17.04.2014

Lecture 6

Beilinson conjectures

$L(H^{n-1}, s)$ $\quad X/\mathbb{Q}$ smooth proj. $\quad H^{n-1} = H^{n-1}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$

$\quad\|$

Hasse Weil L-function $\longleftrightarrow$ Galois repn of $G = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on H^{n-1}

Special values of $L(H^{n-1}, s)$ at $s = q > 0$ $\quad n-1-2q \leq -3$

$\updownarrow$ functional equation?

$L(H^{n-1}, n-s)$ $\quad s = n-q$

$L^*(H^{n-1}, n-q) =$ value of 1-st non-vanishing derivative

$H^n_M(X, \mathbb{Q}(q)) \longleftrightarrow$ cycles of codim q in $X \times \square^{2q-n}$

<u>Step 1</u> $\quad H^n_M(X, \mathbb{Q}(q)) \xrightarrow{reg} Ext^1_{HS}(\mathbb{Z}(0), H^{n-1}_{Betti}(X, \mathbb{Z}(q)))$

$$\cong \left[H^{n-1}(X, \mathbb{C}(q)) \Big/ F^0 H^{n-1}(X, \mathbb{C}(q)) + H^{n-1}(X, \mathbb{Z}(q)) \right]^+$$

① $\operatorname{rk} H^n_M(X, \mathbb{Q}(q)) \overset{?}{=} N - k$ $\quad$ and reg embeds H_M as a complementary lattice in J

(Must work with $\mathcal{X}/\mathbb{Z}$ arithmetic scheme)

$\| \quad \mathbb{R}^N/\mathbb{Z}^k \qquad J(H^{n-1}(X, \mathbb{Z}(q)))$ interm. jacobian

② $Vol\left(J(H^{n-1}(X, \mathbb{Z}(q))) / Im(Reg) \right) \overset{?}{=} L(H^{n-1}, q) \cdot \mathbb{Q}^\times$

$$H^{n-1}_{DR}(X/\mathbb{Q})(q) \Big/ F^0 H^{n-1}_{DR}(q) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \text{Tang. space to } J(H^{n-1}(X, \mathbb{Q}(q)))$$

compatible with real conjugation

$\| \quad H^{n-1}(X, \mathbb{C}(q))/F^0$

Get $\mathbb{Q}$-structure on T_J $\rightsquigarrow$ $\mathbb{Q}$-structure on $\det(T_J)$

hence (up to $\mathbb{Q}^*$-scale) a measure on J

(3) Rank $H_M \overset{?}{=}$ Order of vanishing of $L(H^{n-1}, s = \cancel{}\ n-q)$

(4) $L^*(H^{n-1}, s = n-q) = \mathrm{Vol}'(J/\mathrm{Reg}) \cdot \mathbb{Q}^*$

Vol' alternative choice of volume

<u>Example</u> $X = \mathrm{Spec}\, \mathbb{Q}$ $\quad H^1_M(\mathrm{Spec}\,\mathbb{Q}, \mathbb{Q}(3))$

$$\updownarrow$$

$F^{-3} \supset F^{-2} = 0$ $\quad L(H^0, s = 3) = \zeta(3)$

$$J = \left(\mathbb{C}(3) / 0 + \mathbb{Z}(3)\right)^+ = \left(\mathbb{C} / \mathbb{Z}\cdot(2\pi i)^3\right)^+ = \left(\mathbb{R} \oplus \frac{\mathbb{R}i}{\mathbb{Z}(2\pi i)^3}\right)^+ = \mathbb{R}$$

Beilinson:

$$H^1_M(\mathrm{Spec}\,\mathbb{Q}, \mathbb{Q}(3)) = \mathbb{Q}\cdot\alpha$$

$\mathrm{Reg}(\alpha) \in \mathbb{R}$

gets $\mathbb{Q}$-structure from $H^0_{DR}(\mathrm{Spec}\,\mathbb{Q}) = H^0_{Betti}(\mathrm{Spec}\,\mathbb{Q})$

$\mathrm{Reg}(\alpha) = \zeta(3)\cdot\mathbb{Q}^\times$

$$0 \to \mathbb{Q}(3) \to M \to \mathbb{Q}(0) \to 0$$

Obstruction to splitting is $\zeta(3)$

How can we write down all periods of mixed Tate HS/$\mathbb{Z}$ (Motives with good reduction everywhere)

$x \in \mathbb{Q}^\times$ $\quad K_x \quad 0 \to \mathbb{Q}(1) \to K_x \to \mathbb{Q}(0) \to 0$

Mixed Tate motive Bad reduction at (x) = div of x

Not /$\mathbb{Z}$

$$0 \to \mathbb{Q}(3) \to M_{H^1_M(\mathbb{Q}(3))} \to \mathbb{Q}(0) \to 0 \quad \text{is } /\mathbb{Z}$$

Voevodsky: abelian cat. of mixed Tate motives over number field

Get motives from $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$

① Thm of Beilinson

X variety/$\mathbb{C}$, $a \in X$

$\pi_1(X, a)$ fund. group

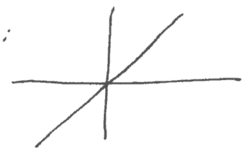
$\mathbb{Q} \twoheadleftarrow \mathbb{Q}[\pi_1(X, a)] \leftarrow I$ = Aug. ideal

$$H^N_{Betti}(A,B) \cong \frac{\left(\mathbb{Q}[\pi_1]/I^{N+1}\right)^\vee}{\mathbb{Q}} \text{ is a motive } = (I/I^{N+1})^\vee$$

$A = X^N$, $B \subset A$:

$$a \times X^{N-1} \cup \Delta_X \times X^{N-2} \cup X \times \Delta_X \times X^{N-3} \cup \dots \cup X^{N-2} \times \Delta_X \cup X^{N-1} \times a$$

$N = 2$:



$\pi_1^{unip} = \overset{\text{affine}}{\text{alg. group}} = \operatorname{Spec}(\mathcal{O})$

π_1 motivic means that $\mathcal{O}$ is an ind motive

$\mathcal{O} = \varinjlim$ motives

Suppose π_1 nice Z^i terms in descending central series. Assume $Z^i/Z^{i+1} \simeq \mathbb{Z}^?$

$\pi_1/Z^N \simeq \mathbb{Z}^p \quad (*)$

$$\begin{tikzcd} \pi_1/Z^N \arrow[r] \arrow[rr, bend right, "\tilde{l}"'] & \mathbb{Q}[\pi_1]/I^N \arrow[r, "l"] & \mathbb{Q} \end{tikzcd}$$

$\tilde{l}$ is a polynomial map via identification $(*)$

$\varinjlim (\mathbb{Q}[\pi_1]/I^N)^\vee = \mathcal{O}$ algebra

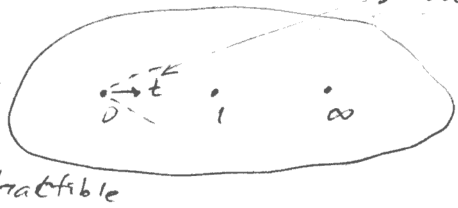
$\pi_1^{unip} = \operatorname{Spec}\mathcal{O}$

F. Brown

$\pi_1^{unip}(\mathbb{P}^1 \setminus \{0,1,\infty\};\ 0, 1)$

Infinitesimal basept.

s = sector

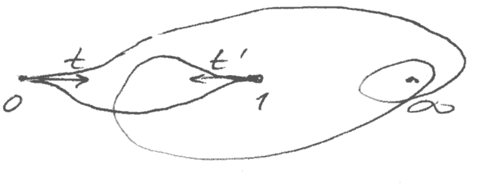


t tangent vector to $\mathbb{P}^1$ at 0

"$\varprojlim$"s

$$\begin{tikzcd} \coprod \text{contractible sets} \arrow[r] & p^{-1}(s) & Y \arrow[d, "p", "\text{ét.}"'] \\ & s & \mathbb{P}^1 - \{0,1,\infty\} \end{tikzcd}$$

$Y/\mathbb{P}^1 \setminus \{0,1,\infty\} \rightsquigarrow \pi_0(p^{-1}(s))$ fibre functor



$\pi_1^{unip}(\mathbb{P}^1\setminus\{0,1,\infty\}; 0, 1)$

$\qquad\qquad \parallel$

$\qquad \operatorname{Spec}(\mathcal{O})$

$\qquad$ ind motive

$\mathcal{O} \qquad \pi_1 = F(u_0, u_1)$

$\qquad$ words $u_0^{a_0} u_1^{a_1} u_0^{b_0} u_1^{b_1} \dots$

$\mathcal{O} = \mathbb{Q}\langle u_0, u_1\rangle = \mathbb{Q}$ vector space spanned by words

Mult. by shuffle

$$t_1 \dots t_k = w(t_1 \dots t_k)$$

$$w(t_1 \dots k_k) * w(t_{k+1}, \dots, t_{k+l}) = \sum w(t_{\sigma^{-1}(1)} \dots t_{\sigma^{-1}(k+l)})$$

$$\pi_1^{unip}(\mathbb{P}^1\setminus\{0,1,\infty\}; 0, 1) = \operatorname{Spec}\mathcal{O}$$

Choice of coh. yields different versions

of $\pi_1(\mathbb{P}^1\setminus\{0,1,\infty\}; 0,1)_B \dots, DR$

$$\pi_1(\mathbb{P}^1\setminus\{0,1,\infty\}; 0,1)_B(\mathbb{C}) \xrightarrow[\text{per}]{\sim} \pi_1(\mathbb{P}^1\setminus\{0,1,\infty\}; 0,1)_{DR}(\mathbb{C})$$

$\qquad \cup$

$\pi_1(\dots)_B(\mathbb{Q})$ $\qquad\qquad$ $\operatorname{per}(dch): \mathcal{O} \longrightarrow \mathbb{C}$

$\quad \in$ $\qquad\qquad\qquad\qquad \parallel$

$\quad dch$ $\qquad\qquad\qquad \mathbb{Q}\langle u_0, u_1\rangle \ni w \longmapsto ?$

$0 \quad 1$

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > \dots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \qquad s_1 \geq 2$$

$$w = u_0^{s_1 - 1} u_1 \dots u_0^{s_k - 1} u_1 \qquad \operatorname{per}(dch)(w) = (-1)^k \zeta(s_1, \dots, s_k)$$

? why $(\mathbb{P}^1 \setminus \{0,1,\infty\})^N \supset \Sigma = a \times X^{N-1} \cup \Delta_x \times X^{N-2} \cup \ldots \cup X^{N-1}_{\times a}$

$dch: [0,1] \longrightarrow \mathbb{P}^1 \setminus \{0,1,\infty\}$

$dch^N: [0,1]^N \longrightarrow (\mathbb{P}^1 \setminus \{0,1,\infty\})^N$

$\cup$

simplex $= \{t_1, \ldots, t_N\}$ $\quad 0 \le t_1 \le t_2 \le \ldots \le t_N \le 1$

$dch^N: (\text{simplex}, \partial\,\text{simplex}) \longrightarrow (X^N, \Sigma)$

$X: \quad \frac{dz}{z}, \frac{dz}{1-z}$

$W \longleftrightarrow$ word in this alphabet on $X^N = \omega$

$\int_{dch^N(\text{simplex})} \omega$ = iterated integral involving $\frac{dz}{z}, \frac{dz}{1-z}$

this is related to $\zeta(s_1 \ldots s_k)$

all $MTM/\mathbb{Z}$ appear in $\mathcal{O}$

per(dch) $\in \mathbb{C}$-point orbit of Tannaka group of MTM acting on this point

$M_{\mathbb{Q}} \subset \pi_1^{unip}(\mathbb{P}^1 \setminus \{0,1,\infty\}; 0,1)_{DR}$

Action of G on M is faithful

$M_{\mathbb{Q}} = \mathrm{Spec}(\mathcal{O}/\mathcal{J})$

$\tilde{\omega} = \zeta_{\tilde{\omega}}$

$\mathcal{O}/\mathcal{J} \longrightarrow \mathbb{C}$

$\zeta_{\tilde{\omega}} \longmapsto \zeta(s_1, \ldots s_k)$

$\Rightarrow$ all elements in Tannaka(G) arise as subquotients of tensor powers of elements in $\mathcal{O}/\mathcal{J}$