Блох лекция, 15 апреля

$\begin{pmatrix} \mathrm{li}_n \\ \vdots \end{pmatrix}$ "The motive" associated to $\begin{pmatrix} 1 & 0 & 0 \\ -\mathrm{li}_1(z) & 2\pi i & 0 \\ -\mathrm{li}_2(z) & 2\pi i \log z & (2\pi i)^2 \end{pmatrix}$,

mixed tate motives/k.

Tannakian category

$G = \mathrm{Spec}\, H$, H comm. Hopf alg.

$\rho : \mathrm{Lie}(G) \to \mathrm{End}(\mathbb{Q}^3)$

k field DGA $\mathcal{N}^*(\cdot)$ cycles on cubes

$\mathcal{N}^p(q) =$ Cod q alg cycles on $(\mathbb{P}^1 - \{1\})^{\overset{=n}{2q-p}}$

$\overset{\shortparallel}{\square}$ Alt. proj by π

$$\cdots \to \mathcal{N}^0(q) \xrightarrow{\partial} \mathcal{N}^1(q) \to \cdots$$

$\pi_n(\pm 1) \ni \rho_n = e_y$

$e_y \xrightarrow{\mathrm{Alt}} (\pm 1)$

$(\pm 1) = (\pm 1)$

$\rho_n \xrightarrow{\mathrm{alt}} (\pm 1)$

$$\begin{tikzcd}
2 & 1 & 0 \\
\mathcal{N}^1 \otimes \mathcal{N}^1 \arrow[r, "\mathrm{mixt}"] & \mathcal{N}^2 & \\
& H^0 & \\
(\mathcal{N}^1 \otimes \mathcal{N}^0) \oplus (\mathcal{N}^0 \otimes \mathcal{N}^1) \arrow[r, "\mathrm{mult}"'] \arrow[uu, "\partial"] & \mathcal{N}^1 / \partial \mathcal{N}^0 \arrow[uu, "\partial"] &
\end{tikzcd}$$

$$I \to H^0(\mathrm{Bar}(\mathcal{N}^\cdot(\cdot))) \to \mathbb{Q} \to 0$$

$\overset{\shortparallel}{\text{Aug. ideal}}$

$T_z = \{(t, 1-t, 1-zt^{-1})\} \in \mathcal{N}^1(2)$ $(x_1 | \ldots | x_n)$.

$$\partial T_z = z \wedge (1-z) \in \Lambda^2 k^*$$

$$\begin{array}{cc} (T_z\,, & z(1-z)) \in I \\ \cap & \updownarrow \\ \eta^1(2) & \eta^1(1)\otimes\eta^1(1)) \end{array}$$

mod I^2 $\quad z|(1-z) \rightsquigarrow z\wedge(1-z)$

$$(T_z, z\wedge(1-z)) \to z\wedge(1-z)$$

$$I/I^2 \longrightarrow \Lambda^2 I/I^2$$

$$\mathcal{L}^\vee \longrightarrow \Lambda^2\mathcal{L}^\vee$$

$$\Lambda^2\mathcal{L} \to \mathcal{L}$$

$$l_1\wedge l_2 \to [l_1, l_2]$$

Rep n of $\mathcal{L} := (I/I^2)^\vee$

$$\begin{pmatrix} 0 & z & T_z \\ 0 & 0 & 1-z \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \mathcal{L} = (I/I^2)$$

Homom. of Lie algebras.

$$\boxed{\partial z = \partial(1-z) = 0}$$

$v, w \in \mathcal{L}$

$$\left[\begin{pmatrix} 0 & z(v) & T_z(v) \\ 0 & 0 & (1-z)(v) \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z(w) & T_z(w) \\ 0 & 0 & (1-z)(w) \\ 0 & 0 & 0 \end{pmatrix}\right] =$$

$$= \begin{pmatrix} 0 & 0 & z(v)\,(1-z)(w) - z(w)\,(1-z)(v) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 & z[v,w] & T_z[v,w] \\ 0 & 0 & (1-z)[v,w] \\ 0 & 0 & 0 \end{pmatrix}$$

$$\partial z = \partial(1-z) = 0$$

$$z : \mathcal{L}/[\mathcal{L},\mathcal{L}] \longrightarrow \mathbb{Q}$$

$$T_z[v,w] := \langle \partial T_z, v\wedge w\rangle = \langle z\wedge(1-z), v\wedge w\rangle$$

$$\| $$

$$z(v)\,(1-z)(w) -$$

Beilinson conj

Hasse Weil L-functions

$X/\mathbb{Q}$ $\quad H^p_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ $= H^p$ etale cohom.

$G = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on H^p

$$G \to G_p \twoheadrightarrow \hat{\mathbb{Z}} = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

G_p = decomp. grp at p

(defined given a lifting of p to $\overline{\mathbb{Q}}$)

$\rho : G \to Aut(H^n)$

ρ unramified if

$$\begin{tikzcd} \rho \mid G_p : G_p \arrow[r] \arrow[rd, two heads] & Aut(H^n) \\ & \hat{\mathbb{Z}} \ni 1 \arrow[u, "\hat{\rho}"'] \end{tikzcd}$$

ρ unramified at almost all p

Assume ρ unramified at p

$f_p := $ geometric frob. $\hat{\rho}(1)^{-1}$

$\det(1 - f_p p^{-s} \mid H^n) = P_p(s)$

$$L(H^n, s) := \prod_p \frac{1}{P_p(s)}$$

If ρ is ramified at p, replace H^n by $(H^n)^{I_p}$

Ex. E ~~El~~ elliptic curve $/\mathbb{Q}$

$H^1_{ét}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ p good $\mathbb{P}_p = 1 - a_p p^{-s} + p^{1-2s}$

$1 - a_p + p = \# E(\mathbb{F}_p)$

p bad reduction  nodal cubic

$f_p : H^1(\alpha) \circlearrowleft$  loop topol.

$=$ defined over $\mathbb{F}_p$

or over $\mathbb{F}_{p^2}$ $\Leftrightarrow$ $f_p \sigma = -\sigma$

cohomol. of special fiber

$\sigma =$ loop

$f_p(\sigma) = \pm \sigma$

$1 - \alpha p^{-s}$ split case $1 - p^{-s}$

non-split case $1 + p^{-s}$.

Beilinson conjecture

$$\mathrm{Rank}\, H^2_M(E, \mathbb{Q}(2)) = \mathrm{ord}_{s=0} L(H^1, s)$$

$\leftarrow$ pole at $s=0$

$$L = \ldots \frac{1}{1-p^{-s}} \ldots \frac{1}{1+p^{-s}}$$

$E/\mathbb{Q}$

$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathbb{Z} \end{tikzcd}$$

$$H^2_M(\mathcal{E}/\mathbb{Z}, \mathbb{Q}(2)) \to H^2_M(E/\mathbb{Q}, \mathbb{Q}(2))$$

localization means removing split multiplicative bad reduction fibres $\Rightarrow$ change H^2_M (removing ~~fibres, &~~ poles, but adding zeroes).

Recall $E/\mathbb{Q}$ symbols $f, g \in \mathbb{Q}(E)^*$

$$\Gamma_{f,g} = \{(x, f(x), g(x))\} \subset E \times \square^2$$

$$\partial \sum n_i \Gamma_{f_i, g_i} = 0 \quad \text{in } E \times \square^1$$

f, g $\quad |(f)|, |(g)| \subset E_{\text{tors}}$

$\alpha \in E_N$ $\quad (f) = N(\alpha) - N(0)$

$$\partial(\underset{\Gamma}{\Gamma_{f,g} + \text{trivial}}) = 0$$

cycle class

$$\begin{tikzcd} 0 \arrow[r] & H^3(E \times (\square^2, \partial\square^2), \mathbb{Q}(2)) \arrow[r] & M \arrow[r] & \mathbb{Q}(0) \arrow[d] \\ & & s(1) \to 1 & 0 \end{tikzcd}$$

$$0 \to H^1(E, \mathbb{Q}(2)) \to M \to \mathbb{Q}(0) \to 0$$

$s(1) - s_F \in H^1(E, \mathbb{Q}(2))$ $\quad s_F \in \Gamma^2$ $\quad \Gamma \mapsto 1$

extension class lies in $H^1(E, \mathbb{C}(2)) / H^1(E, \mathbb{Q}(2))^+$

real conj.

$E/\mathbb{Q}$ $\quad E(\mathbb{C}) \circlearrowleft F_\infty$

$H^1(E, \mathbb{C}) \supset H^i(E, \underset{\underset{\mathbb{Q}(2\pi i)^n}{\shortparallel}}{\mathbb{Q}(n)})$ $\quad$ conj. acts on coeff.

$F_\infty \circ \text{conj} = \text{conj} \circ F_\infty$

$X/\mathbb{Q}$

$$\begin{tikzcd} H^n_{DR}(X/\mathbb{Q}) \otimes \mathbb{C} \arrow[r, "\simeq"] \arrow[loop below, "1 \otimes \text{conj}"] & H^n_{\text{Betti}}(X, \mathbb{C}) \arrow[loop below, "F_\infty \circ \text{conj}."] \end{tikzcd}$$

$X(\mathbb{C}) \xrightarrow{f} \mathbb{C}$ $\quad X/\mathbb{Q} \to Z/\mathbb{Q}$ $\quad C = C($

$$f^{-} = \sum \overline{a_I}\, z^I$$

$$f^{\sim} = \sum a_I\, \overline{z}^I \qquad\qquad f^{\vee} = \sum \overline{a_I}\, \cancel{z}\, \overline{z}^I$$

More generally

X smooth proj $/\mathbb{Q}$

$$H^n_M(X, \mathbb{Q}(p)) \longleftrightarrow L(H^{n-1}, s) \quad \text{at} \quad s = q \quad \text{or} \quad s = n-q$$

$$L(H^{n-1}, s) \longleftrightarrow L(H^{n-1}, n-s)$$

suppose, there is a funct. equation

We will assume that

$$n - 2q \leq -3 \quad \left(n-2q = -2 \,,\; n-2q = -1 \quad (BSD) \right)$$

$$H^n_M(X, \mathbb{Q}(q)) \longrightarrow \operatorname{Ext}^1_{HS}\left(\mathbb{Q}(0), H^{n-1}_{\text{Betti}}(X, \mathbb{Q}(q))\right)^{+} =$$

$$= \left[H^{n-1}(X, \mathbb{C}(q)) \Big/ F^0 H^{n-1}(X, \mathbb{C}(q)) + H^{n-1}(X, \mathbb{Q}(q)) \right]^{+}$$

abelian Lie group.

$$H_M \longrightarrow \mathbb{R}^n/\Lambda \rightarrow \mathbb{R}^k/\Lambda\otimes\mathbb{R} = \frac{\mathbb{R}^{k-h}}{\text{Motivic cohom.}}$$

$\Lambda \cong \mathbb{Z}^h$ (lattice of max. rank in $\mathbb{R}^{k-h}$

Volume of lattices $\longleftrightarrow L(q)$