С. Блох лекция. 10 апреля (1)

$$\mathrm{ext} \in \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^{2p-1}(X, \mathbb{Q}(p)))$$

$$\text{SII}$$

$$[Z] \in J^p := \frac{H^{2p-1}(X, \mathbb{C}(p))}{\underbrace{F^p H^{2p-1}(X, \mathbb{C}(p))}_{F^p H^{2p-1}(X, \mathbb{C})} + H^{2p-1}(X, \mathbb{Z}(p))}$$

$\mathbb{Q}(p)$

$= \mathbb{C}^{2g}/\mathbb{C}^g + \mathbb{Z}^{2g} = \mathbb{C}^g/\mathbb{Z}^{2g} = (S^1)^{2g}$

= compact complex torus

Higher Chow groups

$Z^p(X \times \Delta^n)$ ← Simplicial complex

$\Delta^n \hookrightarrow \Delta^{n+1}$ $\Delta^{n-1} \hookrightarrow \Delta^n$

$\Delta^n \cong \mathbb{A}^n$

$$\to Z^p(X, n) \xrightarrow{\sum(-1)^i \delta_i^*} Z^p(X, n-1) \qquad Z = \sum n_i Z_i$$

on $X \times \Delta^n$.

$X \times (\Delta^n, \partial\Delta^n)$

$CH^p(X, n)$

$\Downarrow$

Z

$$0 \to \overbrace{H^{2p-1}(X \times \Delta^n, X \times \partial\Delta^n; \mathbb{Q}(p))}^{\text{II}} \to$$

$$\to H^{2p-1}(X \times \mathbb{A}^n - |Z|, X \times \Delta^{n-1} - |Z| \cap X \times \Delta^{n-1}; \mathbb{Q}(p)) \to \mathbb{Q}$$

~~cl cl~~

$$cl(Z): 0 \to H^{2p-1-n}(X, \mathbb{Q}(p)) \to H_Z \to \mathbb{Q}(0) \to 0$$

ex. $X = pt$ $\quad n = 2p-1$ $\quad$ cod p cycles on Δ^{2p-1}

$$CH^p(\mathbb{C}, 2p-1) \to \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), \mathbb{Q}(p))$$

$$0 \to \mathbb{Q}(p) \to H_{\mathbb{Q}} \rightleftarrows \mathbb{Q}(0) \to 0$$

s - vector spaces splitting

Totaro cycle $x \in k - \{0,1\}$.

$T_x = \{(t, 1-t, 1-xt^{-1})\}$

$$0 \to \mathbb{C}(p) \to H_{\mathbb{C}} \to \mathbb{C}(0) \to 0$$

$s(1) \mapsto 1$

$$0 \to F^0\mathbb{C}(p) \to F^0H_{\mathbb{C}} \xrightarrow{\cong} F^0\mathbb{C}(0) \to 0$$

$F^0\mathbb{C}(p) = 0$, $\quad s_F \to 1$

$$s(1) - s_F \in \mathbb{C}(p)/\mathbb{Q}(p) = \mathbb{C}/\mathbb{Q}\cdot(2\pi i)^p$$

$$CH^p(pt, 2p-1) \to \mathbb{C}/\mathbb{Q}(2\pi i)^p \cong \mathbb{R}(2\pi i)^{p-1} \oplus \frac{\mathbb{R}(2\pi i)^p}{\mathbb{Q}(2\pi i)^p}$$

$S^1 \otimes \mathbb{Q}$

Nahm's conjecture

$$F_{A,B,C}(q) = \sum_{n \in \mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^tAn + n^tB + C}}{(q)_{n_1} \cdots (q)_{n_r}}$$

$A \in M_r(\mathbb{Q})$ symmetric > 0, $B \in \mathbb{Q}^r$, $C \in \mathbb{Q}$

$$(q)_n := (1-q)(1-q^2)\cdots(1-q^n)$$

When is $F_{A,B,C}(q)$ a modular function

$q = \exp(2\pi i\tau)$ $\quad SL_2(\mathbb{Z})$ acts on τ $\quad \tau \mapsto \frac{a\tau+b}{c\tau+d}$.

Given A, when does there exist B, C

<u>paper</u> Vlasenko – Zwegers Arxiv 1104.4008

<u>lemma</u> $A \in M_r(\mathbb{Q})$ symm > 0

real numbers

Then $\exists!\ Q_i$ $\quad 0 < Q_i < 1$, $1 \le i \le r$

s.t. $1 - Q_i = \prod_{j=1}^{r} Q_j^{A_{ij}}$

Totaro cycle $\qquad \partial T_x = x \otimes (1-x)$

T_{Q_i} $\qquad \partial \sum_{i=1}^{r} T_{Q_i} = \prod_i \left(Q_i \otimes \prod_j Q_j^{A_{ij}} \right)$ — symmetric

$\mathrm{Sym}^2 \mathbb{R}^* \subset \mathbb{R}^* \otimes \mathbb{R}^*$

symmetric

$\Downarrow$

$(H^2(\mathbb{R}, 3))$

$\|$

<u>Cor.</u> $\sum T_{Q_i} \in H^1_M(\mathbb{R}, \mathbb{Q}(2))$

$$\frac{\ker\left(\mathcal{Z}^2(\Delta^3_{\mathbb{R}}) \xrightarrow{\partial} \mathcal{Z}^2(\Delta^2)\right)}{\mathrm{Im}\left(\mathcal{Z}^2(\Delta^4) \to \mathcal{Z}^2(\Delta^3)\right)}$$

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto 2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 0$$

$\otimes$

$\mathbb{C}^* \otimes \mathbb{Q}$

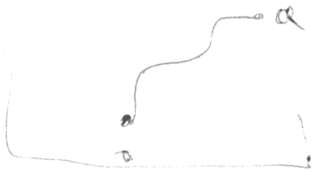
$$\begin{array}{c} \varepsilon(a) \\ \uparrow \end{array}$$

$$0 \to \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C} \otimes \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^* \otimes \mathbb{C}^*_{\mathbb{Q}} \to 0$$

$a \in \mathbb{C} - \{0, 1\}$, $\quad \varepsilon(a) := \log(1-a) \otimes a + 2\pi i \otimes \exp\left(-\frac{1}{2\pi i} \int_0^a \log(1-t) \frac{dt}{t}\right)$

$\|$

$\mathrm{Li}_2(a) \in \mathbb{C} \times \mathbb{C}^*$



$\varepsilon(a)$ well-defined ind. of path $0 \to a$

$$\sum_{i=1}^{r} \left(\varepsilon(Q_i) - \varepsilon(1 - Q_i) \right) \in \mathbb{C}^*_{\mathbb{Q}}$$

$Z^2(\Delta^3)$ $\quad T_x = ((t, 1-t, 1-xt)) \in \square^3 \qquad \square^3 = (\mathbb{P}^1 - \{1\})^3$

$\partial T_x = (x, 1-x) \in \square^2$

$$H^1_M(\mathbb{C}, \mathbb{Q}(2)) \xrightarrow{reg} \mathbb{C}/\mathbb{Q}\cdot(2\pi i)^2 \to \mathbb{R}\cdot 2\pi i \oplus \mathbb{R}/\mathbb{Q}\cdot(2\pi i)^2$$

$\Downarrow$

$\sum T_{Q_i} \longmapsto \qquad \longleftarrow \sum R_2(Q_i) \bmod \mathbb{Q}(\pi^2)$

$$reg\left(\sum T_{Q_i}\right) = \underbrace{\sum \mathcal{E}(Q_i) - \mathcal{E}(1-Q_i)}\cdot 2\pi i$$

$\mathbb{C}^*_{\mathbb{Q}} = \mathbb{C}/2\pi i\mathbb{Q} \longmapsto \mathbb{C}/(2\pi i)^2\mathbb{Q}$

Using this Rogers dilog

$$R_2(\overset{0<x<1}{x}) := \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x)$$

$$R_2(x) + R_2(1-x) = \frac{\pi^2}{6}$$

<u>Lemma</u> $A \in M_r(\mathbb{Q})$ symm. >0

necessary condition for $F_{A,B,C}(q)$

to be modular (look at asympt. behaviour $q \to 1$)

is $\quad \sum R_2(Q_i) \in \mathbb{Q}\cdot\pi^2$

F_{ABC} modular $\Rightarrow reg\left(\sum T_{Q_i}\right) = 0$

<u>Nahm's conj.</u> $\left[\sum T_{Q_i}\right] \in H^1_M(\mathbb{R}, \mathbb{Q}(2)) = 0$

$\Rightarrow \exists\, B \in \mathbb{Q}^r, C \in \mathbb{Q}$ s.t. $F_{A,B,C}(q)$ modular

False as stated, but many examples

Fact (Borel) $H^1_M(k, \mathbb{Q}(2)) = 0$ k totally real number field

Q_i algebraic $1 - Q_i = \prod Q_j^{A_{ij}}$ $Q_i \in \mathbb{R}$

DGA, k field

$\mathcal{N}^*(\cdot)$ DGA

$$\mathcal{N}^p(q) = \mathcal{Z}^q(\square_k^{2q-p})^{\otimes \mathbb{Q}}_{AH}$$ ← project onto all action

S_{2q-p} acts on $\square^{2q-p}$ permutations

$(\pm 1)^{2q-p}$ acts by $z_i \mapsto \frac{1}{z_i}$

wreath product $S_{2q-p} \wr (\pm 1)^{2q-p}$ acts $\mathcal{N}^p(q) \xrightarrow{\partial} \mathcal{N}^{p+1}(q)$

$\downarrow AH$

$\square^p \times \square^{p'} = \square^{p \times p'}$

$\mathcal{N}^p(q) \times \mathcal{N}^{p'}(q') \to \mathcal{N}^{p+p'}(q, q')$

$H^0(\mathrm{Bar}(\mathcal{N}^\cdot))$

$/\!/ ?.$

$\mathcal{O}(G)$ G tannaki group of mixed tate motives $/k$.

$$\begin{array}{c} \vdots \\ \uparrow \\ \mathcal{N}^2 \\ \uparrow \\ \mathcal{N}^1 \\ \uparrow \\ \mathcal{N}^0 \\ \uparrow \\ \mathcal{N}^{-1} \end{array} \quad \text{D.G.A.}$$

next time understand this dilog.