15.04.14

Lecture 5

$\begin{pmatrix} Li_n \\ \vdots \end{pmatrix}$ „The motive" associated to $\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \log z & (2\pi i)^2 \end{pmatrix}$

mixed Tate motives/k

Tannakian category, $G = \operatorname{Spec} H$, H - comm. Hopf alg.

$\rho : \operatorname{Lie}(G) \longrightarrow \operatorname{End}(\mathbb{Q}^3)$

k field, DGA. $\mathcal{N}^*(\cdot)$ cycles on cubes

$\mathcal{N}^p(q) =$ codim q alg. cycles on $(\mathbb{P}^1 \setminus \{1\})^{2q-p}_{\text{Alt, proj.}}$, $= \square$

$\ldots \to \mathcal{N}^0(q) \xrightarrow{\partial} \mathcal{N}^1(q) \to \ldots$

$(\mathbb{Z}/\pm 1) \wr S_n = \mathcal{G}$

$\mathcal{G} \xrightarrow{\text{Alt}} (\pm 1)$

$\pm 1 \mapsto \pm 1$

$S_n \xrightarrow{\text{alt}} (\pm 1)$

$T_z = \{(t, 1-t, 1-zt^{-1})\} \in \mathcal{N}^1(2)$

$\mathcal{N}^1(1) = \mathbb{Z}[\text{pts on } \mathbb{G}_m - \{1\}]$

$$\begin{array}{ccc} \mathcal{N}^1 \otimes \mathcal{N}^1 & \xrightarrow{\text{mult}} & \mathcal{N}^2 \\ \uparrow \partial & & \uparrow \\ (\mathcal{N}^1 \otimes \mathcal{N}^0) \oplus (\mathcal{N}^0 \otimes \mathcal{N}^1) & \xrightarrow{\text{mult}} & \mathcal{N}^1/\partial \mathcal{N}^0 \end{array}$$

aug. ideal

$$0 \longrightarrow I \longrightarrow \underset{\overset{\|}{H}}{H^0(\operatorname{Bar}(\mathcal{N}^\bullet(\cdot)))} \longrightarrow \mathbb{Q} \longrightarrow 0$$

$I/I^2 \cong \mathcal{L}^\vee$ $\quad \mathcal{L}$ Lie algebra

$\mathcal{L} = \operatorname{Lie}(G)$, $G = \operatorname{Spec} H$

$\partial T_z = z \wedge (1-z) \in \Lambda^2 k^\times$

$(T_z, \; z|(1-z)) \in I$

$\mathcal{H}^1(2) \quad \mathcal{H}(1)\otimes\mathcal{H}^1(1)$

Mod I^2: $\quad z|(1-z) \rightsquigarrow z\wedge(1-z)$

$I/I^2 \longrightarrow \Lambda^2 I/I^2$

$(T_z, z\wedge(1-z)) \longmapsto z\wedge(1-z)$

$\mathcal{L}^\vee \longrightarrow \Lambda^2\mathcal{L}^\vee$

$\Lambda^2\mathcal{L} \longrightarrow \mathcal{L}$

$l_1\wedge l_2 \longmapsto [l_1, l_2]$

Rep'n of $\mathcal{L} := (I/I^2)^\vee$

$$\begin{pmatrix} 0 & z & T_z \\ 0 & 0 & 1-z \\ 0 & 0 & 0 \end{pmatrix} \longleftarrow \mathcal{L} = (I/I^2)^\vee$$

homomorphism of Lie algebras:

$v, w \in \mathcal{L}$

$$\left[\begin{pmatrix} 0 & z(v) & T_z(v) \\ 0 & 0 & (1-z)(v) \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z(w) & T_z(w) \\ 0 & 0 & (1-z)(w) \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & z(v)\cdot(1-z)(w) - z(w)(1-z)(v) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \cancel{z[v,w]} & T_z[v,w] \\ 0 & 0 & \cancel{(1-z)[v,w]} \\ 0 & 0 & 0 \end{pmatrix}$$

$\partial z = \partial(1-z) = 0$

$\begin{matrix} z \\ (1-z) \end{matrix} : \; \mathcal{L} \longrightarrow \mathbb{Q}$

$[\mathcal{L},\mathcal{L}]$

$T_z[v,w] = \langle \partial T_z, v\wedge w\rangle = \langle z\wedge(1-z), v\wedge w\rangle = z(v)(1-z)(w) - z(w)(1-z)(v)$

Beilinson conj.

Hasse Weil L-functions

$X/\mathbb{Q}$ $\quad H^n_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = H^n$ étale cohom.

$G = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on H^n

p - prime $\quad G \supset G_p \twoheadrightarrow \hat{\mathbb{Z}} = Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$

decomposition group at p (defined given a lifting of p to $\bar{\mathbb{Q}}$)

$\rho : G \to Aut(H^n)$

ρ unramified at p if $\rho/_{G_p} : G_p \to Aut(H^n)$

$$\begin{tikzcd} G_p \arrow[r] \arrow[d] & Aut(H^n) \\ \hat{\mathbb{Z}} \ni 1 \arrow[ur, dashed, "\tilde{\rho}"'] & \end{tikzcd}$$

ρ unramified for almost all p

Assume ρ unramified at p

$f_p :=$ geometric Frobenius $\tilde{\rho}(1)^{-1}$

$\det(1 - f_p p^{-s} \mid H^n) = P_p(s)$

$L(H^n, s) := \prod_p \frac{1}{P_p(s)}$

If ρ is ramified at p, replace H^n by $(H^n)^{I_p}$, where $I_p = \ker(G_p \to \hat{\mathbb{Z}})$

In this way, we define $f_p(s)$ $\forall p$

Ex E - elliptic curve $/\mathbb{Q}$

$H^1_{et}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ $\quad p$ good $P_p = 1 - a_p p^{-s} + p^{1-2s}$

$L(H^1, s)$ $\quad 1 - a_p + p = \# E(\mathbb{F}_p)$

p bad reduction α nodal curve

$f_p : H^1(\alpha) \circlearrowleft$ σ loop

$f_p(\sigma) = \pm\sigma$

tangents def $/\mathbb{F}_p$

or def. $/\mathbb{F}_{p^2}$

$f_p \sigma = -\sigma$

$1 - \alpha p^{-s}$

split case $1 - p^{-s}$

non-split $1 + p^{-s}$

Beilinson conj. Rank $H^2_M(E, \mathbb{Q}(2)) = \mathrm{ord}_{s=0} L(H^1, s)$

$L = \dots \left(\frac{1}{1-p^{-s}} \dots \frac{1}{1+p^{-s}} \dots\right)$ ← no pole

pole at $s=0$

$E/\mathbb{Q}$ $\quad \mathcal{E} \downarrow \mathbb{Z}$ $\quad H^2_M(\mathcal{E}/\mathbb{Z}, \mathbb{Q}(2)) \to H^2_M(E/\mathbb{Q}, \mathbb{Q}(2))$

localization means removing factors from L

removing split bad fibres introduces new zeros

B. conj. says: get more $H^2_M(E, \mathbb{Q}(2))$

<u>Recall</u> $E/\mathbb{Q}$ symbols $f, g \in \mathbb{Q}(E)^\times$

$\Gamma_{f,g} = \{(x, f(x), g(x))\} \subset E \times \square^2$

$\partial \sum n_i \Gamma_{f_i g_i} = 0$ in $E \times \square^1$

f, g suppose $|(f)|, |(g)| \subset E_{tors}$

$\alpha \in E_N$, $(f) = N(\alpha) - N(0)$

$\partial(\Gamma_{f,g} + \text{trivial}) = 0$

cycle class $\quad 0 \to H^3(E\times \square^2, E\times\partial\square^2; \mathbb{Q}(2)) \to M \to \mathbb{Q}(0) \to 0$

$$0 \to H^1(E, \mathbb{Q}(2)) \to M \to \mathbb{Q}(0) \to 0$$

$s(1) \longmapsto 1$

$s(1) - s_F \in H^1(E, \mathbb{C}(2))/H^1(E, \mathbb{Q}(2)) \qquad s_F \in F^0 M_{\mathbb{C}} \mapsto 1$

extension class lies in $\left(H^1(E, \mathbb{C}(2))/H^1(E, \mathbb{Q}(2))\right)^+$

real conjugation

Story of 3 ~~c~~onjugations

$E/\mathbb{Q} \qquad E(\mathbb{C}) \circlearrowleft F_\infty$

$H^1(E, \mathbb{C}) \supset H^1(E, \mathbb{Q}(n)) \quad$ conj acts on coeff's.

$\mathbb{Q}(n) = \mathbb{Q}\cdot(2\pi i)^n$

$F_\infty \circ \text{conj} = \text{conj} \circ F_\infty \quad$ – third conjugation

$$X/\mathbb{Q} \qquad H^n_{DR}(X/\mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} H^n_{Betti}(X, \mathbb{C})$$

$1 \otimes \tau \qquad\qquad F_\infty \circ \text{conj}$

$X(\mathbb{C}) \xrightarrow{f} \mathbb{C}$

$f = f(z_1, \dots z_n) \qquad f = \sum a_I z^I$

$f^- = \sum \bar{a}_I z^I$

$f^\sim = \sum a_I \bar{z}^I$

$f^* = \sum \bar{a}_I \bar{z}^I$

More generally

X - smooth proj / $\mathbb{Q}$

$H^n_M(X, \mathbb{Q}(p)) \longleftrightarrow L(H^{n-1}, s)$ at $s = q$ or $s = n - q$

$L(H^{n-1}, s) \leftrightarrow L(H^{n-1}, n - s)$

We will assume $n-2q \leq -3$

$(n-2q=-2, \quad n-2q=-1$ (BSD))

$$H^n_M(X,\mathbb{Q}(q)) \longrightarrow \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^{n-1}_{Betti}(X,\mathbb{Q}(q)))^+$$

$$\quad\quad\quad\quad\quad\quad\quad\quad \| $$

$$\left[\frac{H^{n-1}(X,\mathbb{C}(q))}{F^0H^{n-1}(X,\mathbb{C}(q))+H^{n-1}(X,\mathbb{Z}(q))}\right]^+$$

abelian Lie group

$H_M \longrightarrow \mathbb{R}^k/\Lambda \qquad \Lambda \simeq \mathbb{Z}^h$

$\mathbb{R}^k/\Lambda\otimes\mathbb{R} = \mathbb{R}^{k-h}$

Motivic coh.

lattice of max. rank in $\mathbb{R}^{k-h}$

Volume of Lattice $\longleftrightarrow L(q)$