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Galois descent on the Brauer group of varieties

(Joint work with A. N. Skorobogatov)

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For any scheme Y, let $\mathrm{Pic}(Y) \simeq H^1_{ét}(Y, \mathbb{G}_m)$ be its Picard group and let $\mathrm{Br}(Y) = H^2_{ét}(X, \mathbb{G}_m)$ be its Brauer group. These are abelian groups.

Let k be a field, $\overline{k}$ be a separable closure of k and $G = \mathrm{Gal}(\overline{k}/k)$. Let X be a smooth, projective, geometrically integral variety over k and $\overline{X} = X \times_k \overline{k}$.

We have a natural map

$$\alpha : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^G.$$

The kernel of α, denoted $\mathrm{Br}_1(X)$, is called the "algebraic Brauer group" of X.
The image of α, i.e. the quotient $\mathrm{Br}(X)/\mathrm{Br}_1(X)$, is called the "transcendental Brauer group" of X and denoted $\mathrm{Br}_{tr}(X)$.

Main Theorem. *Assume char(k)=0 and X/k smooth, projective, and geometrically integral. The cokernel of α, in other words the quotient $\mathrm{Br}(\overline{X})^G/\mathrm{Br}_{tr}(X)$, is finite.*

Reminders from Grothendieck's *Le groupe de Brauer, I,II,III.*

The Brauer group $\mathrm{Br}(Y)$ of a *smooth* k-variety Y/k is a torsion group.
If k is algebraically closed, for any prime ℓ, the ℓ-primary component of $\mathrm{Br}(Y)$ is a group of cofinite type, extension of a finite group by a finite sum of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$.

To prove the theorem, it is thus enough to prove that *the exponent of Coker*(α) *is finite.*
A corestriction argument shows that to prove this one may allow for a finite field extension of k, hence one may assume $X(k) \neq \emptyset$.

Let X/k be proper, geometrically integral.
The first terms in the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G, H^q_{ét}(\overline{X}, \mathbb{G}_m)) \Longrightarrow H^n_{ét}(X, \mathbb{G}_m)$$

gives a well known seven term exact sequence which if $X(k) \neq \emptyset$ gives

$\mathrm{Pic}(X) \simeq \mathrm{Pic}(\overline{X})^G$ (Galois descent for the Picard group)

and

$\mathrm{Br}_1(X)/\mathrm{Br}(k) \simeq H^1(G, \mathrm{Pic}(\overline{X}))$.

Simple and key observation :

Looking further in the spectral sequence gives a complex

$$\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(\overline{X})^G \to H^2(G, \mathrm{Pic}(\overline{X}))$$

which is an exact sequence as soon as $X(k) \neq \emptyset$ or $H^3(k, \mathbb{G}_m) = 0$ (which holds if k is a number field.)

That sequence, which we shall here call *the basic exact sequence*, is functorial in X.

FIRST PROOF of main theorem, via the Grothendieck-Tsen theorem for curves

The key idea is to consider the restriction of the basic exact sequence to a suitable, finite set of curves.

If $C \to X$ is a k-morphism from a smooth, geometrically integral curve, then since $\mathrm{Br}(\overline{C}) = 0$ (Grothendieck-Tsen theorem), the image of $Coker(\alpha)$ in $H^2(G, \mathrm{Pic}(\overline{X}))$ lies in the kernel of the restriction map

$$H^2(G, \mathrm{Pic}(\overline{X})) \to H^2(G, \mathrm{Pic}(\overline{C})),$$

hence also in the kernel of the composite map

$$H^2(G, \mathrm{Pic}(\overline{X})) \to H^2(G, \mathrm{Pic}(\overline{C})) \to H^2(G, \mathbb{Z})$$

where the last map is induced by the degree $\mathrm{Pic}(\overline{C}) \to \mathbb{Z}$.

After replacing k by a finite extension one may assume
(a) $X(k) \neq \emptyset$, hence $Coker(\alpha) \hookrightarrow H^2(G, \mathrm{Pic}(\overline{X}))$.
(b) The Galois group G acts trivially on the finitely generated Néron-Severi group $NS(\overline{X}) = Num^1(\overline{X})$, hence also on the finitely generated group $Num_1(\overline{X})$.
(c) There exists a finite set of smooth, geometrically integral k-curves $C_i, i = 1, \ldots, n$ with k-morphisms $C_i \to X$ whose images generate the group $Num_1(\overline{X})$.
(d) (Bertini) There exists a smooth, geometrically integral linear curve section $C_0 \subset X$, which then (Lefschetz) satisfies $H^1_{ét}(\overline{X}, \mathbb{Q}/\mathbb{Z}) \hookrightarrow H^1_{ét}(\overline{C}_0, \mathbb{Q}/\mathbb{Z})$.

The maps $C_i \to X$ induce a surjective, G-split map $\mathbb{Z}^n \to Num_1(\overline{X})$.
The following maps

$$NS(\overline{X}) \to Num^1(\overline{X}) \to Hom(Num_1(\overline{X}), \mathbb{Z})) \to \bigoplus_{i=1}^{n} \mathbb{Z}$$

each have kernel killed by a positive integer.
There thus exists a positive integer n_0 which sends the kernel of

$$H^2(G, \mathrm{Pic}(\overline{X})) \to \bigoplus_{i=1}^{n} H^2(G, \mathbb{Z}))$$

to the kernel of

$$H^2(G, \mathrm{Pic}(\overline{X})) \to H^2(G, NS(\overline{X})).$$

Upon using the exact sequence

$$0 \to \mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to NS(\overline{X}) \to 0$$

one is reduced to showing that the kernel of

$$H^2(G, \mathrm{Pic}^0(\overline{X})) \to H^2(G, \mathrm{Pic}(\overline{C}_0))$$

is of finite exponent. This follows from the fact that the map

$$\mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}^0(\overline{C}_0)$$

is injective (Lefschetz) and from Poincaré complete reducibility theorem, which then gives a near-retraction at the level of abelian varieties.

The method just described yields

a bound for the exponent of Coker(α).

If $X(k) \neq \emptyset$, a bound is the product of the following integers :
(a) The exponent ν of the torsion subgroup of $NS(\overline{X})$.
(b) The exponent δ of the finite cokernel of the intersection map

$$Num^1(\overline{X}) \to Hom(Num_1(\overline{X}), \mathbb{Z}).$$

(c) The degree of a "splitting field" for $Num_1(\overline{X})$.
(d) The integer appearing in a near retraction for $H^2(k, \mathrm{Pic}^0(\overline{X})) \to H^2(k, \mathrm{Pic}^0(\overline{C}_0))$. This is no problem if $H^2(k, \mathrm{Pic}^0(X)) = 0$, which is the case if either $H^1(X, O_X) = 0$ or k is a totally imaginary number field.

SECOND PROOF of main theorem, via comparison of numerical and homological equivalence, together with some homological algebra

The surface case

Let X/k be a smooth, projective, geometrically connected surface such that $NS(\overline{X})$ is torsionfree. This last hypothesis implies that $H^3(\overline{X}, \mathbb{Z})_{tors} = 0$, hence that $\mathrm{Br}(\overline{X})$ is divisible.
Intersection on $NS(\overline{X})$ gives an exact sequence

$$0 \to NS(\overline{X}) \to Hom(NS(\overline{X}), \mathbb{Z}) \to D \to 0.$$

Let δ be the exponent of D. For a $\mathbb{Z}_\ell$–module M denote $M^0 = Hom_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell)$. From the commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & NS(\overline{X}) \otimes \mathbb{Z}_\ell & \to & H^2(\overline{X}, \mathbb{Z}_\ell(1)) & \to & T_\ell(\mathrm{Br}(\overline{X})) & \to & 0 \\
 & & \downarrow & & \downarrow \simeq & & & & \\
0 & \to & [NS(\overline{X}) \otimes \mathbb{Z}_\ell]^0 & \leftarrow & [H^2(\overline{X}, \mathbb{Z}_\ell(1))]^0 & & & & \\
 & & \downarrow & & & & & & \\
 & & D\{\ell\} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\tag{1}
$$

where the Tate module $T_\ell(\mathrm{Br}(\overline{X}))$ of $\mathrm{Br}(\overline{X})$ is a torsionfree group, we get a G-equivariant map $\sigma : H^2(\overline{X}, \mathbb{Z}_\ell(1)) \to NS(\overline{X}) \otimes \mathbb{Z}_\ell$ such that the composite with $NS(\overline{X}) \otimes \mathbb{Z}_\ell \to H^2(\overline{X}, \mathbb{Z}_\ell(1))$ is multiplication by δ.

Fir each ℓ, we tensor the upper line of (1) by $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and take direct sums. This gives an exact sequence equipped with the near-retraction σ.

$$0 \to NS(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(\overline{X}, \mathbb{Q}/\mathbb{Z}(1)) \to \mathrm{Br}(\overline{X}) \to 0$$

and then a 2-extension

$$0 \to NS(\overline{X}) \to NS(\overline{X}) \otimes \mathbb{Q} \to H^2(\overline{X}, \mathbb{Q}/\mathbb{Z}(1)) \to \mathrm{Br}(\overline{X}) \to 0.$$

From this 2-extension we get a map

$$\beta : \mathrm{Br}(\overline{X})^G \to H^2(G, NS(\overline{X}))$$

which is annihilated by multiplication by δ.
Expected but nontrivial CLAIM:

Up to a sign, the map β coincides with the composite of the map $\mathrm{Br}(\overline{X})^G \to H^2(G, \mathrm{Pic}(\overline{X}))$ (in the Hochschild-Serre spectral sequence) with the natural map $H^2(G, \mathrm{Pic}(\overline{X})) \to H^2(G, NS(\overline{X}))$.

Assume either $X(k) \neq \emptyset$ or $H^3(k, \mathbb{G}_m) = 0$.
One then concludes that $\delta.Coker(\alpha) \subset H^2(G, \mathrm{Pic}(\overline{X}))$ is in the image of $H^2(G, \mathrm{Pic}^0(\overline{X}))$.
Hence if $H^1(X, O_X) = 0$ or if k is a totally imaginary number field, then

$$\delta.Coker(\alpha) = 0.$$

In the bound, there is no mention any more of the degree of a splitting field for $NS(\overline{X})$. If one allows for torsion in $NS(\overline{X})$, and lets ν denote the exponent of the torsion subgroup, the method gives

$$\delta.\nu^2.Coker(\alpha) = 0.$$

In the above proof for surfaces, one uses Matusaka's theorem that numerical and homological equivalence coincide on divisors.
When extending this second proof to higher dimension, one needs both this statement of Matsusaka for divisors and the analogous statement on numerical and homological equivalence, for 1-cycles.
This last statement fortunately is known, this is a theorem of Lieberman (1968) for which an algebraic proof is available (Kleiman).

For X/k be of arbitrary dimension d, one issue one has to deal with is that the image of the étale cycle map

$$CH_1(\overline{X}) \otimes \mathbb{Z}_\ell \to H^{2d-2}_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(d-1))$$

need not be saturated.

Let ν be the exponent of $NS(\overline{X})_{tors}$, and let γ be the exponent of $H^3(\overline{X}, \mathbb{Z})_{tors}$. (For surfaces, $\gamma = \nu$.)
If either $X(k) \neq \emptyset$ or $H^3(k, \mathbb{G}_m) = 0$, a suitable extension of the argument given above for surfaces yields:

The subgroup $\delta.\nu.\gamma.Coker(\alpha) \subset H^2(G, \mathrm{Pic}(\overline{X}))$ lies in the image of $H^2(G, \mathrm{Pic}^0(\overline{X}))$.

When $H^1(X, O_X) = 0$ or k is a totally imaginary number field, one then gets

$$\delta.\nu.\gamma.Coker(\alpha) = 0.$$

$K3$-**surfaces**

Such a surface X has $H^1(X, O_X) = 0$ and it has no torsion in its Néron-Severi group. Over a number field, one thus finds $\delta.Coker(\alpha) = 0$. A formal argument then shows that the order of $Coker(\alpha)$ divides $\delta^{b_2 - \rho}$.

For a diagonal quartic surface in $\mathbb{P}^3_k$, the second proof shows that any element of odd order in $\mathrm{Br}(\overline{X})^G$ is in the image of $\mathrm{Br}(X)$. More precisely, $Coker(\alpha)$ is a subgroup of $(\mathbb{Z}/8)^2$.

Product of two curves

Following the first proof, one gets the following result.

Let $X = C_1 \times C_2$ be the product of two curves with a rational point. Assume that over $\overline{k}$ here is no nontrivial homomorphism between the jacobians of the two curves.
Then $\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(\overline{X})^G$ is onto.

There exist many elliptic curves $E/\mathbb{Q}$ such that for $X = E \times E$ the map $\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(\overline{X})^G$ is not onto (Skorobogatov and Zarhin).

T. Szamuely has asked us whether the main theorem extends to smooth open varieties: if X/k is a smooth, geometrically integral variety over a field k of char. zero, is $\mathrm{Br}(\overline{X})^G/Im(\mathrm{Br}(X))$ finite ?

We could prove this for k of finite type over $\mathbb{Q}$.

Motivation for the study of $\mathrm{Br}(X)$ *and of* $\mathrm{Br}(X)/\mathrm{Br}_1(X) \subset \mathrm{Br}(\overline{X})$.

The Brauer-Manin set

Let X/k be a variety over a number field.
Let $X(\mathbb{A}_k)$ be the space of adèles of X with its usual topology and let $X(\mathbb{A}_k)_\bullet$ be the analogous space where the connected components at infinity have each been collapsed to one point.
Class field theory induces a pairing

$$X(\mathbb{A}_k)_\bullet \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z},$$

and implies that the closure of the diagonal image of $X(k)$ in $X(\mathbb{A}_k)$ lies in the left kernel $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ of this pairing, which is a closed subset of $X(\mathbb{A}_k)_\bullet$, called the Brauer-Manin set of X.

This is a very slight reformulation of Manin's 1970 observation.

For certain classes of smooth, geometrically connected varieties, one proves or tries to prove:

The closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$ coincides with $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$.

Such a statement includes a Brauer-Manin variant of the Hasse principle. For open varieties it includes a Brauer-Manin variant of strong approximation. For proper varieties it includes a Brauer-Manin variant of weak approximation.

Either to prove or to test the statement above it is useful to be able to compute the groupe $\mathrm{Br}(X)$ or rather the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$.

For the statement

The closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$ coincides with $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$.

there is evidence, both theoretical and numerical, when X is a smooth, geometrically rational surface.
There is also evidence for homogeneous spaces of abelian varieties.

For smooth, projective, geometrically connected varieties, the statement is open for :

• Rationally connected varieties (known for hypersurfaces with many variables, via the circle method)
• Curves of arbitrary genus g (for $g \geq 2$, exploratory work of Scharaschkin, Skorobogatov, Stoll, Bruin, ...)
• $K3$-surfaces, for instance quartics in $\mathbb{P}^3$.

For a smooth, projective, geometrically integral variety X over a number field k, with $X(\mathbb{A}_k) \neq \emptyset$, the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ fits into the exact sequence

$$0 \to H^1(G, \mathrm{Pic}(\overline{X})) \to \mathrm{Br}(X)/\mathrm{Br}(k) \to \mathrm{Br}_{tr}(X) \to 0,$$

where $\mathrm{Br}_{tr}(X) \subset \mathrm{Br}(\overline{X})^G$ is the group

$$\mathrm{Br}(X)/\mathrm{Br}_1(X) = Im[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})] = Im[\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(\overline{X})^G].$$

The group $H^1(k, \mathrm{Pic}(\overline{X}))$ may be infinite. But if $\mathrm{Pic}(\overline{X})$ is free of finite type, as is the case for a rationally connected variety and for a $K3$-surface, then it is finite.

Two basic questions :

For an arbitrary smooth, projective, geometrically connected variety over a finitely generated field k,
(i) Is the group $\mathrm{Br}_{tr}(X) \subset \mathrm{Br}(\overline{X})^G$ finite ?
(ii) Is the group $\mathrm{Br}(\overline{X})^G$ finite ?
The second question is closely connected with the Tate conjecture for divisors.

The Main Theorem of this talk (valid over any field of char. zero) shows that

The two questions boil down to one.

Theorem (Skorobogatov and Zarhin, 2008)
For a $K3$-surface X over a field k finitely generated over $\mathbb{Q}$, the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.

They actually prove that $\mathrm{Br}(\overline{X})^G$ is finite. The proof builds upon elaborate integral variants of the Tate conjecture for abelian varieties (Faltings, Zarhin) and on Deligne's method (Kuga-Satake varieties) for reducing problems on $K3$-surfaces to statements on abelian varieties.

Can we numerically test the validity of the statement

The closure of $X(k)$ *in* $X(\mathbb{A}_k)_\bullet$ *coincides with* $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$

for (suitable families of) $K3$-surfaces ?

Computing $\mathrm{Br}_1(X)/\mathrm{Br}(k) = H^1(k, \mathrm{Pic}(\overline{X}))$ is in principle an algorithmic process. But there seems to be no systematic way to compute the finite group $\mathrm{Br}_{tr}(X)$.

For diagonal quartics in $\mathbb{P}^3_{\mathbb{Q}}$,

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0,$$

some tests have been made on the statement

?? The closure of $X(\mathbb{Q})$ in $X(\mathbb{A}_{\mathbb{Q}})_{\bullet}$ coincides with $X(\mathbb{A}_{\mathbb{Q}})_{\bullet}^{\mathrm{Br}}$

Work of Swinnerton-Dyer; Martin Bright; Ieronymou, Skorobogatov, Zarhin.

There is the Hasse principle aspect.

If the product of the a_i is a square, and the a_i/a_j's are general enough (in $\mathbb{Q}^\times/\mathbb{Q}^{\times 4}$) then under two standard conjectures (Bouniakowsky-Dickson-Schinzel and finiteness of Ш of elliptic curves)
$X(\mathbb{A}_\mathbb{Q})_\bullet^{\mathrm{Br}} \neq \emptyset$ implies $X(\mathbb{Q}) \neq \emptyset$ (Swinnerton-Dyer, 2004)

In his thesis (2002), M. Bright considered quartic surfaces X over $\mathbb{Q}$ given by a diagonal equation with integral coefficients

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0.$$

He listed such surfaces with bounded integer coefficients for which
• There are points in all completions
• There is no Brauer-Manin obstruction coming from the algebraic part of the Brauer group.
He found some surfaces, such as $(7, 15, -2, -6)$, for which he could not find any rational point in the height range where he could search.
The suspicion is then that in these cases there are elements in $\mathrm{Br}_{tr}(X)$ which are responsible for the phenomenon.

Theorem (Bright 2010) *Let H be the subgroup of $\mathbb{Q}^\times/\mathbb{Q}^{\times 4}$ generated by $-1, 4$, and the quotients a_i/a_j.*
Suppose that the following conditions are satisfied:
(1) $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$.
(2) $H \cap \{2,3,5\} = \emptyset$ (Ieronymou, Skorobogatov, Zarhin : this implies $\mathrm{Br}(X) = \mathrm{Br}_1(X)$)
(3) H is of maximal order, i.e. $|H| = 256$.
(4) there is some odd prime p which divides precisely one of the coefficients a_i, and does so to an odd power; moreover, if $p \in \{7, 11, 17, 41\}$, then the reduction of X modulo p is not equivalent to $x^4 + y^4 + z^4 = 0$.
Then $\mathrm{Br}X/\mathrm{Br}\mathbb{Q}$ has order 2, and $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset$.

There is also the weak approximation aspect. Assume there are points everywhere locally.

Since $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite, there is a finite set S_0 of places of k such that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ projects onto $\prod_{v \notin S_0} X(k_v)$. Thus if *the closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$ coincides with $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$*, then weak approximation should hold for X away from S_0 (hence $X(k)$ should be Zariski dense, a well known open problem !).
Skorobogatov and I (2010) have given a precise description for a set S_0 as small as possible. Here is a concrete example, further building on the results of Ieronymou, Skorobogatov, Zarhin on $\mathrm{Br}_{tr}(X)$ for diagonal quartic surfaces.

Let X be the diagonal quartic surface over $\mathbb{Q}$ given by

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0,$$

with $a_i \in \mathbb{Z}$. Let S_0 be the set of primes consisting of 2 and the primes dividing some a_i.

Let Z be the image of the projection

$$X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \to X(\mathbb{R}) \times \prod_{p \in S_0} X(\mathbb{Q}_p).$$

Then $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = Z \times \prod_{p \notin S_0} X(\mathbb{Q}_p)$.

For diagonal quartic surfaces with the product of the a_i a square, work on proving density of $X(\mathbb{Q})$ in various completions $X(\mathbb{Q}_v)$ has been done by Logan, McKinnon, van Luijk, and by Swinnerton-Dyer.