Integral transforms in non-abelian Hodge theory

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Outline

- joint with R.Donagi and C.Simpson

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- applications

Non-abelian Hodge correspondence

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Theorem: [Corlette-Simpson] Let $(X, \mathcal{O}_X(1))$ be a smooth complex projective variety. Then there is a natural equivalence of dg $\otimes$-categories:

$$\mathbf{nah}_X : \left(\begin{array}{l} \text{finite rank } \mathbb{C}\text{-} \\ \text{local systems} \\ \text{on } X \end{array} \right) \xrightarrow{\cong} \left(\begin{array}{l} \text{finite rank } \mathcal{O}_X(1)\text{-} \\ \text{semistable Higgs} \\ \text{bundles on } X \text{ with} \\ ch_1 = 0 \text{ and } ch_2 = 0 \end{array} \right)$$

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Note: The hardest part is the compatibility with pushforwards

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Note: If $f : X \to Y$ is a general morphism, then the application of any of $f_* f_!$, f^*, or $f^!$ to a local system will result in a general (regular holonomic) D-module.

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Conclusion: To achieve general functoriality we must extend the non-abelian Hodge correspondence to objects with singularities. Such an extension - the theory of twistor D-modules - was developed and studied extensively in the works of Sabbah and Mochizuki.

Some applications of functoriality (i)

The compatibility of **nah** with arbitrary pullbacks and pushforwards is a powerful tool: it allows us to convert complicated questions about D-modules to geometric questions about coherent sheaves.

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Example: (Non-degenerate representations of Kähler groups) Simpson used the compatibility of **nah** with pushforwards and a deformation theory calculation for coherent sheaves to construct projective manifolds whose fundamental groups admit a non-rigid, non-factorizable representations which do not come from variations Hodge structures. Such examples are of central importance in understanding the Shafarevich uniformization conjecture.

Some applications of functoriality (ii)

Example: (Construction of Hecke eigensheaves) In a joint work with Donagi we use the general compatibility with pushforwards to construct a geometric Langlands type equivalence as a conjugation of a Fourier-Mukai transform with two non-abelian Hodge correspondences. We also use the compatibbility to show that the the equivalence intertwines Hecke and tensorization symmetries.

Some applications of functoriality (iii)

Example: (Construction of Fourier-Mukai kernels) One can use the general compatibility to construct a Fourier-Mukai kernel for the stratified Mukai flop and to check the Bondal-Orlov orthogonality conditions for this kernel and thus prove the Bondal-Orlov "K-equivalence implies D-equivalence" conjecture in this case.

Tamely ramified non-abelian Hodge theory

Theorem: [Mochizuki] *Let $(X, \mathcal{O}_X(1))$ be smooth projective, and let $D \subset X$ be an effective divisor. Suppose that we have a closed subvariety $Z \subset X$ of codimension $\geqslant 3$, such that $X - Z$ is smooth and $D - Z$ is a normal crossing divisor. Then there is a canonical equivalence of dg $\otimes$-categories:*

$$\left(\begin{array}{l}\text{finite rank tame} \\ \text{parabolic } \mathbb{C}\text{-} \\ \text{local systems on} \\ (X, D)\end{array}\right) \xrightarrow{\mathbf{nah}_{X,D}} \left(\begin{array}{l}\text{finite rank locally abelian} \\ \text{tame parabolic Higgs} \\ \text{bundles on } (X, D) \\ \text{which are } \mathcal{O}_X(1)\text{-} \\ \text{semistable and satisfy} \\ \mathit{parch}_1 = 0 \text{ and } \mathit{parch}_2 = 0\end{array}\right)$$

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- For the compatibility of **nah** with push-forwards we will need an algebraic formula for f_* in terms of Higgs bundles.

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Assume that $\mathbf{nah}((\mathbf{V}_\bullet, \nabla)) = (\mathbf{E}_\bullet, \theta)$ (i.e. these two objects are related by a tame harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - (H \cup K)$).

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Assume that $\mathbf{nah}((\mathbf{V}_\bullet, \nabla)) = (\mathbf{E}_\bullet, \theta)$

Set $U = X - (H \cup K)$, $V = Y - R$, $g : U \to V$, and consider for each k the L^2 push-forward

$$F^k := R^k f_{*,L^2}(\mathbb{L}, h)$$

of $(\mathbf{V}_\bullet, \nabla)$.

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of $(\mathbf{V}_\bullet, \nabla)$. Here $\mathbb{L} = \left[(\mathbf{V}_\alpha)_{|U}\right]^\nabla$. **Note:** This is independent of α.

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General principle: **Jost-Yang-Zuo** The Higgs bundle corresponding to F^k should be given by the L^2 holomorphic push-forward of $(\mathbf{E}_\bullet, \theta)_{|U}$.

An analytic formula

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Theorem: **[Donagi-P-Simpson]** F^k is the harmonic bundle corresponding to the L^2 holomorphic push-forward

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Note: This is essentially contained in Jost-Yang-Zuo. They have an addittional hypothesis (that the local monodromy of ∇ is compact) which is unnecessary.

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Theorem: [Donagi-P-Simpson] F^k is the harmonic bundle corresponding to the tame locally abelian parabolic Higgs bundle $(\mathbf{H}_\bullet, \varphi)$, where

$$\mathbf{H}_{\boldsymbol{\alpha}} = \mathbb{R}^k f_* \left(\Omega^\bullet_{X/Y} \left(W_{-2(\bullet^{\text{ver}})} \mathbf{E}_{0^{\text{ver}}, \boldsymbol{\alpha}^{\text{hor}}} \right), \theta \right).$$

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- Take the monodromy weight-filtration for N and pull it back to $\mathbf{E_c}$ under the natural map $\mathbf{E_c} \to \left(\imath_{H_i *} \oplus {}^i\mathrm{gr}_a\, \mathbf{E_c}\right)$.

Global cohomology

The compatibility of **nah** with pushforwards follows from the higher order Kähler identities: **the D and D'' Laplacians on a harmonic bundle are proportional**.

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- $(E, \theta) = \mathbf{nah}_X(V, D)$ is the corresponding Higgs bundle,

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General case: There non-abelian Hodge correspondence on Y gives a canonical quasi-isomorphism of de Rham and Dolbeault pushforwards:

$$\boxed{\mathbf{nah}_Y : \left(\mathbb{R}^\bullet f_*(\Omega^\bullet_{X/Y}(V), D), \nabla\right) \xrightarrow{\cong} \left(\mathbb{R}^\bullet f_*(\Omega^\bullet_{X/Y}(E), \theta\wedge), \varphi\wedge\right)}$$

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\begin{tikzcd}
 & \mathrm{tot}(f^* T^\vee Y) \arrow[dl, "q"'] \arrow[dr, "g"] & \\
\mathrm{tot}(T^\vee X) & & \mathrm{tot}(T^\vee Y)
\end{tikzcd}
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Here $\omega_g = p^* \Omega^n_{X/Y}$ is the relative dualizing sheaf of g.

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The **stratified Mukai flop** is the birational transformation between the two natural crepant resolutions of X:

$$\boxed{\begin{tikzcd} X' \arrow[rr, dashed] \arrow[dr, "\varepsilon'"'] & & X'' \arrow[dl, "\varepsilon''"] \\ & X & \end{tikzcd}}$$

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Kashiwara proved that if $U \subset Gr(r, V) \times Gr(n-r, V)$ is the subvariety of pairs of transversal subspaces, then the integral transform

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Conjecture: **[Kashiwara]** The complex of Higgs sheaves $\mathbf{nah}(i_{U!}\mathbb{C}_U)$ viewed as a complex of coherent sheaves on $T^\vee Gr(r, V) \times T^\vee Gr(n-r, V) = X' \times X''$ is a Fourier-Mukai kernel supported on $X' \times_X X''$ and giving the D-equivalence of X' and X''.

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Definition: A **torsion free parabolic sheaf** on (X, D) is a collection of torsion free coherent sheaves $\{\mathcal{E}_\alpha\}_{\boldsymbol{\alpha}\in\mathbb{R}^s}$ together with inclusions $\mathcal{E}_{\boldsymbol{\alpha}} \subset \mathcal{E}_{\boldsymbol{\beta}}$ of sheaves of $\mathcal{O}_X$-modules, specified for all $\alpha \leqslant \beta$, satisfying the conditions:

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[support] if $\boldsymbol{\delta}_i : S \to \mathbb{R}$ is the characteristic function of i, then for all $\boldsymbol{\alpha} \in \mathbb{R}^S$ we have $\mathcal{E}_{\boldsymbol{\alpha}+\boldsymbol{\delta}_i} = \mathcal{E}_{\boldsymbol{\alpha}}(D_i)$ (compatibly with the inclusion).

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Note: The sheaf $\mathbf{E}_{\mathbf{c}}$ together with the flags $\{{}^iF_a |\ i \in S, a \in \mathrm{weights}(\mathbf{E}_{\mathbf{c}}, i)\}$ reconstruct the parabolic sheaf $\mathbf{E}_\bullet$.

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Example: A **parabolic line bundle** is a parabolic sheaf $F_{\bullet}$ for which all sheaves F_{α} are invertible.

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Claim: Every parabolic line bundle $F_\bullet$ is isomorphic to $L \otimes \mathcal{O}_X\left(\sum_{i \in S} \mathbf{a}_i D_i\right)_\bullet$ for some $L \in \mathrm{Pic}(X)$, and some $\mathbf{a} \in \mathbb{R}^S$.

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Definition: A parabolic sheaf $F_\bullet$ is a **locally abelian bundle**, if in a Zariski neighborhood of any point $x \in X$ there is an isomorphism between $F_\bullet$ and a direct sum of parabolic line bundles.

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Variant: We can define similarly locally abelian parabolic local systems, Higgs bundles, or more generally locally abelian parabolic λ-connections.

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Note:

(flat 1-connection) = (flat connection with regular singularities)

(flat 0-connection) = (Higgs bundle with logarithmic poles)

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A tame parabolic λ-connection $(E_\bullet, \mathbb{D}^\lambda)$ is locally abelian if in a Zariski neighborhood of any point $x \in X$ there is an ismomorphism between $(E_\bullet, \mathbb{D}^\lambda)$ and direct sum of rank one tame parabolic λ-connections.

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[support] $\Rightarrow$ $\mathcal{E}_\bullet$ is effectively reconstructed by any truncation ${}_{\mathbf{c}}E$. In fact: the numerator of the Iyer-Simpson formula is independent of the choice of truncation.

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$$\boxed{\mathrm{parc}_1 = c_1(\mathbf{E_c}) - \sum_{i \in S} \left(\sum_{a \in \mathrm{weights}(\mathbf{E_c}, i)} a \, \mathrm{rank}\, {}^{i}\mathrm{gr}_a \, \mathbf{E_c} \right) \cdot D_i}$$

Weights and residues (i)

A parabolic λ-connection $(\mathbf{E}_\bullet, \mathbb{D}^\lambda)$ has a collection of numerical invariants which are most conveniently packaged in the so called **KMS spectrum**.

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Kashiwara-Malgrange-Sabbah-Simpson

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A parabolic λ-connection $(\mathbf{E}_\bullet, \mathbb{D}^\lambda)$ has a collection of numerical invariants which are most conveniently packaged in the so called **KMS spectrum**. By definition:

$$\mathrm{KMS}((\mathbf{E}_\bullet, \mathbb{D}^\lambda), i) := \bigcup_{\mathbf{c}} \mathrm{KMS}((\mathbf{E}_\mathbf{c}, \mathbb{D}^\lambda), i) \subset \mathbb{R} \times \mathbb{C}$$

$$\mathrm{KMS}((\mathbf{E}_\mathbf{c}, \mathbb{D}^\lambda), i) := \left\{ (a, \alpha) \,\middle|\, \begin{array}{l} a \in \mathrm{weights}(\mathbf{E}_\mathbf{c}, i), \ \alpha \text{ is} \\ \text{an eigenvalue of } {}^i\mathrm{gr}_a \mathbb{D}^\lambda \text{ on} \\ {}^i\mathrm{gr}_a \mathbf{E}_\mathbf{c} \end{array} \right\}$$

Weights and residues (ii)

A tame harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$ gives rise to a twistor family of parabolic λ-connections $(\mathbf{E}_\bullet^\lambda, \mathbb{D}^\lambda)$ parametrized by $\lambda \in \mathbb{C}$.

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Theorem: **[Mochizuki,Simpson]** For any twistor family $(\mathbf{E}^{\lambda}_{\bullet}, \mathbb{D}^{\lambda})$ of parabolic λ-connections, the map

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{C} & \longrightarrow & \mathbb{R} \times \mathbb{C} \\ (a, \alpha) & \longmapsto & (a + 2\,\mathrm{Re}(\lambda \cdot \bar{\alpha}), \alpha - a \cdot \lambda - \bar{\alpha} \cdot \lambda^2) \end{array}$$

identifies $\mathrm{KMS}((\mathbf{E}^{0}_{\bullet}, \mathbb{D}^{0}), i)$ with $\mathrm{KMS}((\mathbf{E}^{\lambda}_{\bullet}, \mathbb{D}^{\lambda}), i)$, and preserves multiplicities.

Weights and residues (iii)

Thus if a parabolic local system $(\mathbf{V}_\bullet, \nabla)$ corresponds to a parabolic Higgs bundle $(\mathbf{E}_\bullet, \theta)$ under $\mathbf{nah}_{X,D}$, then we have a matching:

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$(\mathbf{E}_\bullet, \theta)$	$(\mathbf{V}_\bullet, \nabla)$
a - parabolic weight along D_i	$b = a + 2\,\mathrm{Re}(\alpha)$ - parabolic weight along D_i
α - eigenvalue of ${}^i\mathrm{gr}_a \operatorname{Res}_{D_i} \theta$	$\beta = -a + \sqrt{-1} \cdot 2\,\mathrm{Im}(\alpha)$ - eigenvalue of ${}^i\mathrm{gr}_a \operatorname{Res}_{D_i} \nabla$

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- $N : \mathrm{gr}^W_{k+j} \to \mathrm{gr}^W_{k-j}$ is an isomorphism.

△