10.04.2014

Lecture 4

X smooth variety $/ \mathbb{C}$ $\quad Z = \sum n_i Z_i$ codim p algebr. cycle

$|Z| = \bigcup_i Z_i \quad U = X \setminus |Z|$

Gysin sequence

$$\begin{tikzcd}
 & & & H^{2p}(X, \mathbb{Q}(p)) \\
0 \arrow[r] & H^{2p-1}(X, \mathbb{Q}(p)) \arrow[r] & H^{2p-1}(U, \mathbb{Q}(p)) \arrow[r] & \bigoplus_i \mathbb{Q}(0) \arrow[u] \\
0 \arrow[r] & H^{2p-1}(X, \mathbb{Q}(p)) \arrow[r] \arrow[u, equal] & H_Z \arrow[r, two heads] \arrow[u, phantom, "\cup"] & \mathbb{Q}(0) \arrow[u, "Z"'] \arrow[uu, bend right, "{0, \text{ if } Z \sim 0}"']
\end{tikzcd}$$

$\underbrace{0 \longrightarrow H^{2p-1}(X, \mathbb{Q}(p)) \longrightarrow H_Z \longrightarrow \mathbb{Q}(0)}_{\text{extension of } HS}$

$\text{Ext}^1_{HS}(\overset{\uparrow}{\mathbb{Q}(0)}, H^{2p-1}(X, \mathbb{Q}(p)))$

$$[Z] \in J^p = \frac{\overset{\cong}{H^{2p-1}(X, \mathbb{C}(p))}}{\underbrace{F^0 H^{2p-1}(X, \mathbb{C}(p))}_{F^p H^{2p-1}(X, \mathbb{C})} + H^{2p-1}(X, \mathbb{Z}(p))}$$

$\nearrow$ compact complex torus

$$\frac{\mathbb{C}^{2g}}{\mathbb{C}^g + \mathbb{Z}^{2g}} = \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} = (S^1)^{2g}$$

Higher Chow groups $\qquad \Delta^{n-1} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \Delta^n \cong \mathbb{A}^n$

$\mathcal{Z}^p(X \times \Delta^n) \longleftarrow$ simplicial complex

$\underset{\cup}{}$

$\ldots \to \mathcal{Z}^p(X, n) \xrightarrow{\sum(-1)^i \delta_i^*} \mathcal{Z}^p(X, n-1)$

$CH^p(X, n) \ni Z \qquad Z = \sum n_i Z_i$ on $X \times \Delta^n$

We may assume that $\delta_i^* Z = 0 \quad \forall i$

$$0 \to H^{2p-1}(\overbrace{X\times\Delta^n, X\times\partial\Delta^n}^{X\times(\Delta^n,\partial\Delta^n)=X\times S^n}; \mathbb{Q}(p)) \to H^{2p-1}(X\times\Delta^n\setminus|Z|, X\times\Delta^{n-1}\setminus|Z|\cap X\times\Delta^{n-1}, \mathbb{Q}(p))$$

$$\longrightarrow \oplus \ldots$$

$$cl(Z):\ 0\to H^{2p-1-n}(X,\mathbb{Q}(p)) \longrightarrow H_Z \longrightarrow \mathbb{Q}(0) \to 0$$

Ex $X = pt \quad n = 2p-1$

codim p cycles on Δ^{2p-1}; ex. Totaro cycle

$x\in k\setminus\{0,1\}$

$T_x = \{(t, 1-t, 1-xt^{-1})\}$

$$CH^p(\mathbb{C}, 2p-1) \to Ext^1_{HS}(\mathbb{Q}(0), \mathbb{Q}(p))$$

$$0\to\mathbb{Q}(p)\to \underset{s(1)\in}{H}_{\mathbb{Q}} \to \mathbb{Q}(0)\to 0$$

$\leftarrow$ s - vector space splitting

$$0\to\mathbb{C}(p)\longrightarrow H_{\mathbb{C}}\longrightarrow \mathbb{C}(0)\longrightarrow 0$$

$$F^{-p} \qquad\qquad \overset{\parallel}{F^0}$$

$$0\to F^0\mathbb{C}(p)\to F^0H_{\mathbb{C}} \xrightarrow{\sim} F^0\mathbb{C}(0)\to 0$$

$$\overset{\parallel}{0} \qquad s_F \longrightarrow \overset{\cup}{1}$$

$$s(1)-s_F \in \mathbb{C}(p)/\mathbb{Q}(p) = \mathbb{C}/\mathbb{Q}\cdot(2\pi i)^p$$

$$CH^p(pt, 2p-1) \longrightarrow \mathbb{C}/\mathbb{Q}\cdot(2\pi i)^p \cong \mathbb{R}(2\pi i)^{p-1} \oplus \underbrace{\frac{\mathbb{R}(2\pi i)^p}{\mathbb{Q}(2\pi i)^p}}$$

$\nwarrow S^1\otimes\mathbb{Q}$

Nahm's conjecture

$$F_{A,B,C}(q) = \sum_{n\in\mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^tAn + n^tB + C}}{(q)_{n_1}\cdot\ldots\cdot(q)_{n_r}} \qquad (q)_n := (1-q)(1-q^2)\cdot\ldots\cdot(1-q^n)$$

$A\in M_r(\mathbb{Q})$ symmetric, >0; $B\in\mathbb{Q}^r$; $C\in\mathbb{Q}$

When is $F_{A,B,C}$ a modular function?

$q = \exp(2\pi i\tau)$

$SL_2(\mathbb{Z})$ acts on τ,

$\tau \mapsto \frac{a\tau+b}{c\tau+d}$

Given A, when does there exist B, C

Vlasenko-Zwegers Arxiv 1104.4008

Lemma $A \in M_r(\mathbb{Q})$ symm. >0

Then $\exists!\, Q_i$ $\quad 0 < Q_i < 1$, $\quad 1 \le i \le r$, $\quad$ s.t.

$$1 - Q_i = \prod_{j=1}^{r} Q_j^{A_{ij}}$$

Totaro cycles T_{Q_i} $\qquad \in \mathrm{Sym}^2 \mathbb{R}^\times \subset \mathbb{R}^\times \otimes \mathbb{R}^\times$, since A is symmetric

$$(*) \quad \partial \sum_{i=1}^{r} T_{Q_i} = \prod_i \Big(Q_i \otimes \prod_j Q_i^{A_{ij}}\Big)$$

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto 2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \to 0 \qquad \otimes (\mathbb{C}^\times \otimes \mathbb{Q}) = \mathbb{C}^\times_{\mathbb{Q}}$$

$$\overset{\varepsilon(a) \in}{}$$

$$0 \to \mathbb{C}^\times_{\mathbb{Q}} \to \mathbb{C} \otimes \mathbb{C}^\times_{\mathbb{Q}} \to \mathbb{C}^\times \otimes \mathbb{C}^\times_{\mathbb{Q}} \to 0$$

$$a \in \mathbb{C} \setminus \{0,1\} \quad \varepsilon(a) = \log(1-a) \otimes a + 2\pi i \otimes \exp\Big(\frac{-1}{2\pi i} \underbrace{\int_0^a \log(1-t) \frac{dt}{t}}_{}\Big)$$

$Li_2(a) = \int_0^a \log(1-t)\frac{dt}{t}$

$\varepsilon(a)$ is well-defined, indep. of path $0 \to a$ $\qquad \in \mathbb{C} \otimes \mathbb{C}^\times_{\mathbb{Q}}$

$$\sum_{i=1}^{r} \big(\varepsilon(Q_i) - \varepsilon(1-Q_i)\big) \in \mathbb{C}^\times_{\mathbb{Q}}$$

Cor $\boxed{\sum T_{Q_i} \text{ + stupid stuff}} \in H^1_M(\mathbb{R}, \mathbb{Q}(2)) \hookrightarrow \dfrac{\ker\big(Z^2(\Delta^3) \to Z^2(\Delta^2)\big)}{\mathrm{Im}\big(Z^2(\Delta^4) \to Z^2(\Delta^3)\big)}$

$$H^1_M(\mathbb{R}, \mathbb{Q}(2)) = CH^2(\mathbb{R}, 3)$$

$$T_x = \{(t, 1-t, 1 - x t^{-1})\} \subset \square^3 \qquad \square^3 = (\mathbb{P}^1 \setminus \{1\})^3$$

$$\partial T_x = (x, 1-x) \in \square^2$$

$$\begin{tikzcd}
H^1_M(\mathbb{C}, \mathbb{Q}(2)) \arrow[r, "reg"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathbb{C}/\mathbb{Q}\cdot(2\pi i)^2 \arrow[r] & \mathbb{R}\cdot(2\pi i) \oplus \mathbb{R}/\mathbb{Q}\cdot(2\pi i)^2 \\
\sum T_{Q_i} \arrow[rr, mapsto, bend right=10, "?"'] & &
\end{tikzcd}$$

$$\mathrm{reg}\Big(\sum T_{Q_i}\Big) = \underbrace{\Big(\sum \varepsilon(Q_i) - \varepsilon(1-Q_i)\Big) 2\pi i}_{\mathbb{C}_{\mathbb{Q}}^{\times} = \mathbb{C}/2\pi i\,\mathbb{Q} \;\mapsto\; \mathbb{C}/(2\pi i)^2\mathbb{Q}}$$

Using this, Roger's dilog:

$0<x<1 \qquad R_2(x) = Li_2(x) + \frac{1}{2}\log(x)\log(1-x)$

$$\sum T_{Q_i} \longmapsto \sum R_2(Q_i) \mod \mathbb{Q}\cdot\pi^2)$$

$$R_2(x) + R_2(1-x) = \frac{\pi^2}{6}$$

<u>Lemma</u> Given $A \in M_r(\mathbb{Q})$ symm., >0 necessary condition for $F_{A,B,C}(q)$ to be modular (look at asymptotic behavior as $q \longrightarrow 1$) is

$$\sum R_2(Q_i) \in \mathbb{Q}\cdot\pi^2$$

Conclusion: $F_{A,B,C}$ modular $\Rightarrow \mathrm{reg}\big(\sum T_{Q_i}\big) = 0$

Nahm's conj. $\left[\begin{array}{l} [\sum T_{Q_i}] \in H_M^1(\mathbb{R}, \mathbb{Q}(2)) = 0 \\ \Rightarrow \exists B \in \mathbb{Q}^r, C \in \mathbb{Q}, \text{ s.t. } F_{A,B,C} \text{ is modular} \end{array}\right.$

This is false as stated, but many examples

Fact (Borel): $H_M^1(K, \mathbb{Q}(2)) = 0$ if K totally real number field.

Q_i algebraic $\quad 1 - Q_i = \prod Q_j^{A_{ij}}; \quad Q_i \in \mathbb{R}$

DGA k - field

$\mathcal{N}^*(\cdot)$ DGA

$\mathcal{N}^p(q) = \mathcal{Z}^q(\square_k^{2q-p})^{\otimes\mathbb{Q}}_{\mathrm{Alt}}$ ← project onto alternating action

$\mathfrak{S}_{2q-p}$ acts on $\square^{2q-p}$ permuting coords.

$(\pm)^{2q-p}$ acts by $z_i \mapsto \frac{1}{z_i}$

Wreath product

$\mathfrak{S}_{2q-p} \wr (\pm)^{2q-p}$ acts

$\mathcal{N}^p(q) \xrightarrow{\partial} \mathcal{N}^{p+1}(q)$

$\square^p \times \square^{p'} = \square^{p+p'}$

$\mathcal{N}^p(q) \times \mathcal{N}^{p'}(q') \longrightarrow \mathcal{N}^{p+p'}(q+q')$

This gives DGA $\mathcal{N}^\cdot$

$H^0(\mathrm{Bar}(\mathcal{N}^\cdot))$

$\overset{?}{\|}$

$\mathcal{O}(G)$, where G - Tannakian group of mixed Tate motives / k

Next tim: understand dilog. from this point of view