Graded matrix factorizations and functor categories

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September 8, 2011

Attributions

Based on joint work with Matthew Ballard (Upenn) and Ludmil Katzarkov (Miami and Wien).

Hodge diamond of cubic fourfold and a $K3$ surface

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

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\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & \color{green}{1} & & 0 & & \\
 & 0 & & \color{green}{0} & & \color{green}{0} & & 0 & \\
0 & & \color{green}{1} & & \color{green}{20} & & \color{green}{1} & & 0 \\
 & 0 & & \color{green}{0} & & \color{green}{0} & & 0 & \\
 & & 0 & & \color{green}{1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Definition

A semi-orthogonal decomposition of a triangulated category, $\mathcal{T}$, is a sequence of full triangulated subcategories, $\mathcal{A}_1, \ldots, \mathcal{A}_m$, in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^{\perp}$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a diagram:

$$
\begin{tikzcd}
0 \arrow[rr] & & T_{m-1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & T_2 \arrow[rr] & & T_1 \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
& A_m \arrow[ul, "|" marking] & & & & A_2 \arrow[ul, "|" marking] & & A_1 \arrow[ul, "|" marking] &
\end{tikzcd}
$$

where all triangles are distinguished and $A_k \in \mathcal{A}_k$. We shall denote a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$.

Theorem (Orlov)

Let X be a hypersurface in $\mathbb{P}^n$ which is the zero locus of a homogeneous polynomial, f, of degree, d.

1. If $n+1-d > 0$, there is a semi-orthogonal decomposition,
$$\mathrm{D^b}(\operatorname{coh} X) = \langle \mathcal{O}_X(d-n), ..., \mathcal{O}_X, \mathrm{MF}(R, f, \mathbb{Z}) \rangle.$$
2. If $n+1-d = 0$, there is an equivalence of triangulated categories,
$$\mathrm{D^b}(\operatorname{coh} X) = \langle \mathrm{MF}(R, f, \mathbb{Z}) \rangle.$$
3. If $n+1-d < 0$, there is a semi-orthogonal decomposition,
$$\mathrm{MF}(R, f, \mathbb{Z}) \cong \left\langle k, \ldots, k(n+2-d), \mathrm{D^b}(\operatorname{coh} X) \right\rangle.$$

Example

Consider a cubic 4-fold, X, defined by f. Orlov's theorem gives:

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) = \langle \mathrm{MF}(\mathbb{C}[x_0, \ldots, x_6], f, \mathbb{Z}), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle.$$

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Work of Kuznetsov shows that when f is Pfaffian then $\mathrm{MF}(\mathbb{C}[x_0, \ldots, x_6], f, \mathbb{Z})$ is equivalent to the derived category of a $K3$ surface and when X contains a plane, then $\mathrm{MF}(\mathbb{C}[x_0, \ldots, x_6], f, \mathbb{Z})$ is equivalent to the derived category of a twisted $K3$ surface.

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So when f is Pfaffian, we have a $K3$ surface, Y, and:

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In Hochschild homology, given a semi-orthogonal decomposition, we get a splitting:

$$HH_*(X) = HH_*(Y) \oplus HH_*(\langle \mathcal{O} \rangle) \oplus HH_*(\langle \mathcal{O}(1) \rangle) \oplus HH_*(\langle \mathcal{O}(2) \rangle)$$

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The last three things are just the chern characters of these vector bundles, telling us that $HH_*(Y)$ is the orthogonal to these three chern characters, after making the appropriate identifications using the Hochschild-Kostant-Rosenberg isomorphism.

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When, the cubic potential is a sum of cubic potentials in 3 variables, $f(x, y, z) + g(u, v, w)$, this noncommutative $K3$ surface, can be realized as a $\mathbb{Z}_3$ quotient of the product of the two elliptic curves defined by f and g. Let us state the general theorem.

Theorem

Let M, M' be finitely generated abelian groups. Let $R = k[x_0, ..., x_n], R' = k[y_0, ..., y_{n'}]$ be M, M' graded rings with x_i, y_i homogeneous. Let $f \in R_d, f' \in R_{d'}$ be homogeneous functions such that $f \in df, f' \in df'$ and $d \in M, d' \in M'$ are not torsion. The full sub(dg)category of compact objects in the category of functors from $\mathrm{MF}(R, f, M)$ to $\mathrm{MF}(R', f', M')$ is equivalent to $\mathrm{MF}(R \otimes R', f \otimes 1 - 1 \otimes f', M \oplus M'/(d, -d'))$.

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Remark

This is a graded version of a result of Dyckerhoff. Independently, Polishchuk and Vaintrob prove this theorem in the case where the singularities are isolated and $M \otimes_{\mathbb{Z}} \mathbb{Q}, M' \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

Consider a collection of hypersurfaces, $X_i \subseteq \mathbb{P}^{n_i}$ defined by polynomials f_i of degree d_i for $1 \leq i \leq s$. Let R_i be the coordinate rings of the $\mathbb{P}^{n_i}$. Consider the free abelian group of rank s, $\mathbb{Z}^s$, with basis $\mathbf{e}_i$, $1 \leq i \leq s$. Let L be the subgroup generated by $d_i\mathbf{e}_i = d_j\mathbf{e}_j$ and $M := \mathbb{Z}^s / L$. Denote by H the torsion subgroup of M. Explicitly, letting d_{ij} be the greatest common divisor of d_i and d_j, H is the finite subgroup of M generated by the images of $\frac{d_i}{d_{ij}}\mathbf{e}_i - \frac{d_j}{d_{ij}}\mathbf{e}_j$. One has $M/H \cong \mathbb{Z}$. Let m be the least common multiple of the d_i. In this setting the degree map $\deg : M \to \mathbb{Z}$ can be identified with the mapping which takes $\mathbf{e}_i$ to $\frac{d}{d_i}$. Let δ be an element of degree 1.

The dual group to M can be identified with the set, $D := \{(\lambda_1, ..., \lambda_s) | \lambda_i^{d_i} = \lambda_j^{d_j} \forall i,j\} \subseteq (k^*)^s$ and acts on $\mathbb{A}^{n_1+\dots n_s+s} \backslash 0$ by multiplication by λ_i on the coordinates, $x_{d_1+\dots+d_{i-1}}$ through $x_{d_1+\dots+d_i}$. Let Y denote the hypersurface in $\mathbb{A}^{n_1+\dots n_s+s} \backslash 0$ defined by the zero locus of $f_1 + \dots f_s$ and consider the global quotient stack, $Z := [Y/D]$.

Theorem (Orlov)

Let $\mathcal{A} = \mathrm{MF}(R_1 \otimes \ldots \otimes R_s, f_1 + \ldots + f_s, M)$. (which by our theorem is equivalent to $(\mathrm{MF}(R_1, f_1, \mathbb{Z}) \hat{\otimes}_k \ldots \hat{\otimes}_k \mathrm{MF}(R_s, f_s, \mathbb{Z}))_{\mathrm{pe}}$).

1. If $a > 0$, there is a semi-orthogonal decomposition,
$$\mathrm{D^b}(\mathrm{coh}\, Z) \cong \langle \bigoplus_{h \in H} \mathcal{O}_Z((-a+1)\delta h), \ldots, \bigoplus_{h \in H} \mathcal{O}_Z(h), \mathcal{A} \rangle.$$
2. If $a = 0$, there is an equivalence of triangulated categories,
$$\mathrm{D^b}(\mathrm{coh}\, Z) \cong \mathcal{A}.$$
3. If $a < 0$, there is a semi-orthogonal decomposition,
$$\mathcal{A} \cong \langle \bigoplus_{h \in H} k(h), \ldots, \bigoplus_{h \in H} k((a+1)\delta h), \mathrm{D^b}(\mathrm{coh}\, Z) \rangle.$$

Example

In the simple case of one variable, Orlov's theorem in the context of algebras yields an equivalence between $\mathrm{MF}(k[x], x^d, \mathbb{Z})$ and $\mathrm{D^b}(A_{d-1})$. Therefore,

$$
\begin{aligned}
&(\mathrm{MF}(k[x], x^p, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(k[y], y^q, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(k[z], z^r, \mathbb{Z}))_{\mathrm{pe}} \\
\cong &(\mathrm{D^b}(A_{p-1}) \hat{\otimes}_k \mathrm{D^b}(A_{q-1}) \hat{\otimes}_k \mathrm{D^b}(A_{r-1}))_{\mathrm{pe}} \\
\cong &\, \mathrm{D^b}(A_{p-1} \otimes_k A_{q-1} \otimes_k A_{r-1}).
\end{aligned}
$$

Applying Orlov's theorem, this category is compared via a semi-orthogonal decomposition to the weighted projective line with weight sequence (p, q, r) (the parameter is $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$).

Example

Let $f(x,y,z) = x(x-z)(x-\lambda z) - zy^2$ and $g(u,v,w) = u(u-w)(u-\gamma w) - wv^2$ define two smooth elliptic curves, E and F respectively. Then $f+g$ defines a smooth cubic fourfold containing at least three planes by setting $z = w = 0$. By work of Kuznetsov, the category $\mathrm{MF}(k[x,y,z,u,v,w], f+g, \mathbb{Z})$ is, in this case, equivalent to the derived category of a certain gerby $K3$ surface, Y. On the other hand, letting $M = \mathbb{Z} \oplus \mathbb{Z}/(3,-3)$ with x,y,z in degree $(1,0)$ and u,v,w in degree $(0,1)$, we have $\mathrm{MF}(k[x,y,z,u,v,w], f+g, M) \cong (\mathrm{MF}(k[x,y,z], f, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(k[u,v,w], g, \mathbb{Z}))_{\mathrm{pe}}$. From Orlov, we have $\mathrm{MF}(k[x,y,z], f, \mathbb{Z}) \cong \mathrm{D^b}(\mathrm{coh}\, E)$ and $\mathrm{MF}(k[u,v,w], g, \mathbb{Z}) \cong \mathrm{D^b}(\mathrm{coh}\, F)$. Hence $\mathrm{MF}(k[x,y,z,u,v,w], f+g, M) \cong \mathrm{D^b}(\mathrm{coh}\, E \times_k F)$. In this way, $\mathrm{D^b}(\mathrm{coh}\, E \times_k F)$ is a $\mathbb{Z}_3$-cover of $\mathrm{D^b}(\mathrm{coh}\, Y)$ (this can be made precise using orbit categories).

Remark

Furthermore, on each elliptic curve, E, F the autoequivalence (1) is a composition of Dehn twists. Hence this autoequivalence can be viewed as a symplectic automorphism of the mirror. The action of $\mathbb{Z}_3$ on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} E \times_k F)$ is given by $(1, -1)$. This can therefore be considered as a product of sympletic automorphisms of the product of the two mirrors. The relationship between the surfaces $E \times_k F$ and Y can then be seen by viewing the mirror of $E \times_k F$ as a three to one symplectic cover of the mirror of Y.

Relating cycles directly

Theorem

Let Y be a $K3$ surface such that $\mathrm{D^b}(\operatorname{coh} Y)$ is equivalent to the $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle^{\perp}$ of the Fermat cubic fourfold. The Hodge conjecture over $\mathbb{Q}$ holds for n-fold products of Y.

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Idea of the proof:

Similarly to the previous example, due to Orlov's theorem and results of Kuznetsov, $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y) \cong \mathrm{MF}(k[x_0, \ldots, x_5], x_0^3 + \ldots + x_5^3, \mathbb{Z})$. Therefore by our theorem, $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y^n)$ is a $\mathbb{Z}_3^{n-1}$-cover of $\mathrm{MF}(k[x_0, \ldots, x_{6n-1}, x_0^3 + \ldots + x_{6n-1}^3, \mathbb{Z})$. On any Fermat of prime degree, a basis for the cycles can be obtained by considering all partitions of the variables into sets of cardinality two and reducing the associated grading back to $\mathbb{Z}$. The cycles which are not obtained by induction on the $K3$ are the invariant ones under the $\mathbb{Z}_3^{n-1}$-action.

Generation

Fix a triangulated category, $\mathcal{T}$, and a subcategory $\mathcal{I}$. Four ways to build new objects from old:

- Take X and shift by i to get $X[i]$.
- Take a (finite) coproduct of copies of X, $\bigoplus_{i \in I} X$.
- If $X \cong Y \oplus Z$, then split off Y and Z.
- From a morphism, $a : X \to Y$, take the cone $C(a)$, with $Y \in \mathcal{I}$.

We denote by $\langle \mathcal{I} \rangle_n$ the smallest full subcategory of $\mathcal{T}$ which is closed under shifts, sums, summands, and taking at most n-cones.
We say an object, G, is a **generator** of $\mathcal{T}$ if the smallest full subcategory containing G and closed under these operations is equivalent to $\mathcal{T}$. We say that G is a **strong generator** if $\langle G \rangle_n$ is equivalent to $\mathcal{T}$ for some n. The **generation time** of $\mathcal{I}$ is the minimal n for which $\langle G \rangle_n \cong \mathcal{T}$; it is denoted by $\clock(G)$.

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Definition

The Orlov spectrum of a triangulated category $\mathcal{T}$ is the set of all generation times of all strong generators.

Definition

The Rouquier dimension of a triangulated category $\mathcal{T}$ is the minimum of the Orlov spectrum i.e. the minimal generation time achieved by a strong generator.

Theorem (Rouquier)

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Theorem (Orlov)

The above conjecture holds for smooth curves. More generally, if C is a smooth curve, then the spectrum of $\mathrm{D}^{\mathrm{b}}(C)$ contains $\{1, 2\}$ with equality if and only if $C = \mathbb{P}^1$.

Proposition

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Example

Let $(d_0, .., d_n)$ be a weight sequence with $\sum_{i=1}^{s} \frac{1}{d_i} \leq 1$ containing either $\{2\}$, $\{3,3\}$, $\{3,4\}$, or $\{3,5\}$. Let k be a field whose characteristic does not divide any of the d_i then Orlov's Conjecture holds for the weighted fermat hypersurface defined by f. Similarly, the Rouquier dimension of $\mathrm{D}^{\mathrm{b}}(A_{d_0-1} \otimes ... \otimes A_{d_n-1})$ is equal to $n-2$. In general, the upper bound on the product category is $n-1$ and is achieved when $\sum_{i=1}^{s} \frac{1}{d_i} \leq \frac{1}{2}$.

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Example

Orlov's conjecture holds for the product, $E \times E$ of two Fermat elliptic curves and the $K3$ surface obtained as a $\mathbb{Z}_3$ quotient and other similar examples.

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Let f be a homogeneous polynomial of degree d defining a smooth projective hypersurface X in projective space. Consider the category $\mathrm{MF}(R, f, \mathbb{Z})$, here R is the coordinate ring of projective space. For any $r \in R_i$ and $B \in \mathrm{MF}(R, f, \mathbb{Z})$ we have a morphism $r : B \to B(i)$. Let I be the homogeneous ideal which annihilates all $B \in \mathrm{MF}(R, f, \mathbb{Z})$. We define the scheme theoretic support of B as the ideal in R/I which annihilates B. Note that R/I is a quotient of the Jacobian ring.

Roughly, the image of the Chern character map can be identified with polynomials in R/I of degree $d(n-1)+d-n-1$.

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Roughly, the image of the Chern character map can be identified with polynomials in R/I of degree $d(n-1) + d - n - 1$.

Theorem

Let X be a smooth hypersurface in $\mathbb{P}^n$, defined by a homogeneous polynomial, f, of degree d, $k \in \mathrm{MF}(R, f, \mathbb{Z})$ be the residue field, and I be the ideal of polynomials in J_f which are homotopic to zero for all matrix factorizations in $\mathrm{MF}(R, f, \mathbb{Z})$. For any homogeneous ideal $J \subseteq R/I$, the generation time of $\mathrm{MF}(J)$, the category of $\mathbb{Z}$-graded matrix factorizations scheme theoretically supported on J, is bounded below by one less than the nilpotent order of J in R/I with equality if J is principal.

Let $\mathfrak{m}$ be the maximal ideal in R/I. One can show that $\mathrm{MF}(\mathfrak{m})$ is equal to the additive category spanned by all internal and homological shifts of the residue field.

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Corollary

Let X be a smooth hypersurface of even dimension in $\mathbb{P}^n$. Suppose the algebraic classes form a full sublattice of $\mathrm{H}^{\frac{n-1}{2},\frac{n-1}{2}}(X,\mathbb{C})$ i.e. there is a basis of the lattice which forms a $\mathbb{C}$-basis of $\mathrm{H}^{\frac{n-1}{2},\frac{n-1}{2}}(X,\mathbb{C})$. For any ideal $J \subseteq J_f$, generated by homogeneous polynomials of degree i, the generation time of $\mathrm{MF}(J)$ is bounded below by $\lfloor \frac{(n+1)(d-2)}{2i} \rfloor$ in both $\mathrm{MF}(R,f)$ and $\mathrm{MF}(R,f,\mathbb{Z})$ with equality when J is principal.

Corollary

If $\text{⌚}(\bigoplus_{i=0}^{d-1} k(i)) < \frac{(n+1)(d-2)}{2}$, then X has no primitive algebraic classes.

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Remark

If M is any finite generated abelian group of rank 1, and R is M-graded with the usual $\mathbb{Z}$-grading a quotient of M with $f \in R_d$. We have

$$\text{⏱}(\bigoplus_{i=0}^{d-1} k(i)) = \text{⏱}(\bigoplus_{m \in M/d} k(m))$$

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Questions

Does equality hold for all ideals? Is the converse to the corollary true?