Balanced line bundles

Rational points on algebraic varieties

Let X be a smooth projective variety over a number field F.

Hope

Global geometric properties are reflected in arithmetic properties of X.

Rational points on algebraic varieties

Let X be a smooth projective variety over a number field F.

Hope

Global geometric properties are reflected in arithmetic properties of X.

For example:

- First order: Abundance of rational points should be related to abundance of rational curves.

Rational points on algebraic varieties

Let X be a smooth projective variety over a number field F.

Hope

Global geometric properties are reflected in arithmetic properties of X.

For example:

- First order: Abundance of rational points should be related to abundance of rational curves.
- Higher order: Spaces of rational curves

$$\{\mathbb{P}^1 \to X\}$$

exhibit uniform behavior, their geometry carries arithmetic information.

Standard conjectures

- Potential density: there exists a finite extension F of the ground field such that $X(F)$ is Zariski dense.

Standard conjectures

- Potential density: there exists a finite extension F of the ground field such that $X(F)$ is Zariski dense.
- Asymptotic formulas: Let $\mathcal{L} = (L, \|\cdot\|)$ be an ample, adelically metrized, line bundle on X and $\mathrm{H}_{\mathcal{L}}$ the associated height. Then there exists a Zariski open $X^\circ \subset X$ such that

$$\#\{x \in X^\circ(F) \mid \mathrm{H}_{\mathcal{L}}(x) \leq \mathrm{B}\} \sim c(X^\circ, \mathcal{L})\mathrm{B}^{a(X,L)} \log(\mathrm{B})^{b(X,L)-1},$$

as $\mathrm{B} \to \infty$.

Standard conjectures

- Potential density: there exists a finite extension F of the ground field such that $X(F)$ is Zariski dense.
- Asymptotic formulas: Let $\mathcal{L} = (L, \|\cdot\|)$ be an ample, adelically metrized, line bundle on X and $\mathrm{H}_{\mathcal{L}}$ the associated height. Then there exists a Zariski open $X^\circ \subset X$ such that

$$\#\{x \in X^\circ(F) \mid \mathrm{H}_{\mathcal{L}}(x) \leq \mathrm{B}\} \sim c(X^\circ, \mathcal{L}) \mathrm{B}^{a(X,L)} \log(\mathrm{B})^{b(X,L)-1},$$

as $\mathrm{B} \to \infty$. Here $a(X, L)$ and $b(X, L)$ are certain *geometric* constants and $c(X^\circ, \mathcal{L})$ is a Tamagawa-type number.

Counting problems

Depend on:

- an adelic metrization of L, i.e., projective embedding

$$X \hookrightarrow \mathbb{P}^n$$

  and a choice of a height on $\mathbb{P}^n(F)$;
- a choice of $X^\circ \subset X$.

Counting problems

Depend on:

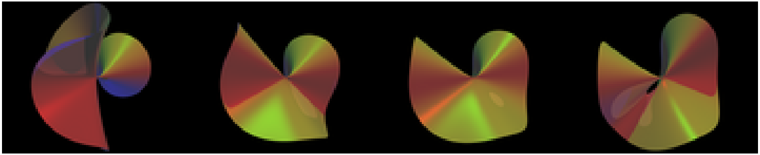
- an adelic metrization of L, i.e., projective embedding

$$X \hookrightarrow \mathbb{P}^n$$

  and a choice of a height on $\mathbb{P}^n(F)$;
- a choice of $X^\circ \subset X$.

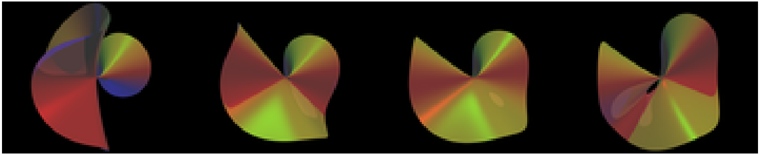
Are there any preferred choices? Yes, in the group-theoretic framework. No, in general.

Singular cubic surfaces



Deforming singularities on cubic surfaces / Madore

Singular cubic surfaces



Deforming singularities on cubic surfaces / Madore

In all cases, we expect $B \log(B)^6$ rational points of bounded height, on the complement of lines.

The geometric framework: Manin's conjecture

Manin (1989)

Let $X \subset \mathbb{P}^n$ be a smooth projective Fano variety over a number field F, in its anticanonical embedding.

The geometric framework: Manin's conjecture

Manin (1989)

Let $X \subset \mathbb{P}^n$ be a smooth projective Fano variety over a number field F, in its anticanonical embedding. Then there exists a Zariski open subset $X^\circ \subset X$ such that

$$\mathrm{N}(X^\circ, -\mathcal{K}_X, \mathrm{B}) \sim c \cdot \mathrm{B} \log(\mathrm{B})^{b-1}, \quad \mathrm{B} \to \infty,$$

where $b = \operatorname{rk} \operatorname{Pic}(X)$.

The geometric framework: Manin's conjecture

Manin (1989)

Let $X \subset \mathbb{P}^n$ be a smooth projective Fano variety over a number field F, in its anticanonical embedding. Then there exists a Zariski open subset $X^\circ \subset X$ such that

$$\mathrm{N}(X^\circ, -\mathcal{K}_X, \mathrm{B}) \sim c \cdot \mathrm{B} \log(\mathrm{B})^{b-1}, \quad \mathrm{B} \to \infty,$$

where $b = \mathrm{rk}\,\mathrm{Pic}(X)$.

We do not know, in general, whether or not $X(F)$ is Zariski dense, even after a finite extension of F. Potential density of rational points has been proved for some families of Fano varieties, but is still open, e.g., for hypersurfaces $X_d \subset \mathbb{P}^d$, with $d \geq 5$.

The Batyrev–Manin conjecture

$$\mathrm{N}(X^\circ, \mathcal{L}, \mathrm{B}) = c(X^\circ, \mathcal{L}) \cdot \mathrm{B}^{a(X,L)} \cdot \log(\mathrm{B})^{b(X,L)-1}(1 + o(1)), \quad \mathrm{B} \to \infty$$

The Batyrev–Manin conjecture

$$\mathrm{N}(X^\circ, \mathcal{L}, \mathrm{B}) = c(X^\circ, \mathcal{L}) \cdot \mathrm{B}^{a(X,L)} \cdot \log(\mathrm{B})^{b(X,L)-1}(1 + o(1)), \quad \mathrm{B} \to \infty$$

- $a(X, L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$,

The Batyrev–Manin conjecture

$$\mathrm{N}(X^\circ, \mathcal{L}, \mathrm{B}) = c(X^\circ, \mathcal{L}) \cdot \mathrm{B}^{a(X,L)} \cdot \log(\mathrm{B})^{b(X,L)-1}(1 + o(1)), \quad \mathrm{B} \to \infty$$

- $a(X, L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$,
- $b(X, L) =$ codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(X, L)[L] + [K_X]$,

The Batyrev–Manin conjecture

$$N(X^\circ, \mathcal{L}, B) = c(X^\circ, \mathcal{L}) \cdot B^{a(X,L)} \cdot \log(B)^{b(X,L)-1}(1 + o(1)), \quad B \to \infty$$

- $a(X, L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$,
- $b(X, L) =$ codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(X, L)[L] + [K_X]$,
- $c(X^\circ, -\mathcal{K}_X) = c(X, -\mathcal{K}_X) = \alpha(X) \cdot \beta(X) \cdot \tau(\mathcal{K}_X)$ – "volume" of the effective cone, nontrivial part of the Brauer group, Peyre's Tamagawa type constant,

The Batyrev–Manin conjecture

$$\mathrm{N}(X^\circ, \mathcal{L}, \mathrm{B}) = c(X^\circ, \mathcal{L}) \cdot \mathrm{B}^{a(X,L)} \cdot \log(\mathrm{B})^{b(X,L)-1}(1 + o(1)), \quad \mathrm{B} \to \infty$$

- $a(X, L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$,
- $b(X, L) =$ codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(X, L)[L] + [K_X]$,
- $c(X^\circ, -\mathcal{K}_X) = c(X, -\mathcal{K}_X) = \alpha(X) \cdot \beta(X) \cdot \tau(\mathcal{K}_X)$ – "volume" of the effective cone, nontrivial part of the Brauer group, Peyre's Tamagawa type constant,
- $c(X^\circ, \mathcal{L}) = \sum_y c(X_y, \mathcal{L}|_{X_y})$, where $X \to Y$ is a "Mori fiber space" – $\mathcal{L}$-primitive fibrations of Batyrev–T.

The Batyrev–Manin conjecture

$$\mathrm{N}(X^\circ, \mathcal{L}, \mathrm{B}) = c(X^\circ, \mathcal{L}) \cdot \mathrm{B}^{a(X,L)} \cdot \log(\mathrm{B})^{b(X,L)-1}(1 + o(1)), \quad \mathrm{B} \to \infty$$

- $a(X, L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$,
- $b(X, L) =$ codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(X, L)[L] + [K_X]$,
- $c(X^\circ, -\mathcal{K}_X) = c(X, -\mathcal{K}_X) = \alpha(X) \cdot \beta(X) \cdot \tau(\mathcal{K}_X)$ – "volume" of the effective cone, nontrivial part of the Brauer group, Peyre's Tamagawa type constant,
- $c(X^\circ, \mathcal{L}) = \sum_y c(X_y, \mathcal{L}|_{X_y})$, where $X \to Y$ is a "Mori fiber space" – $\mathcal{L}$-primitive fibrations of Batyrev–T.

Fujita (1985): $\kappa\epsilon(X, L) := -a(X, L)$, the Kodaira energy.
Sommese (1986): $\sigma(X, L) := \dim(X) + 1 - a(X, L)$, the spectral value.

Results

Many recent theoretical results on asymptotics of points of bounded height on cubic surfaces and other Del Pezzo surfaces, via (uni)versal torsors (Browning, de la Breteche, Derenthal, Fouvry, Heath-Brown, Moroz, Salberger, Swinnerton-Dyer, Wooley, ...)

Results

Many recent theoretical results on asymptotics of points of bounded height on cubic surfaces and other Del Pezzo surfaces, via (uni)versal torsors (Browning, de la Breteche, Derenthal, Fouvry, Heath-Brown, Moroz, Salberger, Swinnerton-Dyer, Wooley, ...)

Counterexamples to Manin's conjecture for cubic surface bundles (Batyrev-T.). These are compactifications of affine spaces.

Results

Integral points

A wealth of results by Duke, Rudnick, Sarnak, Eskin, McMullen, Mozes, Shah, Oh, Gorodnik, Maucourant, Nevo, Weiss, and others... on $X^\circ = G/H$.

Results

Integral points

A wealth of results by Duke, Rudnick, Sarnak, Eskin, McMullen, Mozes, Shah, Oh, Gorodnik, Maucourant, Nevo, Weiss, and others... on $X^\circ = G/H$. Recent results by Chambert-Loir–T. for partial equivariant compactifications of additive groups and tori.

Results

Integral points

A wealth of results by Duke, Rudnick, Sarnak, Eskin, McMullen, Mozes, Shah, Oh, Gorodnik, Maucourant, Nevo, Weiss, and others... on $X^\circ = G/H$. Recent results by Chambert-Loir–T. for partial equivariant compactifications of additive groups and tori.

Rational points

(Franke-Manin-T.) G/P; (Strauch) twisted products of G/P; (Batyrev-T.) $X \supset T$; (Strauch-T.) $X \supset G/U$; (Chambert-Loir-T.) $X \supset \mathbb{G}_a^n$; (Shalika-T.) $X \supset U$ (bi-equivariant); (Shalika-Takloo-Bighash-T.) $X \supset G$, De Concini-Procesi varieties

Results

Integral points

A wealth of results by Duke, Rudnick, Sarnak, Eskin, McMullen, Mozes, Shah, Oh, Gorodnik, Maucourant, Nevo, Weiss, and others... on $X^\circ = G/H$. Recent results by Chambert-Loir–T. for partial equivariant compactifications of additive groups and tori.

Rational points

(Franke-Manin-T.) G/P; (Strauch) twisted products of G/P; (Batyrev-T.) $X \supset T$; (Strauch-T.) $X \supset G/U$; (Chambert-Loir-T.) $X \supset \mathbb{G}_a^n$; (Shalika-T.) $X \supset U$ (bi-equivariant); (Shalika-Takloo-Bighash-T.) $X \supset G$, De Concini-Procesi varieties

In all cases, Manin's conjecture, and its refinements by Batyrev-Manin, Peyre, Batyrev-T. hold. Chambert-Loir–T. proposed a framework interpolating the theories of rational and integral points; e.g., a log-version of Peyre's constant, the constants a and b.

Batyrev-Manin conjecture + refinements

The Batyrev-Manin's conjecture, and its refinements (by Peyre and Batyrev-T.), should be viewed as a strong, quantitative, version of density of rational points.

Batyrev-Manin conjecture + refinements

The Batyrev-Manin's conjecture, and its refinements (by Peyre and Batyrev-T.), should be viewed as a strong, quantitative, version of density of rational points. Implicitly, it asserts that in the case when $L = -K_X$, no proper subvariety Y of X, which is not contained in the exceptional set $X \setminus X^\circ$, contributes a positive proportion to the main term of the asymptotic formula.

Batyrev-Manin conjecture + refinements

The Batyrev-Manin's conjecture, and its refinements (by Peyre and Batyrev-T.), should be viewed as a strong, quantitative, version of density of rational points. Implicitly, it asserts that in the case when $L = -K_X$, no proper subvariety Y of X, which is not contained in the exceptional set $X \setminus X^\circ$, contributes a positive proportion to the main term of the asymptotic formula. Internal consistency would imply that for all such Y we have

$$(a(Y, L|_Y)), b(Y, L|_Y)) < (a(X, L), b(X, L)), \tag{1}$$

in the lexicographic ordering.

Batyrev-Manin conjecture + refinements

The Batyrev-Manin's conjecture, and its refinements (by Peyre and Batyrev-T.), should be viewed as a strong, quantitative, version of density of rational points. Implicitly, it asserts that in the case when $L = -K_X$, no proper subvariety Y of X, which is not contained in the exceptional set $X \setminus X^\circ$, contributes a positive proportion to the main term of the asymptotic formula. Internal consistency would imply that for all such Y we have

$$(a(Y, L|_Y)), b(Y, L|_Y)) < (a(X, L), b(X, L)), \qquad (1)$$

in the lexicographic ordering. However, there exist varieties of dimension ≥ 3 where this property fails.

Batyrev-Manin conjecture + refinements

The Batyrev-Manin's conjecture, and its refinements (by Peyre and Batyrev-T.), should be viewed as a strong, quantitative, version of density of rational points. Implicitly, it asserts that in the case when $L = -K_X$, no proper subvariety Y of X, which is not contained in the exceptional set $X \setminus X^\circ$, contributes a positive proportion to the main term of the asymptotic formula. Internal consistency would imply that for all such Y we have

$$(a(Y, L|_Y)), b(Y, L|_Y)) < (a(X, L), b(X, L)), \tag{1}$$

in the lexicographic ordering. However, there exist varieties of dimension ≥ 3 where this property fails. No counterexamples are known in the equivariant context, when X is an equivariant compactification of a linear algebraic group G or of a homogeneous space $H\backslash G$.

Adelic vs. accumulating constants

Example / Elsenhans (2010)

Consider the $(1, 2)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ given by

$$x_0 y_0^2 + x_1 y_1^2 + (2x_0 - x_1) y_2^2 + (-6x_0 + x_1) y_3^2 = 0.$$

This is a quadric surface bundle over $\mathbb{P}^1$.

Adelic vs. accumulating constants

Example / Elsenhans (2010)

Consider the $(1,2)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ given by

$$x_0 y_0^2 + x_1 y_1^2 + (2x_0 - x_1) y_2^2 + (-6x_0 + x_1) y_3^2 = 0.$$

This is a quadric surface bundle over $\mathbb{P}^1$. Exclude accumulating subvarieties and singular quadrics. Count points of height $\mathrm{B} \leq 10^8$.

Adelic vs. accumulating constants

Example / Elsenhans (2010)

Consider the $(1,2)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ given by

$$x_0 y_0^2 + x_1 y_1^2 + (2x_0 - x_1) y_2^2 + (-6x_0 + x_1) y_3^2 = 0.$$

This is a quadric surface bundle over $\mathbb{P}^1$. Exclude accumulating subvarieties and singular quadrics. Count points of height $\mathrm{B} \leq 10^8$. Excluding the infinitely many split quadrics we have

$$\mathrm{N(B)} \sim c \cdot \mathrm{B} \log(\mathrm{B}), \quad c = c_\infty(X) \cdot \prod_p c_p \quad = 0.6524.$$

Adelic vs. accumulating constants

Example / Elsenhans (2010)

Consider the $(1,2)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ given by

$$x_0 y_0^2 + x_1 y_1^2 + (2x_0 - x_1) y_2^2 + (-6x_0 + x_1) y_3^2 = 0.$$

This is a quadric surface bundle over $\mathbb{P}^1$. Exclude accumulating subvarieties and singular quadrics. Count points of height $\mathrm{B} \leq 10^8$. Excluding the infinitely many split quadrics we have

$$\mathrm{N}(\mathrm{B}) \sim c \cdot \mathrm{B} \log(\mathrm{B}), \quad c = c_\infty(X) \cdot \prod_p c_p \quad = 0.6524.$$

On the split quadrics

$$\mathrm{N}'(\mathrm{B}) \sim c' \cdot \mathrm{B} \log(\mathrm{B}), \quad c' = \sum_{\mathbf{x} \in \mathbb{P}^1(\mathbb{Q}), Q_\mathbf{x} \text{ split}} c(Q_\mathbf{x}) = 0.0903.$$

a – basic properties

Let X be a smooth projective variety.

Assume that K_X is not pseudo-effective and L is big.

a – basic properties

Let X be a smooth projective variety.

Assume that K_X is not pseudo-effective and L is big.

- $a(X, L) \in \mathbb{Q}_{\geq 0}$.

a – basic properties

Let X be a smooth projective variety.

Assume that K_X is not pseudo-effective and L is big.

- $a(X, L) \in \mathbb{Q}_{\geq 0}$.
- Let $\beta : \tilde{X} \to X$ be a birational morphism of projective varieties and put $\tilde{L} = \beta^* L$. Then

$$a(X, L) = a(\tilde{X}, \tilde{L}).$$

b – basic properties

BCHM

Assume that $-K_X$ is ample. Then $\Lambda_{\text{eff}}(X)$ (the pseudo-effective cone) is finite polyhedral, and is generated by effective divisors.

b – basic properties

BCHM

Assume that $-K_X$ is ample. Then $\Lambda_{\mathrm{eff}}(X)$ (the pseudo-effective cone) is finite polyhedral, and is generated by effective divisors.

There are other such situations:

- When X is an equivariant compactification of $X^\circ := H\backslash G$, where G is a connected linear algebraic group, and H is closed subgroup of G such that X° is affine.

b – basic properties

BCHM

Assume that $-K_X$ is ample. Then $\Lambda_{\mathrm{eff}}(X)$ (the pseudo-effective cone) is finite polyhedral, and is generated by effective divisors.

There are other such situations:

- When X is an equivariant compactification of $X^\circ := H\backslash G$, where G is a connected linear algebraic group, and H is closed subgroup of G such that X° is affine.
- Let $\beta : \tilde{X} \to X$ be a birational morphism of projective varieties and put $\tilde{L} = \beta^* L$. Assume that $K_{\tilde{X}}$ is not pseudo-effective and $b(\tilde{X}, \tilde{L})$ is defined. Then

$$b(X, L) = b(\tilde{X}, \tilde{L}).$$

b – basic properties

Proposition

Let X be a smooth projective variety and L a big line bundle. Assume that $\Lambda_{\mathrm{eff}}(X)$ is finite polyhedral and that

$$D = a(X, L)L + K_X$$

is semi-ample. Let $\pi : X \to Y$ be the semi-ample fibration of D. Then

$$b(X, L) = \mathrm{rk}\,\mathrm{NS}(X) - \mathrm{rk}\,\mathrm{NS}_\pi(X),$$

where $\mathrm{NS}_\pi(X)$ is the lattice generated by π-vertical divisors, i.e., divisors $M \subset X$ such that $\pi(M) \subsetneq Y$.

Balanced line bundles

Definition

A line bundle L on X is weakly balanced with respect to an irreducible subvariety $Y \subset X$ if

- $a(Y, L|_Y) \leq a(X, L)$;
- if $a(Y, L|_Y) = a(X, L)$ and $b(X, L)$ is defined then $b(Y, L|Y)$ is defined and $b(Y, L|_Y) \leq b(X, L)$.

Balanced line bundles

Definition

A line bundle L on X is weakly balanced with respect to an irreducible subvariety $Y \subset X$ if

- $a(Y, L|_Y) \leq a(X, L)$;
- if $a(Y, L|_Y) = a(X, L)$ and $b(X, L)$ is defined then $b(Y, L|Y)$ is defined and $b(Y, L|_Y) \leq b(X, L)$.

A line bundle is called balanced with respect to Y if it is weakly balanced and one of the two inequalities is strict.

Balanced line bundles: examples

- Every ample line bundle on $\mathbb{P}^2$ is balanced.

Balanced line bundles: examples

- Every ample line bundle on $\mathbb{P}^2$ is balanced.
- On $X = \mathbb{P}^1 \times \mathbb{P}^1$, an ample L is balanced if and only if L is proportional to $-K_X$.

Balanced line bundles: examples

- Every ample line bundle on $\mathbb{P}^2$ is balanced.
- On $X = \mathbb{P}^1 \times \mathbb{P}^1$, an ample L is balanced if and only if L is proportional to $-K_X$.
- Cubic surface: $-K_X$ is not balanced on exceptional curves (lines) but is balanced on all other curves.

Balanced line bundles: examples

- Every ample line bundle on $\mathbb{P}^2$ is balanced.
- On $X = \mathbb{P}^1 \times \mathbb{P}^1$, an ample L is balanced if and only if L is proportional to $-K_X$.
- Cubic surface: $-K_X$ is not balanced on exceptional curves (lines) but is balanced on all other curves.

Proposition

Let G be a connected semi-simple algebraic group, $P \subset G$ a parabolic subgroup and $X = P\backslash G$ the associated generalized flag variety. Let L be a line bundle on X whose class is not proportional to $-K_X$ and is contained in the interior of the effective cone $\Lambda_{\mathrm{eff}}(X)$. Then L is not balanced (on some subvarieties $Y \subset X$).

Del Pezzo surfaces

Let X be a smooth projective surface with ample $-K_X$, i.e., a Del Pezzo surface. These are classified by the degree of the canonical class $d := (K_X, K_X)$. Let L be a big line bundle on X. When is it balanced?

Del Pezzo surfaces

Let X be a smooth projective surface with ample $-K_X$, i.e., a Del Pezzo surface. These are classified by the degree of the canonical class $d := (K_X, K_X)$. Let L be a big line bundle on X. When is it balanced?

The only subvarieties of X on which we need to test the values of a and b are rational curves $C \subset X$, and $b(C, L|_C) = 1$.

Del Pezzo surfaces

A line bundle L on a Del Pezzo surface X is balanced on all nonexceptional curves iff $a(L)L + K_X$ is rigid effective.

Del Pezzo surface fibrations

Let f, g be general cubic forms on $\mathbb{P}^3$ and

$$X := \{sf + tg = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3$$

the Fano threefold obtained by blowing up the base locus of the pencil. The projection onto the first factor exhibits a cubic surface fibration

$$\pi : X \to \mathbb{P}^1,$$

so that $-K_X$ restricts to $-K_Y$, for every smooth fiber Y of π. Thus

$$a(Y, -K_Y) = a(X, -K_X) = 1.$$

The Picard rank of a smooth fiber of π is 7. On the other hand, by Lefschetz theorem, we have $\mathrm{rk}\,\mathrm{Pic}(X) = 2$. It follows that

$$7 = b(Y, -K_Y) > b(X, -K_X) = 2,$$

i.e., $-K_X$ is not balanced on X. This gives counterexamples to Manin's conjecture and its refinement by Peyre.

Del Pezzo surface fibrations

Let f, g be general cubic forms on $\mathbb{P}^3$ and

$$X := \{sf + tg = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3$$

the Fano threefold obtained by blowing up the base locus of the pencil. The projection onto the first factor exhibits a cubic surface fibration

$$\pi : X \to \mathbb{P}^1,$$

so that $-K_X$ restricts to $-K_Y$, for every smooth fiber Y of π. Thus

$$a(Y, -K_Y) = a(X, -K_X) = 1.$$

The Picard rank of a smooth fiber of π is 7. On the other hand, by Lefschetz theorem, we have $\mathrm{rk}\,\mathrm{Pic}(X) = 2$. It follows that

$$7 = b(Y, -K_Y) > b(X, -K_X) = 2,$$

i.e., $-K_X$ is not balanced on X. This gives counterexamples to Manin's conjecture and its refinement by Peyre. Note that $-K_X$ is balanced with respect to every rational curve on an appropriate Zariski open X°.

Mori fiber spaces

Monodromy (de Fernex-Hacon 2009)

Let $\pi : X \to B$ be a Fano fibration from a smooth Fano variety. Take a smooth fiber X_b, and assume that the monodromy action on $\mathrm{N}^1(X_b)$ is trivial. Then

$$\operatorname{rk}\operatorname{Pic}(X_y) < \operatorname{rk}\operatorname{Pic}(X).$$

Mori fiber spaces

Monodromy (de Fernex-Hacon 2009)

Let $\pi : X \to B$ be a Fano fibration from a smooth Fano variety. Take a smooth fiber X_b, and assume that the monodromy action on $\mathrm{N}^1(X_b)$ is trivial. Then

$$\mathrm{rk}\,\mathrm{Pic}(X_y) < \mathrm{rk}\,\mathrm{Pic}(X).$$

Rigidity (de Fernex-Hacon 2009)

Let $\pi : \mathcal{X} \to B$ be a flat family of Fano varieties over a connected smooth curve B and $\mathcal{L}$ be a π-big line bundle on $\mathcal{X}$. Then

$$a(X_b, \mathcal{L}_b) = a(X_{b'}, \mathcal{L}_{b'}), \qquad b(X_b, \mathcal{L}_b) = b(X_{b'}, \mathcal{L}_{b'}),$$

for all $b, b' \in B$.

Fano varieties

Codimension one

Let X be a smooth Fano variety. Then

- $-K_X$ is weakly balanced on curves (outside some closed locus),

Fano varieties

Codimension one

Let X be a smooth Fano variety. Then

- $-K_X$ is weakly balanced on curves (outside some closed locus),
- if $\mathrm{rk}\,\mathrm{Pic}(X) = 1$ then $-K_X$ is balanced on irreducible divisors $Y \subset X$.

Fano varieties

Codimension one

Let X be a smooth Fano variety. Then

- $-K_X$ is weakly balanced on curves (outside some closed locus),
- if $\operatorname{rk}\operatorname{Pic}(X) = 1$ then $-K_X$ is balanced on irreducible divisors $Y \subset X$.

Cubic threefolds

If X is a cubic threefold then $-K_X$ is weakly balanced but not balanced (on lines, these dominate).

Fano threefolds

Smooth projective threefolds X with ample $-K_X$; finitely many families, classification completed by Iskovskikh and Mori–Mukai.

Fano threefolds

Smooth projective threefolds X with ample $-K_X$; finitely many families, classification completed by Iskovskikh and Mori–Mukai. The basic invariants are:

- the rank of the Picard group, i.e., $b(X, -K_X)$;
- the index $r = r(X)$, which is the maximal integer such that K_X is divisible by r in $\mathrm{Pic}(X)$;
- the degree $d(X) := (-K_X)^3$;
- the Mori invariant $m(X)$ which is the smallest integer m such that through every point of X passes a rational curve C with $(-K_X, C) \leq m$.

Fano threefolds

Every smooth Fano threefold (over an algebraically closed field of characteristic zero) is isomorphic to one of the following:

(1) a generalized flag variety $P\backslash G$;

(2) a variety X with $m(X) = 2$ (i.e., there is a rational curve of degree $\leq m(X)$ through every point of X);

(3) a blowup of varieties of type (1) or (2);

(4) a direct product of $\mathbb{P}^1$ and a del Pezzo surface.

Fano threefolds

Every smooth Fano threefold (over an algebraically closed field of characteristic zero) is isomorphic to one of the following:

(1) a generalized flag variety $P \backslash G$;

(2) a variety X with $m(X) = 2$ (i.e., there is a rational curve of degree $\leq m(X)$ through every point of X);

(3) a blowup of varieties of type (1) or (2);

(4) a direct product of $\mathbb{P}^1$ and a del Pezzo surface.

Finer classification:

$\mathrm{rk}\,\mathrm{Pic}(X) = 1$: We have the following possibilities for X:

- $\mathbb{P}^3$, or a quadric, or
- $r(X) = 2$ and $d(X) \in \{8, 16, 24, 32, 40\}$, or
- $r(X) = 1$ and $d(X) \in \{2, 4, 6, 8, 10, 12, 14, 16, 18, 22\}$.

Fano threefolds

The first two cases are are flag varieties.

Fano threefolds

The first two cases are are flag varieties. We saw that $-K_X$ is balanced with respect to divisors on X and we need to focus on curves:

- varieties with $r(X) = 2$ are dominated by $-K_X$-conics and there are no curves of smaller degree;

Fano threefolds

The first two cases are are flag varieties. We saw that $-K_X$ is balanced with respect to divisors on X and we need to focus on curves:

- varieties with $r(X) = 2$ are dominated by $-K_X$-conics and there are no curves of smaller degree;
- varieties with $r(X) = 1$ are also dominated by $-K_X$-conics but are not dominated by $-K_X$-lines;

thus, $-K_X$ is weakly balanced but not balanced on curves on X°.

Fano threefolds

The first two cases are are flag varieties. We saw that $-K_X$ is balanced with respect to divisors on X and we need to focus on curves:

- varieties with $r(X) = 2$ are dominated by $-K_X$-conics and there are no curves of smaller degree;
- varieties with $r(X) = 1$ are also dominated by $-K_X$-conics but are not dominated by $-K_X$-lines;

thus, $-K_X$ is weakly balanced but not balanced on curves on X°.

Number fields

$-K_X$-conics dominating X are surfaces of general type which embed into their Albanese varieties. By Faltings' theorem, conics defined over a fixed number field are contained in a proper subvariety, and cannot dominate X.

Fano threefolds

$\operatorname{rk}\operatorname{Pic}(X) \geq 2$: 12 minimal families and all others are blowups.

Fano threefolds

$\operatorname{rk}\operatorname{Pic}(X) \geq 2$: 12 minimal families and all others are blowups.

- 3 are flag varieties,
- 2 are toric,
- all others are conic bundles over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

On all of these, $-K_X$ is balanced on curves in X°.

Balanced line bundles

Definition

It is weakly balanced on X if there exists a Zariski closed subset $Z \subset X$ such that L is weakly balanced with respect to every irreducible subvariety Y not contained in Z. The subset Z will be called exceptional.

Balanced line bundles

Definition

It is weakly balanced on X if there exists a Zariski closed subset $Z \subset X$ such that L is weakly balanced with respect to every irreducible subvariety Y not contained in Z. The subset Z will be called exceptional. A line bundle is called balanced on X if it is weakly balanced on X and if it is balanced with respect to every irreducible subvariety not contained in Z.

Balanced line bundles

Summary

Let X be a Fano threefold. Then $-K_X$ is balanced **more or less** when X is a homogeneous space or an equivariant compactification of a $H\backslash G$.

Equivariant geometry

Let G a connected linear algebraic group, $H \subset G$ a closed subgroup, and X a projective equivariant compactification of $X^\circ := H \backslash G$. We will assume that X is smooth and that the boundary

$$\cup_{\alpha \in \mathcal{A}_X} D_\alpha = X \setminus X^\circ$$

is a divisor with normal crossings. If H is a parabolic subgroup of a semi-simple group G, then there is no boundary, i.e., $\mathcal{A} = \mathcal{A}_X$ is empty, and $H \backslash G$ is a generalized flag variety. Throughout, we will assume that $\mathcal{A}$ is not empty.

Equivariant geometry

Let $\mathfrak{X}(G)^*$ be the group of algebraic characters of G and

$$\mathfrak{X}(G,H)^* = \{\, \chi : G \to \mathbb{G}_m \,|\, \chi(hg) = \chi(g), \quad \forall h \in H \,\}$$

the subgroup of characters whose restrictions to H are trivial.

Equivariant geometry

Let $\mathfrak{X}(G)^*$ be the group of algebraic characters of G and

$$\mathfrak{X}(G, H)^* = \{\, \chi : G \to \mathbb{G}_m \,|\, \chi(hg) = \chi(g), \quad \forall h \in H \,\}$$

the subgroup of characters whose restrictions to H are trivial.

Let $\mathrm{Pic}^G(X)$ be the group of equivalence classes of G-linearized line bundles on X and $\mathrm{Pic}(X)$ the Picard group of X. For $L \in \mathrm{Pic}^G(X)$, the subgroup $H \subset G$ acts linearly on the fiber L_x at $x = H \in H\backslash G$. This defines a homomorphism

$$\mathrm{Pic}^G(X) \to \mathfrak{X}(H)^*.$$

Let $\mathrm{Pic}^{(G,H)}(X)$ be the kernel of this map.

Equivariant geometry

Let X be a smooth projective equivariant compactification of $X^\circ := H\backslash G$, where G is a connected linear algebraic group and H a connected closed subgroup of G. Assume that $X^\circ = H\backslash G$ is affine.

1. We have an exact sequence

$$0 \to \mathfrak{X}(G,H)^*_{\mathbb{Q}} \to \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}} \to 0.$$

Equivariant geometry

Let X be a smooth projective equivariant compactification of $X^\circ := H\backslash G$, where G is a connected linear algebraic group and H a connected closed subgroup of G. Assume that $X^\circ = H\backslash G$ is affine.

1. We have an exact sequence

$$0 \to \mathfrak{X}(G, H)^*_{\mathbb{Q}} \to \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}} \to 0.$$

2. $\oplus_{\alpha \in \mathcal{A}} \mathbb{Q} D_\alpha \cong \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}}$.

Equivariant geometry

Let X be a smooth projective equivariant compactification of $X^\circ := H\backslash G$, where G is a connected linear algebraic group and H a connected closed subgroup of G. Assume that $X^\circ = H\backslash G$ is affine.

1. We have an exact sequence
$$0 \to \mathfrak{X}(G,H)^*_{\mathbb{Q}} \to \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}} \to 0.$$
2. $\oplus_{\alpha \in \mathcal{A}} \mathbb{Q} D_\alpha \cong \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}}$.
3. $\Lambda_{\mathrm{eff}}(X) = \sum_{\alpha \in \mathcal{A}} \mathbb{R}_{\geq 0} D_\alpha$.

Equivariant geometry

Let X be a smooth projective equivariant compactification of $X^\circ := H\backslash G$, where G is a connected linear algebraic group and H a connected closed subgroup of G. Assume that $X^\circ = H\backslash G$ is affine.

1. We have an exact sequence
$$0 \to \mathfrak{X}(G,H)^*_{\mathbb{Q}} \to \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}} \to 0.$$
2. $\oplus_{\alpha\in\mathcal{A}} \mathbb{Q} D_\alpha \cong \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}}$.
3. $\Lambda_{\mathrm{eff}}(X) = \sum_{\alpha\in\mathcal{A}} \mathbb{R}_{\geq 0} D_\alpha$.
4. $-K_X = \sum_{\alpha\in\mathcal{A}} \kappa_\alpha D_\alpha$, with $\kappa_\alpha \geq 1$.

Equivariant geometry

Assume that $\mathfrak{X}(G,H)^*$ is trivial, i.e.,

$$\mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}} = \mathrm{Pic}(X).$$

Let

$$L = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha D_\alpha, \quad \lambda_\alpha \in \mathbb{Q}_{>0},$$

be a big line bundle. Then

$$a(X,L) = \max_\alpha \frac{\kappa_\alpha}{\lambda_\alpha}$$

Equivariant geometry

Assume that $\mathfrak{X}(G,H)^*$ is trivial, i.e.,

$$\mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}} = \mathrm{Pic}(X).$$

Let

$$L = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha D_\alpha, \quad \lambda_\alpha \in \mathbb{Q}_{>0},$$

be a big line bundle. Then

$$a(X,L) = \max_\alpha \frac{\kappa_\alpha}{\lambda_\alpha}$$

and

$$b(X,L) = \#\{\alpha \in \mathcal{A} \,|\, a(X,L) = \frac{\kappa_\alpha}{\lambda_\alpha}\}.$$

Balanced line bundles

Hassett-Tanimoto-T. 2011

Let G be a connected linear algebraic group, $H \subset G$ a closed subgroup such that the quotient space $X^\circ : H\backslash G$ is affine. Let X be a smooth projective G-equivariant compactification of X°. Let $M \subset G$ be a closed subgroup of X containing H and such that $M\backslash G$ is not projective. Let $Y \subset X$ be the induced equivariant compactification of $H\backslash M$. Then $-K_X$ is balanced with respect to Y.

Balanced line bundles

Hassett-Tanimoto-T. 2011

Let G be a connected linear algebraic group, $H \subset G$ a closed subgroup such that the quotient space $X^\circ : H\backslash G$ is affine. Let X be a smooth projective G-equivariant compactification of X°. Let $M \subset G$ be a closed subgroup of X containing H and such that $M\backslash G$ is not projective. Let $Y \subset X$ be the induced equivariant compactification of $H\backslash M$. Then $-K_X$ is balanced with respect to Y.

Example

Let $G = \mathbb{G}_a^n$ and $M = \mathbb{G}_a^d \subset G$. Let X be a smooth projective equivariant compactification of G and Y the induced compactification of M. If $a(Y, -K_X|_Y) = 1$ then the number of irreducible boundary components of Y is strictly smaller than the number of boundary components of X.

Equivariant geometry

Let $G = \mathrm{PGL}_2$, $M = B$, the Borel subgroup of G and $H = 1$. Let $X = \mathbb{P}^3 \supset G$, with boundary $D := \mathbb{P}^1 \times \mathbb{P}^1$. Then $Y = \mathbb{P}^2$; with boundary $D_Y = Y \setminus B = \ell_1 \cup \ell_2$, a union of two intersecting lines. Put $X' := B \backslash G = \mathbb{P}^1$. Then $\pi : X \dashrightarrow X'$ has indeterminacy along $D_Y := D \cap Y$. Resolving the indeterminacy, we obtain a fibration

$$\tilde{\pi} : \tilde{X} \to \mathbb{P}^1.$$

We have

$$a(\tilde{X}, -K_{\tilde{X}}) = a(\tilde{Y}, -K_{\tilde{X}}|_{\tilde{Y}}) = a(\tilde{Y}, -K_{\tilde{Y}}).$$

The proof of Theorem shows that

$$\#\mathcal{A}_{\tilde{Y}} = \#\mathcal{A}_{\tilde{X}} = \mathrm{rk}\,\mathrm{Pic}(\tilde{X}).$$

However, $\mathfrak{X}(B)^* = \mathbb{Z}$, and in particular, $\mathrm{rk}\,\mathrm{Pic}(\tilde{Y}) = \#\mathcal{A}_{\tilde{Y}} - 1$ so that

$$b(\tilde{Y}, -K_{\tilde{X}}|_{\tilde{Y}}) < b(\tilde{X}, -K_{\tilde{X}}).$$

Tamagawa numbers / Peyre (1995)

Let X be a smooth projective Fano variety of dimension d over a number field F. Assume that $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood U_x. In U_x, a section of the canonical line bundle has the form $\mathsf{s} := \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. Put

$$\omega_{\mathcal{K}_X, v} := \|\mathsf{s}\|_v \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. This local measure globalizes to $X(F_v)$.

Tamagawa numbers / Peyre (1995)

Let X be a smooth projective Fano variety of dimension d over a number field F. Assume that $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood U_x. In U_x, a section of the canonical line bundle has the form $\mathsf{s} := \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. Put

$$\omega_{\mathcal{K}_X, v} := \|\mathsf{s}\|_v \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. This local measure globalizes to $X(F_v)$. For almost all v,

$$\int_{X(F_v)} \omega_{\mathcal{K}_X, v} = \frac{X(\mathbb{F}_q)}{q^d}.$$

Tamagawa numbers / Peyre

Choose a finite set of places S, and put

$$\omega_{\mathcal{K}_X} := \mathsf{L}_S^*(1, \mathrm{Pic}(\bar{X})) \cdot |\mathrm{disc}(F)|^{-1} \cdot \prod_v \lambda_v \omega_{\mathcal{K}_X, v},$$

with $\lambda_v = \mathsf{L}_v(1, \mathrm{Pic}(\bar{X}))^{-1}$ for $v \notin S$ and $\lambda_v = 1$, otherwise. Put

$$\tau(\mathcal{K}_X) := \int_{\overline{X(F)} \subset X(\mathbb{A}_F)} \omega_{\mathcal{K}_X}.$$

Tamagawa numbers / Peyre

Choose a finite set of places S, and put

$$\omega_{\mathcal{K}_X} := \mathsf{L}_S^*(1, \mathrm{Pic}(\bar{X})) \cdot |\mathrm{disc}(F)|^{-1} \cdot \prod_v \lambda_v \omega_{\mathcal{K}_X, v},$$

with $\lambda_v = \mathsf{L}_v(1, \mathrm{Pic}(\bar{X}))^{-1}$ for $v \notin S$ and $\lambda_v = 1$, otherwise. Put

$$\tau(\mathcal{K}_X) := \int_{\overline{X(F)} \subset X(\mathbb{A}_F)} \omega_{\mathcal{K}_X}.$$

This constant appears in the constant $c = c(-\mathcal{K}_X)$ in Manin's conjecture above.

Tamagawa numbers / local theory

Let X be a smooth projective variety over a local field F, D an effective divisor on X, f_D the canonical section of $\mathcal{O}_X(D)$, and $U = X \setminus |D|$.

Tamagawa numbers / local theory

Let X be a smooth projective variety over a local field F, D an effective divisor on X, f_D the canonical section of $\mathcal{O}_X(D)$, and $U = X \setminus |D|$.

A form $\omega \in \Omega^d(U)$ defines a measure $|\omega|$ as before.

Tamagawa numbers / local theory

Let X be a smooth projective variety over a local field F, D an effective divisor on X, f_D the canonical section of $\mathcal{O}_X(D)$, and $U = X \setminus |D|$.

A form $\omega \in \Omega^d(U)$ defines a measure $|\omega|$ as before.

A metrization of the canonical line bundle K_X gives a global measure on $X(F)$

$$\tau_X = |\omega| / \|\omega\|.$$

Tamagawa numbers / local theory

Let X be a smooth projective variety over a local field F, D an effective divisor on X, f_D the canonical section of $\mathcal{O}_X(D)$, and $U = X \setminus |D|$.

A form $\omega \in \Omega^d(U)$ defines a measure $|\omega|$ as before.

A metrization of the canonical line bundle K_X gives a global measure on $X(F)$

$$\tau_X = |\omega|/\|\omega\|.$$

A metrization of $K_X(D)$ defines a measure on $U(F)$

$$\tau_{(X,D)} = |\omega|/\|\omega f_D\|$$

Example

When X is an equivariant compactification of an algebraic group G and ω a left-invariant differential form on G, we have $\operatorname{div}(\omega) = -D$, so that $K_X(D)$ is a trivial line bundle, equipped with a canonical metrization. We may assume that its section ωf_D has norm 1. Then

$$\tau_{(X,D)} = |\omega| / \|\omega f_D\| = |\omega|$$

is a Haar measure on $\mathsf{G}(F)$.

Height balls

Let L be an effective divisor with support $|D| = X \setminus U$, equipped with a metrization. Then

$$\{u \in U(F) \mid \|f_L(u)\| \geq 1/B\}$$

is a height ball, i.e., it is compact of finite measure $\mathrm{vol}(B)$.

Height balls

Let L be an effective divisor with support $|D| = X \setminus U$, equipped with a metrization. Then

$$\{u \in U(F) \mid \|f_L(u)\| \geq 1/B\}$$

is a height ball, i.e., it is compact of finite measure $\mathrm{vol}(B)$.

To compute the volume, for $B \to \infty$, we use the Mellin transform

$$Z(s) := \int_0^\infty t^{-s} \mathrm{dvol}(t)$$

Height balls

Let L be an effective divisor with support $|D| = X \setminus U$, equipped with a metrization. Then

$$\{u \in U(F) \mid \|f_L(u)\| \geq 1/B\}$$

is a height ball, i.e., it is compact of finite measure $\mathrm{vol}(B)$.

To compute the volume, for $B \to \infty$, we use the Mellin transform

$$Z(s) := \int_0^\infty t^{-s} \mathrm{dvol}(t) = \int_{U(F)} \|f_L\|^s \tau_{(X,D)},$$

combined with a Tauberian theorem.

Assume that over F

$$|D| = \cup_{\alpha \in \mathcal{A}} D_\alpha,$$

where D_α are geometrically irreducible, smooth, and intersecting transversally.

Assume that over F

$$|D| = \cup_{\alpha\in\mathcal{A}} D_\alpha,$$

where D_α are geometrically irreducible, smooth, and intersecting transversally. For $A \subset \mathcal{A}$ let

$$D_A := \cap_{\alpha\in A}, \quad D_A^\circ = D_A \setminus \cup_{A'\supset A} D_{A'}.$$

Igusa zeta functions / local theory

Assume that over F

$$|D| = \cup_{\alpha\in\mathcal{A}} D_\alpha,$$

where D_α are geometrically irreducible, smooth, and intersecting transversally. For $A \subset \mathcal{A}$ let

$$D_A := \cap_{\alpha\in A}, \quad D_A^\circ = D_A \setminus \cup_{A'\supset A} D_{A'}.$$

By the transversality assumption, $D_A \subset X$ is smooth, of codimension $\#A$ (or empty).

Assume that over F

$$|D| = \cup_{\alpha \in \mathcal{A}} D_\alpha,$$

where D_α are geometrically irreducible, smooth, and intersecting transversally. For $A \subset \mathcal{A}$ let

$$D_A := \cap_{\alpha \in A}, \quad D_A^\circ = D_A \setminus \cup_{A' \supset A} D_{A'}.$$

By the transversality assumption, $D_A \subset X$ is smooth, of codimension $\# A$ (or empty). Write

$$D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha.$$

Local computations

The Mellin transform $Z(s)$ can be computed in charts, via partition of unity. In a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_{\alpha} \|\mathsf{f}_{D_\alpha}\|(x)^{\lambda_\alpha s - \rho_\alpha} \,\mathrm{d}\tau_X(x) = \int \prod_{\alpha \in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \phi(x; y; s) \prod_{\alpha} \mathrm{d}x_\alpha \,\mathrm{d}y.$$

Local computations

The Mellin transform $Z(s)$ can be computed in charts, via partition of unity. In a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_\alpha \|\mathsf{f}_{D_\alpha}\|(x)^{\lambda_\alpha s - \rho_\alpha} \,\mathrm{d}\tau_X(x) = \int \prod_{\alpha \in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \phi(x; y; s) \prod_\alpha \mathrm{d}x_\alpha \,\mathrm{d}y.$$

Essentially, this is a product of integrals of the form

$$\int_{|x| \leq 1} |x|^{s-1} \mathrm{d}x.$$

Igusa zeta functions / local theory

Analytic properties of $Z(s)$ are encoded in the combinatorics of the stratification (D_A).

Analytic properties of $Z(s)$ are encoded in the combinatorics of the stratification (D_A).

Abscissa of convergence $= \max_{\substack{D_\alpha(F)\neq\emptyset \\ \lambda_\alpha>0}} \dfrac{\rho_\alpha - 1}{\lambda_\alpha}$;

Analytic properties of $Z(s)$ are encoded in the combinatorics of the stratification (D_A).

Abscissa of convergence $= \max_{\substack{D_\alpha(F)\neq\emptyset \\ \lambda_\alpha>0}} \dfrac{\rho_\alpha - 1}{\lambda_\alpha}$;

Order of pole $=$ number of α that achieve equality;

Igusa zeta functions / local theory

Analytic properties of $Z(s)$ are encoded in the combinatorics of the stratification (D_A).

Abscissa of convergence $= \max\limits_{\substack{D_\alpha(F)\neq\emptyset \\ \lambda_\alpha>0}} \dfrac{\rho_\alpha - 1}{\lambda_\alpha}$;

Order of pole $=$ number of α that achieve equality;

Leading coefficient $=$ sum of integrals over all D_A of minimal dimension where A consists only of such αs.

Global theory

Let X be a smooth projective variety over a number field F, D an effective divisor on X, $U = X \setminus |D|$.

Global theory

Let X be a smooth projective variety over a number field F, D an effective divisor on X, $U = X \setminus |D|$. Fix an adelic metric on $K_X(D)$; this defines measures $\tau_{(X,D),v}$ on $U(F_v)$ for all v.

Global theory

Let X be a smooth projective variety over a number field F, D an effective divisor on X, $U = X \setminus |D|$. Fix an adelic metric on $K_X(D)$; this defines measures $\tau_{(X,D),v}$ on $U(F_v)$ for all v. Assume that

$$\mathrm{H}^1(X, \mathscr{O}_X) = \mathrm{H}^2(X, \mathscr{O}_X) = 0.$$

Let

$$\mathrm{EP}(U) = \Gamma(U_{\bar{\mathbb{F}}}, \mathscr{O}_X^*)/\bar{\mathbb{F}}^* - \mathrm{Pic}(U_{\bar{\mathbb{F}}})/\textbf{torsion}$$

be the virtual Galois module.

Global theory

Let X be a smooth projective variety over a number field F, D an effective divisor on X, $U = X \setminus |D|$. Fix an adelic metric on $K_X(D)$; this defines measures $\tau_{(X,D),v}$ on $U(F_v)$ for all v. Assume that

$$\mathrm{H}^1(X, \mathscr{O}_X) = \mathrm{H}^2(X, \mathscr{O}_X) = 0.$$

Let

$$\mathrm{EP}(U) = \Gamma(U_{\bar{\mathbb{F}}}, \mathscr{O}_X^*)/\bar{\mathbb{F}}^* - \mathrm{Pic}(U_{\bar{\mathbb{F}}})/\textbf{torsion}$$

be the virtual Galois module. Put

$$\lambda_v = \mathsf{L}_v(1, \mathrm{EP}(U)), \quad v \nmid \infty, \quad \lambda_v = 1, \quad v \mid \infty.$$

We have a global measure on $U(\mathbb{A}_F)$ given by

$$\tau_{(X,D)} = \mathsf{L}^*(1, \mathrm{EP}(U))^{-1} \cdot \prod_v \lambda_v \tau_{(X,D),v}$$

Height on the adelic space $U(\mathbb{A}_F)$

Let $\mathcal{L} = (L, (\|\cdot\|_v))$ be an adelically metrized effective divisor supported on $|D|$. This defines a height function on $U(\mathbb{A}_F)$

$$\mathrm{H}_{\mathcal{L}}((x_v)) = \prod_v \|\mathrm{f}_L(x_v)\|_v^{-1}.$$

Height on the adelic space $U(\mathbb{A}_F)$

Let $\mathcal{L} = (L, (\|\cdot\|_v))$ be an adelically metrized effective divisor supported on $|D|$. This defines a height function on $U(\mathbb{A}_F)$

$$\mathsf{H}_{\mathcal{L}}((x_v)) = \prod_v \|\mathsf{f}_L(x_v)\|_v^{-1}.$$

To compute the volume of the height ball

$$\mathrm{vol}(B) := \{x \in U(\mathbb{A}_F) \mid \mathsf{H}_{\mathcal{L}}(x) \leq B\},$$

for $\mathcal{L}$ and $\tau_{(X,D)}$, we use the adelic Mellin transform:

$$Z(s) = \int_0^\infty t^{-s} \mathrm{dvol}(t) = \int_{U(\mathbb{A}_F)} \mathsf{H}_{\mathcal{L}}(x)^{-s} \, \mathrm{d}\tau_{(X,D)}(x) = \prod_v \int_{U(F_v)} \cdots.$$

Denef's formula

Recall that

$$D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha.$$

Choosing adelic metrics on $\mathscr{O}_X(D_\alpha)$ one has:

Denef's formula

Recall that

$$D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha.$$

Choosing adelic metrics on $\mathscr{O}_X(D_\alpha)$ one has:

$$Z_v(s) = \int_{X(F_v)} \prod_\alpha \|\mathsf{f}_{D_\alpha}\|_v^{s\lambda_\alpha - \rho_\alpha} \,\mathrm{d}\tau_{X,v}(x).$$

Denef's formula

Recall that

$$D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha.$$

Choosing adelic metrics on $\mathscr{O}_X(D_\alpha)$ one has:

$$Z_v(s) = \int_{X(F_v)} \prod_\alpha \|\mathsf{f}_{D_\alpha}\|_v^{s\lambda_\alpha - \rho_\alpha} \, \mathrm{d}\tau_{X,v}(x).$$

By the local analysis, this converges absolutely for

$$\Re(s) > \max((\rho_\alpha - 1)/\lambda_\alpha).$$

Denef's formula

Recall that

$$D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha.$$

Choosing adelic metrics on $\mathscr{O}_X(D_\alpha)$ one has:

$$Z_v(s) = \int_{X(F_v)} \prod_\alpha \|\mathsf{f}_{D_\alpha}\|_v^{s\lambda_\alpha - \rho_\alpha} \, \mathrm{d}\tau_{X,v}(x).$$

By the local analysis, this converges absolutely for

$$\Re(s) > \max((\rho_\alpha - 1)/\lambda_\alpha).$$

For almost all v and $\Re(s) > (\rho_\alpha - 1)/\lambda_\alpha$, one has

$$Z_v(s) = \sum_A \frac{\# D_A^\circ(\mathbb{F}_q)}{q^{\dim X}} \prod_{\alpha \in A} \frac{q-1}{q^{s\lambda_\alpha - \rho_\alpha + 1} - 1}.$$

Analyzing the Euler product (Chambert-Loir-T.)

Let $a := \max(\rho_\alpha/\lambda_\alpha)$ and let $A(L, D)$ be the set of α where equality is achieved; put $b = \#A(L, D)$.

Analyzing the Euler product (Chambert-Loir-T.)

Let $a := \max(\rho_\alpha/\lambda_\alpha)$ and let $A(L, D)$ be the set of α where equality is achieved; put $b = \#A(L, D)$. Let E be the divisor $aL - D$; it is effective with $|E| \subseteq |D|$. Then

Analyzing the Euler product (Chambert-Loir-T.)

Let $a := \max(\rho_\alpha/\lambda_\alpha)$ and let $A(L,D)$ be the set of α where equality is achieved; put $b = \#A(L,D)$. Let E be the divisor $aL - D$; it is effective with $|E| \subseteq |D|$. Then

$$\lim_{s\to a} Z(s)(s-a)^b \prod_{\alpha\in A(L,D)} \lambda_\alpha = \int_{X(\mathbb{A}_F)} \mathrm{H}_E(x)^{-1}\,\mathrm{d}\tau_X(x).$$

Analyzing the Euler product (Chambert-Loir-T.)

Let $a := \max(\rho_\alpha/\lambda_\alpha)$ and let $A(L,D)$ be the set of α where equality is achieved; put $b = \#A(L,D)$. Let E be the divisor $aL - D$; it is effective with $|E| \subseteq |D|$. Then

$$\lim_{s\to a} Z(s)(s-a)^b \prod_{\alpha\in A(L,D)} \lambda_\alpha = \int_{X(\mathbb{A}_F)} \mathrm{H}_E(x)^{-1}\,\mathrm{d}\tau_X(x).$$

A Tauberian theorem implies the volume asymptotics with respect to $\mathcal{L}$ and $\tau_{(X,D)}$, for $B \to \infty$, of the form

$$\mathrm{B}^a \log(\mathrm{B})^{b-1} \left(a(b-1)! \prod_{\alpha\in A(L,D)} \lambda_\alpha \right)^{-1} \int_{X(\mathbb{A}_F)} \mathrm{H}_E(x)^{-1}\,\mathrm{d}\tau_X(x).$$

Arithmetic applications

Conclusion

The constants $a = a(X, L)$ and $b = b(X, L)$ characterize asymptotics of height balls with respect to natural Tamagawa measures.

Arithmetic applications

Conclusion

The constants $a = a(X, L)$ and $b = b(X, L)$ characterize asymptotics of height balls with respect to natural Tamagawa measures.

How is this used?

Arithmetic applications

Conclusion

The constants $a = a(X, L)$ and $b = b(X, L)$ characterize asymptotics of height balls with respect to natural Tamagawa measures.

How is this used?

Basic principle

The number of rational or integral points of bounded height is approximated by the volumes of height balls.

Arithmetic applications

Conclusion

The constants $a = a(X, L)$ and $b = b(X, L)$ characterize asymptotics of height balls with respect to natural Tamagawa measures.

How is this used?

Basic principle

The number of rational or integral points of bounded height is approximated by the volumes of height balls. Failure of the balanced property should be viewed as characterizing an accumulating subvariety, i.e., a variety accumulating rational points.

Arithmetic applications

Gorodnik–Takloo-Bighash–T. 2011

Let X be a smooth projective equivariant compactification of $G\backslash G^n$. Then Manin's conjecture holds for X.

Arithmetic applications

Gorodnik–Takloo-Bighash–T. 2011

Let X be a smooth projective equivariant compactification of $G\backslash G^n$. Then Manin's conjecture holds for X.

The $n = 2$ case has been treated by Shalika–Takloo-Bighash-T. using spectral techniques and by Gorodnik-Maucourant-Oh using ergodic theory.