F_j\cdot\gamma \cap U)/(N(F_j\cdot\gamma) \cap \Gamma_x)$$

onto a neighborhood of v in V_i. We have the factorization (1.7, remark)

$$\sigma_{F_i} = \sigma_{F_i, F_j\cdot\gamma} \circ \sigma_{F_j\cdot\gamma} .$$

Let now f be a holomorphic function around v on V_i. The above remarks imply that $f\circ\sigma'_{F_i}$ extends by continuity to an $\mathfrak{A}$-function near v in V, whose restriction to V_i around v is equal to f, and whose restriction to V_j near w, lifted to $F_j\cdot\gamma$, is equal to

$$f\circ\pi\circ\sigma_{F_i, F_j\cdot\gamma} .$$

10.4. THEOREM. *We keep the assumptions and notation of* 10.1, 10.2. *Then* $(V^*, \mathfrak{A})$ *is an irreducible normal analytic space, in which each* V_i *is embedded as a locally closed analytic space.*

To prove the theorem, it is enough to check that the conditions (i) to (iv) of 9.2, with V and V_0 replaced by V^* and V respectively, hold true in the present situation.

Conditions (i), (ii) and (iii) are consequences of 4.11, 4.15 and 10.3, respectively.

It remains to check the separation of points by $\mathfrak{A}$-functions. Let $v \in V_i$ and $x \in F_i$ be such that $\pi(x) = v$. By 8.8, 8.9 there exists an integral automorphic form E, of some weight l, such that $\Phi_{F_i}E(x) \neq 0$. Let U be a good neighborhood of x in X^*, on which the extension of E does not take the value zero, and let $U' = \pi(U)$. Let $p', q' \in U'$ and j, k be the indices such that $p' \in V_j$, $q' \in V_k$. Let $p \in F_j \cap \pi^{-1}(p')$, $q \in F_k \cap \pi^{-1}(q')$. Assume $\dim F_j \geqq \dim F_k$. Since, by construction we have either $j = k$, or $F_j \not\subset F_k\cdot\Gamma$, there exists (8.8, 8.9) a multiple $l' = l\cdot m$ of l, and an integral automorphic form E' of weight l' for Γ, such that

$$\Phi_{F_j}E'(p) \neq 0 , \qquad \Phi_{F_k}E'(q) = 0 .$$

The quotient E'/E^m is then an $\mathfrak{A}$-function on U' which separates p' from q'. Thus 9.2 (iv) also holds true in V^*.

10.5. COROLLARY. *Assume that* G *has no normal* $\mathbf{Q}$*-subgroup of dimension* 3. *Let* U *be open in* V^*. *Then every meromorphic function on* $U \cap V$ *is the restriction of a meromorphic function on* U. *In particular, the restriction to* V *yields an isomorphism of the field of meromorphic functions on* V^* *onto the field of meromorphic functions on* V.

The assumption on G implies, by 3.15, that $\dim (V^* - V) \leqq \dim V^* - 2$.

Therefore 10.5 follows from 10.4 and a well-known extension theorem on normal analytic spaces.

We recall the map $f \mapsto \pi \circ f$ identifies the meromorphic functions on V with the Γ-invariant meromorphic functions on X, i.e., with the automorphic functions for Γ.

10.6. Lemma. *We keep the notation of* 10.1, 10.2. *There exist a weight l and finitely many integral automorphic forms $E_0, \cdots, E_N$ of weight l such that the forms $\Phi_{F_i} \cdot E_j$ are nowhere simultaneously zero* $(0 \leqq i \leqq m)$.

Given $x \in F_i$, there exists a weight l_x, and an integral automorphic form E_x of weight l such that $\Phi_{F_i} E_x(x) \neq 0$, (8.8, 8.9). There is then a good neighborhood $N(x)$ of x such that the extension of E to $N(x)$ is nowhere zero. By compactness, V is covered by the images of finitely many such neighborhoods $N(x_j)$. The lemma follows then by taking for l the l.c.m. of the l_{x_j} and for E_i's suitable powers of the E_{x_j}.

10.7. Let E_j $(0 \leqq j \leqq N)$ be as in 10.6. If we trivialize the bundle $\xi^l_{F_i}$, the forms $\Phi_{F_i} E_j$ are identified with holomorphic functions which are nowhere simultaneously zero; their values at $x \in F_j$ are the coordinates of a point in $\mathbf{C}^{N+1} - 0$. If we change the trivialization, these coordinates are all multiplied by the same non-zero constant, hence define the same point in the associated projective space $\mathbf{P}(N, \mathbf{C})$. Thus, to $x \in F$ there is associated a well-defined point in $\mathbf{P}(N, \mathbf{C})$, whose homogeneous coordinates will be denoted by $\Phi_{F_i} E_j(x)$. Since the E_j are automorphic forms of the same weight, two points x and $x \cdot \gamma (\gamma \in \Gamma)$ will have the same image in $\mathbf{P}(N, \mathbf{C})$, whence a map $f\colon V^* \to \mathbf{P}(N, \mathbf{C})$ defined by

$$f(\pi(x)) = (\Phi_{F_i} E_0(x), \cdots, \Phi_{F_i} E_N(x)) \qquad (x \in F_i,\ i = 0, \cdots, m)\ .$$

Since the quotient of two integral automorphic forms is an $\mathcal{O}$-function outside the set of zeros of the denominator, f is a holomorphic mapping.

10.8. Lemma. *We keep the notation of* 10.1, 10.2. *There exist a weight l and finitely many integral automorphic forms $E_0, \cdots, E_N$ of weight l for Γ, satisfying* 10.5, *such that the map $f\colon V^* \to \mathbf{P}(N, \mathbf{C})$ associated to the E_i's is a homeomorphism of V^* onto $f(V^*)$.*

The proof is essentially the same as in the symplectic case [2], and will be described briefly. It is enough to show that for suitable E_i's the map f is injective.

Let D and Δ be the diagonals in $V^* \times V^*$ and $\mathbf{P}(N, \mathbf{C}) \times \mathbf{P}(N, \mathbf{C})$ respectively, and $S = (f \times f)^{-1}(\Delta)$. Then S is an analytic subset of $V^* \times V^*$ containing D, which is equal to D if and only if f is injective. Since, in a compact analytic space, a decreasing sequence of analytic subsets is stationary, it is enough to show that if $S \neq D$, then there exists a similar map $f'\colon V^* \to \mathbf{P}(N', \mathrm{C})$ associated to integral automorphic forms of some weight l' for which $S' = (f' \times f')^{-1}(\Delta) \subsetneqq S$.

Let $x \in F_i$, $y \in F_j$ ($\dim F_i \geqq \dim F_j$), $x' = \pi(x)$, $y' = \pi(y)$, be such that $(x', y') \in S - D$. Then $x \notin y \cdot \Gamma$; by 8.8, 8.9 there exists a multiple l' of l and an integral automorphic form E of weight l such that $\Phi_{F_i}E(x) \neq 0$, $\Phi_{F_j}E(y) = 0$. We then take as E_j''s all the monomials of degree l'/l in the E_j's, and E. We have $S' \subset S$, and $(x, y) \notin S'$, hence $S' \neq S$.

10.9. Let $\mathfrak{A}(\Gamma)$ be the graded ring of automorphic forms of positive weight for Γ on X. It may be identified with the set of invariants of Γ in a ring $B = \sum_{i \geqq 0} B_i$ of holomorphic functions, on which $H(X)$ operates by

$$(f \circ g^{-1})(x) = J(x, g)^i \cdot f(x \cdot g) \qquad (f \in B_i;\ x \in X)\ ,$$

where J is the functional determinant in some realization of X as a domain in euclidean space. Since X is connected, it follows that $\mathfrak{A}(\Gamma)$ is integrally closed [35, Exp. 17, No. 5]. We claim that the subring $\mathfrak{A}'(\Gamma)$ of integral automorphic forms is also integrally closed. Since $\mathfrak{A}(\Gamma)$ is, this amounts to showing that, if h is an automorphic form of weight l which verifies an integral dependence relation

$$(1) \qquad h^n + \sum_{0 < i \leqq n} a_i(x) \cdot h^{n-i} = 0\ ,$$

where a_i is an integral automorphic form of weight $l \cdot i$, then h is integral. Let $x \in X^*$ and U be a good neighborhood of x. We may identify h and the a_i's with holomorphic functions on $X \cap U$. Moreover the a_i's extend by continuity to continuous functions whose restrictions to $F \cap U$ are holomorphic for any rational boundary component F. The relation (1) and the condition 9.2 (ii) imply again, as in 9.4, that h extends by continuity to a continuous function on U. If follows then from 9.3 that h is analytic on $F \cap U$ for every F. Thus h is integral.

10.10. Let $(E_i)_{0 \leqq i \leqq N}$ be a set of integral automorphic forms verifying 10.8, A be the subring of $A'(\Gamma)$ generated by the E_i's, and $\widetilde{A}$ its integral closure. The latter is a finitely generated algebra over C [15, Ch. 5, § 3, No. 2] and is contained in $A'(\Gamma)$ by 10.9. It is elementary that there exists an integer d such that the subring $\widetilde{A}^{(d)}$ of elements in $\widetilde{A}$ whose degree is a multiple of d is generated by $\widetilde{A}_d$ [15, Ch. 3, § 1, No. 3, Prop. 3]. Moreover, $\widetilde{A}^{(d)}$ is also integrally closed [15, Ch. 5, § 1, No. 8, Cor. 3]. Therefore $\widetilde{A}^{(d)}$ is a normally projective algebra over $\mathbf{C}$, in the sense of [35, Exp. 17]. Let E_i $(0 \leqq i \leqq M)$ be a basis of $\widetilde{A}_d$. Then $\widetilde{A}^{(d)} \cong C[T_0, \cdots, T_M]/I$, where I is the ideal of the relations between the E_i. The projective variety $V(\widetilde{A}^{(d)}) \subset \mathbf{P}(M, \mathbf{C})$ defined by I is then normally projective. The map f: $V^* \longrightarrow \mathbf{P}(M, \mathbf{C})$ associated to the E_i's is well-defined, injective; its image is an analytic, hence algebraic, variety, contained in $V(\widetilde{A}^{(d)})$. It is in fact equal to $V(\widetilde{A}^{(d)})$ since otherwise there would exist a polynomial $P \in \mathbf{C}[T_0, \cdots, T_M]$, not contained in I, such that $P(E_0, \cdots, E_M)$ would be identically zero on X, in contradiction with the definition of I. Thus f is a bijective holomorphic map of V^* onto $V(\widetilde{A}^{(d)})$. Since both V^* and $V(\widetilde{A}^{(d)})$ are normal analytic spaces, f is an isomorphism of analytic spaces. Thus we have proved the following:

10.11. THEOREM. *We keep the notation of* 10.1, 10.2. *There exist a weight l and finitely many integral automorphic forms E_i of weight l whose extensions to X^* are nowhere simultaneously zero, such that the associated map f: $V^* \longrightarrow \mathbf{P}(N, \mathbf{C})$ is an isomorphism of V^* onto a normally projective subvariety of* $\mathbf{P}(N, \mathbf{C})$.

10.12. COROLLARY. *Assume that G has no normal $\mathbf{Q}$-subgroup of dimension* 3. *Then the field of automorphic functions for Γ is canonically isomorphic with the field of rational functions on $f(V^*)$. In particular, it is an algebraic function field of transcendence degree equal to* $\dim_{\mathbf{C}} X$. *Every automorphic function is the quotient of two integral automorphic forms of the same weight.*

This follows from 10.5, and from the fact that a meromorphic function on a projective variety is rational by Chow's theorem.

10.13. Let ρ: $K^0 \longrightarrow \mathbf{GL}(E)$ be a finite dimensional unitary representation of K^0. It defines on X a complex vector bundle ξ_ρ, the bundle associated by ρ to $G^0_{\mathbf{R}}$, viewed as principal K^0-bundle by left translations. The total space is therefore the quotient $G^0_{\mathbf{R}} \times_{K^0} E$ of $G^0_{\mathbf{R}} \times E$ by the equivalence relation

$$(g, v) \approx (k \cdot g, \rho(k) \cdot v) \qquad (k \in K^0,\ g \in G^0_{\mathbf{R}},\ v \in E).$$

It can also be written as $P^- \cdot K^0_{\mathbf{C}} \cdot G^0_{\mathbf{R}} \times_{P^- \cdot K^0_{\mathbf{C}}} E$, where ρ is extended in the obvious fashion to a representation of $P^- \cdot K^0_{\mathbf{C}}$ which is trivial on P^-; hence, it is a holomorphic vector bundle. An automorphic form of type ξ_ρ is a Γ-invariant, holomorphic cross-section of ξ_ρ. These forms correspond in a canonical fashion to the holomorphic V-valued functions on X which satisfy the relation

$$f(x \cdot \gamma) = \mu_\rho(x, \gamma)^{-1} \cdot f(x) \qquad (x \in X, \gamma \in \Gamma)\ ,$$

where μ_ρ is the automorphy factor introduced in 5.6. We let $\mathcal{A}_\rho$ be the sheaf of germs of automorphic functions of type ξ_ρ, for Γ on X/Γ. It is reflexive, torsionless, and is known to be an analytic coherent sheaf [34, Exp. XX].

REMARK. We have tacitly assumed that Γ operates on ξ_ρ. This is certainly the case if $\Gamma \subset G^0_{\mathbf{R}}$. Otherwise we assume that ρ extends to a subgroup K' of finite index of $K \cap H(X)$ such that $\Gamma \subset K' \cdot G^0_{\mathbf{R}}$. Replacing K^0 and $K^0_{\mathbf{C}}$ by K' and $K'_{\mathbf{C}}$ respectively in the above construction, we see easily that the action of $G^0_{\mathbf{R}}$ extends to one of $K'G^0_{\mathbf{R}}$.

10.14. THEOREM. *Assume that G has no normal* $\mathbf{Q}$*-subgroup of dimension* 3, *and let $\mathfrak{A}_\rho$ be as in* 10.13. *Then the direct image $i_*\mathfrak{A}_\rho$ in V^* of the sheaf of germs of automorphic forms of type ξ_ρ is an algebraic coherent sheaf. In particular, if U is an open subset of V^*, every holomorphic section of $\mathfrak{A}_\rho$ over $U \cap V$ extends to a holomorphic section over U. The space of automorphic forms of type ξ_ρ is canonically isomorphic to the space of holomorphic cross-sections of $i_*\mathfrak{A}_\rho$ over V^*, and is finite dimensional. The ring of automorphic forms of positive weight is finitely generated.*

We identify V^* with its image under the map of 10.11. Then the restriction to V of the line bundle $\mathcal{O}$ of $\mathbf{P}(N, \mathbf{C})$ attached to the divisor of a hyperplane is the sheaf $\mathfrak{A}_d$ of germs of automorphic forms of weight d. We know (5.11) that if m is large enough, the product $J^{d\cdot m}\cdot\mu_\rho$ is an automorphy factor which satisfies the condition allowing one to construct Poincaré series. Therefore, Theorem 3 of [19] applies. It shows that given $x \in X/\Gamma$, there exist finitely many analytic cross-sections of the sheaf $\mathfrak{A}_\rho \otimes \mathcal{O}^n$ which generate the fibre of $\mathfrak{A}_\rho$ at x. Since $V^* - V$ has codimension $\geqq 2$ (3.15), Serre's extension theorem [36] applies, and yields the theorem, except for the last assertion. We now know that $A(\Gamma) = A'(\Gamma)$. Let l and E_i be as in 10.11. The automorphic forms of weight $m \cdot l$ (m a positive integer) may be identified with the holomorphic cross-sections of $\mathcal{O}^m$. They are therefore the polynomials of degree m in the E_i's. This means that the algebra $A(\Gamma)^{(l)}$ is generated by the E_i's. Each space $A(\Gamma)_i$ is finite dimensional. Therefore, in order to establish the second assertion, it suffices to show the existence of an integer n_0 such that

$$A(\Gamma)_l \cdot A(\Gamma)_{p+n\cdot l} = A(\Gamma)_{p+(n+1)\cdot l} \qquad (n \geqq n_0,\ p \geqq n_0)\ . \tag{1}$$

Since the sheaves involved extend to algebraic coherent sheaves on a projective variety, the proof of (1) given by Serre [34, XX, nos. 9, 10] when X/Γ is compact applies without change to our case.

APPENDIX

11. Connected components of automorphism groups

In this section, we collect some partly known remarks on connected components, whose use in the preceding sections has allowed for some slight technical simplification. X is a bounded symmetric domain, $H(X)$ the group of complex analytic homeomorphisms (i.e., of automorphisms) of X, and $\mathrm{Is}(X)$ the group of isometries of X with respect to the riemannian structure defined by the Bergman metric.

11.1. As is well-known, $\mathrm{Is}(X)$ and $H(X)$ are semi-simple Lie groups, with finitely many connected components, X is the quotient of $\mathrm{Is}(X)$ by a maximal compact subgroup K, and $\mathrm{Is}(X)^0 = H(X)^0$ is a non-compact semi-simple Lie group with center reduced to $\{e\}$. Thus $\mathrm{Is}(X)^0 = \mathrm{Ad}\,\mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of $\mathrm{Is}(X)$. Furthermore, it is known that $\mathrm{Is}(X) \cong \mathrm{Aut}\,\mathfrak{g}$.

Assume now X to be irreducible. Then *its isometries are either holomorphic or anti-holomorphic, and $H(X)$ has index two in* $\mathrm{Is}(X)$. In fact, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ associated to the Lie algebra $\mathfrak{k}$ of K. Then $\mathrm{Is}(X) = K \cdot P$ $(P = \exp \mathfrak{p})$ and $H(X) = (K \cap H(X)) \cdot P$. The identity component S of the center of the identity component of K is one-dimensional, and an automorphism $\mathrm{Int}\,k$ $(k \in K)$ is either the identity or the inversion $s \mapsto s^{-1}$ on S. On the other hand, the multiplication by $\sqrt{-1}$ in the tangent space X_0 of X at K is induced by $\mathrm{Ad}\,s_0$, where s_0 is an element of order 4 in S. Therefore $\mathrm{Int}\,k(s_0)$ is equal either to s_0 or to s_0^{-1}. The transformation k is holomorphic in the first case, anti-holomorphic in the second one. In particular

$$H(X) \cap K = Z(S) , \tag{1}$$

and $H(X)$ has index $\leqq 2$ in $\mathrm{Is}(X)$. On the other hand, there is clearly a linear orthogonal transformation A on X_0 which carries the given complex structure onto its conjugate. By standard facts on simply connected riemannian symmetric spaces, A extends to an isometry of X, which is then anti-holomorphic, hence $\mathrm{Is}(X) \neq H(X)$.

If X is the product of r irreducible components, it is clear that $\mathrm{Is}(X)$ (resp. $H(X)$) is generated by products of isometries (resp. automorphisms) of the different factors, and permutations of isomorphic factors.

These remarks have already been made by E. Cartan, who has also given the structure of $H(X)$ in all irreducible cases; it is connected, except in the cases mentioned in 11.4, where $H(X)^0$ has index two in $H(X)$, [16, p. 152].

11.2. Proposition. *Let $\bar{D}$ be the natural compactification of X (1.4). The action of $H(X)$ on X extends by continuity to a continuous action on $\bar{D}$, and the restriction of $h \in H(X)$ to any boundary component is holomorphic.*

It suffices to prove this for $K \cap H(X)$ since $H(X)$ is generated by this group and by $H(X)^0$. But, if $k \in K \cap H(X)$, then k commutes with the element s_0 considered in 11.1, and the extension to $\mathfrak{p}_{\mathbf{C}}$ of $\mathrm{Ad}_{\mathfrak{p}} k$ leaves the two subspaces $\mathfrak{p}^+$, $\mathfrak{p}^-$ stable; hence, the action of k on X extends to a linear transformation of $\mathfrak{p}^+$. Therefore k operates continuously on $\bar{D}$. Furthermore, the boundary components are open subsets of complex affine subspaces of $\mathfrak{p}^+$; hence, the restriction of k to such a component is holomorphic.

11.3. PROPOSITION. *Let G be a connected simple algebraic group defined over $\mathbf{R}$ such that the symmetric space of non-compact type X of $G_{\mathbf{R}}$ is a bounded symmetric domain. Then $H(X) \cap G_{\mathbf{R}} = G^0_{\mathbf{R}}$. The group $G_{\mathbf{R}}$ has either one or two connected components.*

Since G is simple, X is irreducible, and we may identify $G_{\mathbf{R}}$ with a subgroup of $\mathrm{Aut}\, \mathfrak{g}_{\mathbf{R}}$, namely $\mathrm{Ad}\, \mathfrak{g}_{\mathbf{C}} \cap \mathrm{Aut}\, \mathfrak{g}_{\mathbf{R}}$. We keep the notation of 11.1; in particular, $K \cap G_{\mathbf{R}}$ is a maximal compact subgroup of $G_{\mathbf{R}}$, and $G_{\mathbf{R}} = (K \cap G_{\mathbf{R}}) \cdot P$.

The second assertion follows from the first one and 11.1; in view of 11.1 (1), the first assertion is equivalent to: $G_{\mathbf{R}} \cap Z(S)$ is connected, which we now prove.

Being the centralizer of a torus in $G_{\mathbf{C}}$, the group $Z(S)_{\mathbf{C}}$ is connected, since G is (cf. [9, § 18]). It is defined over $\mathbf{R}$, and its Lie algebra is $\mathfrak{k}_{\mathbf{C}}$. Therefore $\mathfrak{k}$ is a compact real form of $\mathfrak{k}_{\mathbf{C}}$, and K^0 is the identity component of a maximal compact subgroup L of $Z(S)_{\mathbf{C}}$. Since $Z(S)_{\mathbf{C}}$ is connected, so must be L, whence $Z(S) \cap K = K^0$.

11.4. REMARK. In the type IV_n (IV refers to Siegel's notation; it is III in [16]) of bounded symmetric domains, $G = \mathbf{PSO}(n + 2, \mathbf{C})$ is the quotient of the special orthogonal group in $n + 2$ variables by its center, $G_{\mathbf{R}} = \mathbf{PSO}(n, 2)$, and $K \cap G_{\mathbf{R}}$ is the group of elements of determinant one in $\mathbf{O}(n) \times \mathbf{O}(2)$ (divided by $\{\pm 1\}$ if n is even). From this we see readily that

(a) if n is odd, $G_{\mathbf{R}} = \mathrm{Is}\,(X)$, $G^0_{\mathbf{R}} = H(X)$, and

(b) if n is even, $\mathrm{Is}\,(X)/\mathrm{Is}\,(X)^0 = \mathbf{Z}_2 + \mathbf{Z}_2$, $G_{\mathbf{R}}/G^0_{\mathbf{R}} = \mathbf{Z}_2$, $H(X)/H(X)^0 = \mathbf{Z}_2$.

The situation (b) also occurs for type $I_{n,n}$ $(n \geqq 2)$.

11.5. LEMMA. *Let G be a connected semi-simple group defined over $\mathbf{R}$. Assume that $G^0_{\mathbf{R}}$ has a center reduced to $\{e\}$, and has the same rank as its maximal compact subgroups. Then the center of $G_{\mathbf{C}}$ is reduced to $\{e\}$.*

Let $\mathfrak{t}$ be a Cartan subalgebra of a maximal compact subgroup K of $G_{\mathbf{R}}$. The assumption implies that $\mathfrak{t}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}$. It is the Lie algebra of a maximal torus $T_{\mathbf{C}}$ of $G_{\mathbf{C}}$, which is defined over $\mathbf{R}$, and whose subgroup of real points is compact. The latter is then necessarily a maximal compact subgroup of $T_{\mathbf{C}}$, and is connected and equal to $\exp \mathfrak{t}$. Let now z be in the center of $G_{\mathbf{C}}$. It belongs to $T_{\mathbf{C}}$ and is of finite order; hence, z and $\exp \mathfrak{t}$ generate a compact subgroup of $T_{\mathbf{C}}$. Thus $z \in \exp \mathfrak{t} \cap G^0_{\mathbf{R}}$, and $z = e$.

REMARK. If we drop the assumption on the rank of K, the lemma becomes false as is shown by the case where $G_{\mathbf{R}} = \mathbf{SO}(p, q)$, ($p$, q odd).

11.6. We now revert to the notation of 1.3, 1.5, and prove that $G(F)$ is connected, as asserted in 1.5. An obvious reduction shows that it suffices to do this when X is irreducible. In view of 1.5 (1), (2) this amounts to proving that the group P_b/Z_b of 1.3 is connected.

The group $Q_{\mathbf{C}} = P_{\mathbf{C}}/Z_{b,\mathbf{C}}$ is almost simple, connected, defined over $\mathbf{R}$, and $Q^0_{\mathbf{R}} = \mathrm{Ad}\, l_b$. On the other hand, the symmetric space F_b of non-compact type of L_b is a bounded symmetric domain; hence, L_b has the same rank as its maximal compact subgroups. By 11.5, we have then $Q_{\mathbf{C}} = \mathrm{Ad}\,\mathfrak{q}_{\mathbf{C}}$, which implies that $Q_{\mathbf{R}}$ is a *subgroup* of $\mathrm{Is}\,(F_b)$. By 11.4, all elements of P_b induce complex analytic homeomorphisms of F_b; therefore (11.3), the image of P_b in $Q_{\mathbf{R}}$ is connected, equal to $\mathrm{Ad}\,\mathfrak{l}_b$. The kernel of the homomorphism $P_b \to Q_{\mathbf{R}}$ is $P_b \cap Z_{b,\mathbf{C}}$. This is a normal subgroup of P_b, with Lie algebra $\mathfrak{z}_b$, which contains Z_b. It is therefore equal to Z_b (see 1.3), whence the result.

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