Real valued polylogarithms

e.g. $\log|z|$ $\quad \mathrm{Im}\, Li_2(z) - \log|z| \arg(1-z)$, ..

Are these single-valued. Pure Thought: $H(z)$ on $\mathbb{P}^1 - \{0,1,\infty\}$

Note the Hodge structure $W.\, H(z)_{\mathbb{Q}}$, $F^{\cdot} H(z) \otimes \mathcal{O}$ well-defined

Abstract construction: $H(z) = \bigoplus J^{p,q}(z)$,

semi-simple $T: H \to H$, .. $e^{i\delta_T}$, $\overline{T}$

$\delta \in \mathrm{End}(H(z))$ $\quad \delta$ is defined in terms of basis

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th place.}$$

Elliptic polylog. (Levin, Beilinson)

$$\begin{array}{c} X \\ \downarrow p \\ B \end{array}$$ family of elliptic curves

$\mathbb{G}_m$ $\qquad 0 \to \mathbb{Q}(1)_{\mathbb{G}_m} \to K_z \to \mathbb{Q}_{\mathbb{G}_m} \to 0$

$$\begin{pmatrix} 1 & 0 \\ \log z & 2\pi i \end{pmatrix}$$

sheaf $\underline{\log}$ on X $\qquad \underline{\log}_x$ = homotopy classes of maps in fibre $p^{-1}px$ from 0 to x

sheaf $\mathcal{H}$ on B

$\mathcal{H}_b = \pi_1(p^{-1}(b)) = \mathbb{Z}^2$ $\qquad \underline{\log}$ torsor for $p^*\mathcal{H}$

Consider group algebra $\mathbb{Q}[\mathcal{H}] \to \mathbb{Q}$

$h \longmapsto 1$

Ker = I - augmentation ideal

Γ

$$\mathcal{R} = \varprojlim_n \mathbb{Q}[\mathcal{H}]/I^n$$

$I \subset \hat{\mathcal{R}}$ $\quad gr_I \hat{\mathcal{R}} = \mathbb{Q}[[\mathcal{H}]]$

$\mathbb{G}_m$ case : $\varprojlim_n \mathrm{Sym}^n K_{\mathbb{Z}}$ $\quad \mathrm{Sym}^n K_{\mathbb{Z}} \to \mathrm{Sym}^{n-1} K_{\mathbb{Z}}$

<u>Elliptic case:</u> $p^* \mathcal{R} \otimes \mathbb{Q}[\underline{\log}] =$ completion of $\mathbb{Q}[\log]$ by powers of $p^* I$.

$G = p^* \mathbb{Q}[\mathcal{H}]$

$$\mathcal{R} \xrightarrow{\cong} \mathbb{Q}[[\mathcal{H}]]$$

$\cup \qquad \cup$

$\mathcal{H} \ni a \longmapsto \exp(a) = 1 + a + a^2/2 + \ldots$

$h \in \mathcal{H}$ $\quad h = 1 - (1-h)$ $\quad \log(h) = -\sum \frac{(1-h)^n}{n}$, $1-h \in \Gamma$

Universal property of $\underline{\log}$

$\mathcal{F}$ unipotent local system on X

$\mathcal{H}om(\underline{\log}, \mathcal{F}) \cong \mathcal{O}^* \mathcal{F}$ on B

$$\begin{tikzcd} U := X - O(B) \arrow[r, phantom, "\subset"] \arrow[dr, "p_U"'] & X \arrow[d, "p"] \\ & B \end{tikzcd}$$

$G^{(n)} := G / p^* I^{n+1} G$

<u>Lemma</u> $R^1 p_{U*} G_U^{(n)}(1) \cong I/I^{n+2}$

(Important, But omitted).

<u>Cor.</u> $\mathrm{Ext}^1(p_U^* \mathcal{L}, G_U(1)) \cong \mathcal{H}om(\mathcal{L}, I)$, $\mathcal{L}$ local syst on B

Cor.

$$\begin{tikzcd} 0 \arrow[r] & G_U(1) \arrow[r] & \tilde{P} \arrow[r] & p_U^* I \arrow[r] & 0 \\ 0 \arrow[r] & G_U(1) \arrow[r] \arrow[u] & P \arrow[r] \arrow[u] & p_U^* \mathcal{H} \arrow[u, "\log"'] & \end{tikzcd}$$

↖ elliptic polylogs.

Analog of matrix polylogs

entries

q-averages of polylogs

$$\sum_{r=-\infty}^{\infty} Li_n(q^r z)$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "p"'] & \bar{X} \arrow[d, "\bar{p}"] \\ B \arrow[r, hook] & \bar{B} \end{tikzcd}$$

← Neron model

affine curve

$\bar{B} - B + \infty$

D disk around ∞ in $\bar{B}$

$\{x \in \mathbb{C} \mid |x| < 1\} = D^* = D - \infty$

X^* ← Tate curve.

$\downarrow$

U^*

$$X_q^* = \mathbb{C}^* / q^{\mathbb{Z}}$$

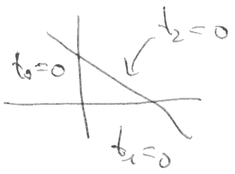
$q \in D$

Geometry which yields "interesting periods"

(Beilinson Conjs periods ⟷ Special values of L-functions)

Motivic Cohomology via algebraic cycles.

$$\Delta^n_k := \operatorname{Spec}\left(k[t_0, \dots, t_n] / (\Sigma t_i - 1)\right) \cong \mathbb{A}^n$$



$t_2 = 1 - t_0 - t_1$

$\Delta^{n-1} \overset{\iota_i}{\hookrightarrow} \Delta^n$ $\vdots$ $\hookrightarrow$

$$\mathcal{Z}^p(X \times \Delta^n)$$

cod. p alg. cycles

$\sum_{\text{finite}} n_i Z_i$ $\quad Z_i \subset X \times \Delta^n$ cod. p irred. subvariety.

$$Z^p(X\times\Delta^n) \xrightarrow{\sum(-1)^i \eta_i^*} Z^p(X\times\Delta^{n-1})$$

$$\cup$$

$$\to Z^p(X,n) \xrightarrow{\sum(-1)^i \eta_i^*} Z^p(X,n-1)$$

cycles in good position w.r.t. faces.

Complex $Z^p(X,\cdot)$ in negative degrees

$$CH^p(X,n) = H_n(Z^p(X,\cdot))$$

Motivic cohomology $H^a_M(X,\mathbb{Z}(b)) := CH^b(X, a-2b)$

Examples 1) $CH^p(X,0) := \mathrm{Coker}\left(Z^p(X,1) \xrightarrow{\eta_1^*-\eta_0^*} Z^p(X)\right) =: CH^p(X)$

$$\| \quad H^{2p}_M(X,\mathbb{Z}(p))$$

2) X = alg. surface ($\dim_{\mathbb{C}} X = 2$)

$$\underset{\sum(C_i,f_i)}{CH^2(X,1)} = H^3_M(X,\mathbb{Z}(2))$$

Represented by $C_i \subset X$ curves $f_i \in k(C_i)^*$

(f_i) = 0-cycle on X given by (f_i) on C_i

$\sum(C_i, f_i)$ $\quad \sum(f_i) = 0$ as 0-cycle on X.

2 rational curve

$(f) = (p)-(q)$, $(g) = (q)-(p)$.

Note periods from here gives the transcendental part of H^2.

3) X curve $\quad CH^2(X, 2) \quad X \times \Delta^2 \supset$ curves.

$\square := \mathbb{P}^1 - \{1\}$ $\quad \square^n = (\square)^n$

Compute these groups using cubical cycles.

$X \times \square^2 \quad f, g \in k(X)^* \quad \Gamma_{f,g} := \{(x, f(x), g(x))\} \subset X \times \square^2$

$CH^2(X, 2) \longleftrightarrow K_2(X) \longleftrightarrow H^2_M(X, \mathbb{Z}(2))$

periods $\longleftrightarrow L(H^1(X), s=2)$

$X = \operatorname{Spec} k$

$CH^2(\operatorname{Spec} k, 3)$

$\downarrow$

$H^1_M(\operatorname{Spec} k, \mathbb{Z}(2))$

Totaro cycles. $\quad$ Curves $T_x \subset \square^3$

$T_x := \{(t, 1-t, 1-xt^{-1})\}$.

$T_x \in Z^2(\square^3) \xrightarrow{\partial} Z^2(\square^2) \xrightarrow{\partial} \cancel{Z^2(\square^1)}$
$\quad \cancel{Z^2(\square^1)} = 0$

$x \in k - \{0, 1\}$

$\partial T_x = \pm (x, 1-x) \in Z^2(\square^2)$

$H^2(\operatorname{Spec} k, \mathbb{Z}(2)) \cong K_2^M(k)$

gives Steinberg relation

Hodge structures associated to H_M^*

Classically: $\overset{\dim_{\mathbb{C}} X = d}{X}$ smooth variety $/\mathbb{C}$

$Z = \sum n_i Z_i$ cod. p cycle on X

$|Z| = \cup Z_i$

Gysin sequence. $\quad U = X - |Z|$ $\quad$ → how to explain this residue for singular Z!

$H_{\underbrace{2d-2p+1}_{> \dim_{\mathbb{R}} |Z|,\ =0}}(|Z|, \mathbb{Q}(p-d)) \to H^{2p-1}(X, \mathbb{Q}(p)) \to H^{2p-1}(U, \mathbb{Q}(p)) \xrightarrow{?} H_{2d-2p}(|Z|, \mathbb{Q}(p-d)) \to$

$\to H^{2p}(X, \mathbb{Q}(p)) \xleftarrow{1 \mapsto [Z]} \bigoplus_i \mathbb{Q}(0) \leftarrow \mathbb{Q}(0)$ $\quad$ assume $= 0$

$[Z] = 0 \qquad Z \mapsto [Z]_H \in \mathrm{Ext}^1_{HS}$

$$0 \to H^{2p-1}(X, \mathbb{Q}(p)) \to M_Z \to \mathbb{Q}(0) \to 0$$

I.e. cycle class $[Z] = 0$ in H^{2p}

then get $\quad [Z]_H \in \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^{2p-1}(X, \mathbb{Q}(p)))$.