Блох лекции 12 апреля

$H^n_M(X, \mathbb{Q}(p)) \leftrightarrow$ of cod. q cycles on $X \times \square^{2q-n}$

Step 1 $\quad H^n_M(X, \mathbb{Q}(q)) \xrightarrow{reg} \mathrm{Ext}^1_{HS}(\mathbb{Z}(0), H^{n-1}_{Betti}(X, \mathbb{Z}(q)))$

$$\text{\textbackslash}\!\!\cong \quad H^{n-1}(X, \mathbb{C}(q)) \Big/ F^0 H^{n-1}(X, \mathbb{C}(q)) + H^{n-1}(X, \mathbb{Z}(q))\Big]^+$$

$$\cong \quad \mathbb{R}^N / \mathbb{Z}^k$$

① $\mathrm{rk}\, H^n_M(X/\mathbb{Q}, \mathbb{Q}(p)) \overset{?}{=} N-k$ and reg embeds H_M as a complem. lattice in J

(must work with $\mathcal{X}/\mathbb{Z}$ arithmetic scheme)

$J/\mathrm{Reg} \cong \mathbb{R}^N/\mathbb{Z}^N$ compact finite volume.

② $\mathrm{Vol}\left(J(H^{n-1}(X, \mathbb{Z}(q)) / \mathrm{Im}(\mathrm{Reg})\right) \cdot \mathbb{Q}^*$

$$\overset{?}{=} L(H^{n-1}, q).$$

$X = \operatorname{Spec} \mathbb{Q}$ $\qquad H^1_M(\operatorname{Spec}\mathbb{Q}, \mathbb{Q}(3)) \hookrightarrow L(H^0, s=3) = \zeta(3)$

$$J = \left(\mathbb{C}(3)/0 + \mathbb{Z}(3)\right)^+ = \left(\mathbb{C}/\mathbb{Z}\cdot(2\pi i)^3\right)^+$$

$F^{-3} \supset F^{-2} = 0$

$$\parallel$$

$$\left(\mathbb{R} \oplus \mathbb{R}i / \mathbb{Z}(2\pi i)^3\right)^+ = \mathbb{R}$$

<u>Beilinson</u> $\quad H^1_M(\operatorname{Spec}\mathbb{Q}, \mathbb{Q}(3)) = \mathbb{Q}\cdot\alpha$

$\operatorname{Reg}(\alpha) \in \mathbb{R} \times$ ~~power of~~ π

gets $\mathbb{Q}$-structure from

$= H^0_{DR}(\operatorname{Spec}\mathbb{Q}) = H^0_{Betti}(\operatorname{Spec}\mathbb{Q})$

$\operatorname{Reg}(\alpha) = \zeta(3)\cdot\mathbb{Q}^*$!!

$$0 \to \mathbb{Q}(3) \to M \to \mathbb{Q}(0) \to 0$$

Obstruction to splitting is $\zeta(3)$

How can we write down all periods of mixed Tate HS / $\mathbb{Z}$. (Motives with good reduction everywhere).

$x \in \mathbb{Q}^*$ $\quad K_x \quad\quad 0 \to \mathbb{Q}(1) \to K_x \to \mathbb{Q}(0) \to 0$

Mixed Tate motive. Bad reduction at $(x) = \mathrm{div}$ of x.

not $/\mathbb{Z}$

$$0 \to \mathbb{Q}(3) \to M \to \mathbb{Q}(0) \to 0 \quad \text{is } /\mathbb{Z}.$$

$$H^1_M(\mathbb{Q}(3))$$

<u>Voevodsky</u> Abelian cat. of MT motives / numb. field.

Get motives from $\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$

1) thm of Beilinson

X variety $/\mathbb{C}$, $a \in X$

$\pi_1(X, a)$ fundamental group

$\mathbb{Q} \Leftarrow \mathbb{Q}[\pi(X,a)] \supset I =$ Augm. ideal

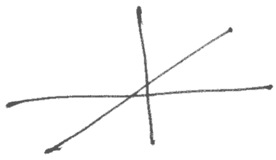
$$H^N_{Betti}(A,B) \cong \frac{(\mathbb{Q}[\pi_1]/I^{N+1})^\vee}{\mathbb{Q}} = (I/I^{N+1})^\vee \quad \text{is a motive.}$$

$A = X^N$

$B \subset A \quad\quad a \times X^{N-1} \cup \Delta_X \times X^{N-2} \cup X \times \Delta_X \times X^{N-3} \cup \ldots$

$X^{N-2} \times \Delta_X \cup X^{N-1} \times a.$

$N = 2$



proof is in the paper of Deligne and Goncharov

$\pi_1^{unip} = \overset{\text{affine}}{\text{alg}}.\ \text{group} = \operatorname{Spec}(\mathcal{O})$

π_1 motivic means that $\mathcal{O}$ is an ind motive

$$\mathcal{O} = \varinjlim \text{ motives}$$

π_1 nice Z^i terms in descending central series

Assume $Z^i/Z^{i+1} \cong \mathbb{Z}$? $\qquad \pi_1/Z^N \overset{(*)}{\cong} \mathbb{Z}^p$

$$\begin{tikzcd} & \mathbb{Q}[\pi_1]/I^N \arrow[r, "\ell"] & \mathbb{Q} \\ \pi_1 \arrow[ur] \arrow[urr, "\tilde{\ell}"'] & & \end{tikzcd}$$

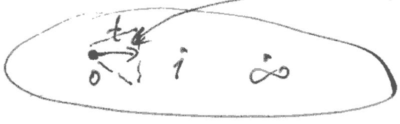
$\tilde{\ell}$ is a poly. map via identification $(*)$

$\varinjlim \left(\mathbb{Q}[\pi_1]/I^N\right)^\vee = \mathcal{O}$ algebra

$\pi_1^{unip} = \operatorname{Spec} \mathcal{O}$

F. Brown $\qquad \pi_1^{unip}(\mathbb{P}^1 - \{0,1,\infty\};\, 0,1) = \operatorname{Spec}(\mathcal{O})$ ind motive

Infinitesimal basept.



S = sector $\quad t$ falls in S

"$\varprojlim$"s

t tangent vector to $\mathbb{P}^1$ at 0

$p^{-1}(S)$ ⨿ contractible

$$\begin{array}{c} Y \\ \downarrow \\ \mathbb{P}^1 - \{0,1,\infty\} \end{array} \quad p \text{ f et}$$

$Y/\mathbb{P}^1 - \{0,1,\infty\} \rightsquigarrow \pi_0(p^{-1}(S))$

fibre functor.

Want to understand $\mathcal{O}$

$\pi_1 = F(u_0, u_1)$

$0, 1$ words $\quad u_0^{a_0} u_1^{a_1} u_0^{b_0} u_1^{b_1} \dots$

$\mathcal{O} = \mathbb{Q}\langle u_0, u_1\rangle = \mathbb{Q}$ v.s. spanned by words

mult. by shuffle $\qquad t_1 \dots t_k = w(t_1, \dots, t_k)$

$$w(t_1, \dots, t_k) * w(t_{k+1}, \dots, t_{k+l}) = \sum w\big(t_{\sigma^{-1}(1)} \dots t_{\sigma^{-1}(k+l)}\big).$$

$$\pi_1^{unip}(\mathbb{P}^1 - \{0,1,\infty\}; 0, 1) = \operatorname{Spec}(\mathcal{O})$$

Choice of cohom. yields different versions of $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, 0, 1)_B, \dots, DR.$

Betti $\qquad$ DeRham

$$\pi_1(\mathbb{P}^1 - \{0,1,\infty\}; 0, 1)_B(\mathbb{C}) \xrightarrow[per]{\cong} \pi_1(\mathbb{P}^1 - \{0,1,\infty\}; 0, 1)_{DR}(\mathbb{C})$$

$\cup \qquad\qquad \in$

$dch \in \pi_1 \dots_B(\mathbb{Q}) \qquad per(dch) : \mathcal{O} \to \mathbb{C}$

$0 \longrightarrow 1 \qquad\qquad \mathcal{O} = \mathbb{Q}\langle u_0, u_1\rangle \ni w \mapsto$

$$\zeta(s_1, \dots, s_k) := \sum_{n_1 > \dots > n_k > 0} \frac{1}{n_1^{s_1} \dots n_k^{s_k}} \qquad s_1 \geq 2$$

$$w = u_0^{s_1 - 1} u_1 \dots u_0^{s_k - 1} u_1$$

$$per(dch)(w) = (-1)^k \zeta(s_1, \dots, s_k)$$

Why is that? $\qquad (\mathbb{P}^1 - \{0, 1, \infty\})^N \supset \Sigma = a \times X^{N-1} \cup \Delta_X \times X^{N-2} \cup \dots$

$dch : [0,1] \longrightarrow \mathbb{P}^1 - \{0,1,\infty\} \qquad dch^N : [0,1]^N \to (\mathbb{P}^1 - \{0,1,\infty\})^N$

$\cup$

Simplex. $= \{t_1, \dots, t_N\}$

$0 \leq t_1 \leq t_2 \leq \dots \leq t_N \leq 1$

$$dch^N : (\text{Simplex}, \partial\,\text{Simplex}) \longrightarrow (X^N, \Sigma)$$

w ⟷ word in this alphabet on X^N

$X \quad \frac{dz}{z}, \quad \frac{dz}{1-z}$

integrated integral involving $\frac{dz}{z}$ and $\frac{dz}{1-z}$

$$\int_{dch^\vee(Simp)} w$$

F. Brown

all MTM/$\mathbb{Z}$ appear in $\mathcal{O}$

per(dch) $\in \mathbb{C}$ - pt

closure of orbit of Tannaki group G of MTM acting on this point

$M_{\mathbb{Q}} \subset \pi_1^{unip}(\mathbb{P}^1 - \{0,1,\infty\}, 0, 1)_{DR}$

$Spec(\mathcal{O}/\widetilde{J})$

Action of G on M is faithful

$\tilde{w}$

$\hat{w} = J_{\tilde{w}}$ $\qquad J_{\tilde{w}} \longmapsto J(s_1, \dots, s_k)$

$\mathcal{O}/J \longrightarrow \mathbb{C}$

$\Rightarrow$ all ell's in Tannaki (G) arise as

S weights of (*) of ell's in $\mathcal{O}/J$