8.04.14

Lecture 3

Real valued polylogarithms

e.g. $\log|z|$, $\operatorname{Im} Li_2(z) - \log|z| \arg(1-z)$, ...

Are these single-valued? yes

Pure Thought: $H(z)$ on $\mathbb{P}^1 - \{0, 1, \infty\}$

Note: the Hodge structure $W_\cdot H(z)_{\mathbb{Q}}$; $F^\cdot H(z)\otimes\mathbb{C}$ are well-defined

Abstract construction: $H(z) = \oplus J^{p,q}(z)$, semi-simple T, ...

$e^{i\tilde{\delta}}$ - measures difference between T, $\overline{T}$

$\delta \in \operatorname{End}(H(z))$ - well-defined

$\tilde{\delta}$ is defined in terms of basis $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ i-th place

δ - single-valued

Elliptic polylog: (Levin, Beilinson)

$\begin{array}{c} X \\ p \downarrow \uparrow 0 \\ B \end{array}$ family of elliptic curves

On $\mathbb{G}_m$: $0 \to \mathbb{Q}(1)_{\mathbb{G}_m} \to K_2 \to \mathbb{Q}_{\mathbb{G}_m} \to 0$

$\underline{\log}$ on X $\quad \underline{\log}_x$ = homotopy classes of maps in fibre $p^{-1}p(x)$ from 0 to x

Sheaf $\mathcal{H}$ on B $\quad \mathcal{H}_b = \pi_1(p^{-1}(b)) = \mathbb{Z}^2$

$\underline{\log}$ - torsor for $p^*\mathcal{H}$

Consider group alg. $\mathbb{Q}[\mathcal{H}]$

$$\begin{aligned} \mathbb{Q}[\mathcal{H}] &\to \mathbb{Q} \\ h &\mapsto 1 \end{aligned} \qquad \ker = I \text{ - augmentation ideal}$$

$$R = \varprojlim \mathbb{Q}[\mathcal{H}]/I^n \qquad I \subset R$$

$$\widehat{gr}_I R = \mathbb{Q}[[\mathcal{H}]]$$

$\mathbb{G}_m$ case: $\varprojlim_n \mathrm{Sym}^n K_{\mathbb{Z}}$

<u>Elliptic case</u>: $\underbrace{p^* R \otimes_{p^*\mathbb{Q}[\mathcal{H}]} \mathbb{Q}[\log]}_{\mathcal{G}\, =}$ = completion of $\mathbb{Q}[\log]$ by powers of $p^* I$

$$\begin{array}{ccc} R & \xrightarrow{\sim} & \mathbb{Q}[[\mathcal{H}]] \\ \cup & & \cup \\ \mathcal{H} \ni a & \mapsto & \exp(a) = 1 + a + \frac{1}{2}a^2 + \dots \end{array}$$

$h \in \mathcal{H}$ $\quad h = 1 - (1-h) \quad \log(h) = -\sum \frac{(1-h)^n}{n}$

$1 - h \in I$

Universal property of $\log$

$\mathcal{F}$ - local system on X, unipotent

$\mathrm{Hom}(\log, \mathcal{F}) \cong 0^* \mathcal{F}$ on B

$$\begin{array}{ccc} U := X \setminus 0(B) & \subset & X \\ & {\scriptstyle p_U} \searrow & \downarrow {\scriptstyle p} \\ & & B \end{array} \qquad \mathcal{G}^{(n)} = \mathcal{G}/p^* I^{n+1} \mathcal{G}$$

<u>Lemma</u>: $R^1 p_{U*} \mathcal{G}^{(n)}_U(1) \cong I/I^{n+2}$ (Important, but omitted)

<u>Cor</u>: $\mathcal{L}$ - local system on B, then

$$\mathrm{Ext}^1(p_U^* \mathcal{L}, \mathcal{G}_U(1)) \cong \mathrm{Hom}(\mathcal{L}, I)$$

Cor: $0 \to G_u(1) \to \tilde{P} \to p_u^* I \to 0$

$$\begin{tikzcd}
0 \arrow[r] & G_u(1) \arrow[r] & \tilde{P} \arrow[r] & p_u^* I \arrow[r] & 0 \\
0 \arrow[r] & G_u(1) \arrow[r] \arrow[u] & P \arrow[r] \arrow[u] & p_u^* \mathcal{H} \arrow[r] \arrow[u, "\log"'] & 0
\end{tikzcd}$$

P — elliptic polylog

Analog of matrix of polylogs

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "p"] & \bar{X} \arrow[d, "\bar{p}"] \\
B \arrow[r, hook] & \bar{B}
\end{tikzcd}$$

$\bar{X}$ — neron model

B — affine curve

$\bar{B} = B + \infty$

D — disk around ∞ in $\bar{B}$

$D^* = D \setminus \infty$

$$\begin{tikzcd}
X^* \arrow[d] \\
U
\end{tikzcd}$$

X^* — Tate curve $X_q^* = \mathbb{C}^* / q^{\mathbb{Z}}$

$q \in D^*$

entries — q-averages of polylogs $\sum_{r=-\infty}^{\infty} Li_n(q^r z)$

Geometry which yields "interesting" periods

(Beilinson's conj.; periods $\leftrightarrow$ special val. of L-funct.)

Motivic cohomology via algebraic cycles

$$\Delta_k^n := \mathrm{Spec}\left(k[t_0, \dots, t_n]/(\textstyle\sum t_i - 1)\right) \simeq \mathbb{A}^n$$

$\Delta^{n-1} \overset{\partial_1}{\underset{\partial_{n+1}}{\rightrightarrows}} \Delta^n \qquad Z^p(X \times \Delta^n)$

codim. p alg. cycles $\sum_{\text{finite}} n_i Z_i$

Z_i — codim p irred subvariety

$$Z^p(X \times \Delta^n) \xrightarrow{\sum (-1)^i \partial_i^*} Z^p(X \times \Delta^{n-1})$$

$$\cup \qquad\qquad\qquad \cup$$

$$Z^p(X, n) \xrightarrow{\sum (-1)^i \partial_i^*} Z^p(X, n-1) \qquad \text{Complex } Z^p(X, \cdot)$$

cycles in good positions w.r.t. faces

$CH^p(X,n) = H_n(Z^p(X,\cdot))$

Motivic cohomology $H^a_M(X, \mathbb{Z}(b)) := CH^b(X, a-2b)$

Ex. 1 $CH^p(X,0) := \operatorname{coker}\left(Z^p(X,1) \xrightarrow{i_1^* - i_0^*} Z^p(X)\right) =: CH^p(X)$

$\quad \parallel$

$H^{2p}_M(X, \mathbb{Z}(p))$

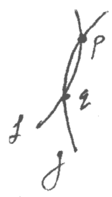
Ex. 2 X = surface ($\dim_{\mathbb{C}} X = 2$)

$CH^2(X,1) = H^3_M(X, \mathbb{Z}(2))$

Represented by $C_i \subset X$ curves $f_i \in k(C_i)^*$

(f_i) = 0-cycle on X given by (f_i) on C_i

(C_i, f_i) $\quad \Sigma(f_i) = 0$ as 0-cycle on X



2 rational curves

$(f) = (p) - (q) \qquad (g) = (q) - (p)$

Ex. 3 X - curve $\quad CH^2(X,2)$

$X \times \Delta^2 \supset$ curves

$\square := \mathbb{P}^1 \setminus \{1\} \qquad \square^n = (\square)^n$

Compute these groups using cubical cycles

$X \times \square^2 \quad f, g \in k(X)^* \qquad H^2_M(X, \mathbb{Z}(2))$

$\qquad\qquad\qquad\qquad\qquad\qquad \updownarrow$

$\Gamma_{f,g} := \{(x, f(x), g(x))\} \subset X \times \square^2 \qquad CH^2(X,2) \longleftrightarrow K_2(X)$

periods $\longleftrightarrow L(H^1(X), s=2)$

Ex. 4 $X = \operatorname{Spec} k$

$CH^2(\operatorname{Spec} k, 3) \longleftrightarrow H^1_M(\operatorname{Spec} k, \mathbb{Z}(2))$

Totaro cycles. Curves $T_x \subset \square^3$

$T_x := \{(t, 1-t, 1-xt^{-1})\}$ $\qquad H^2(\operatorname{Spec} k, \mathbb{Z}(2)) \cong K_2^M(k)$

$T_x \in Z^2(\square^3) \xrightarrow{\partial} Z^2(\square^2) \xrightarrow{\partial} \cancel{Z^2(\square^1)} \to 0$ $\quad x \in k \setminus \{0,1\}$

$\partial T_x = \pm(x, 1-x) \in Z^2(\square^2)$

Hodge structures associated to H^*_M

Classically: X smooth variety $/ \mathbb{C}$, $\dim_{\mathbb{C}} X = d$

$Z = \sum n_i Z_i$ codim p cycle on X

$|Z| = \cup Z_i$

Gysin sequence: $\quad U := X \setminus |Z|$

$$\underbrace{H_{2d-2p+1}}_{>\dim_{\mathbb{R}}|Z|}(|Z|, \mathbb{Q}(p-d)) = 0 \to H^{2p-1}(X, \mathbb{Q}(p)) \to H^{2p-1}(U, \mathbb{Q}(p)) \xrightarrow{?} H_{2d-2p}(|Z|, \mathbb{Q}(p-d)) \to H^{2p}(X, \mathbb{Q}(p))$$

$\mathbb{Q}(0) \to \oplus \mathbb{Q}(0) \subseteq H_{2d-2p}(|Z|, \mathbb{Q}(p-d))$, $\quad 1 \mapsto |Z|$, suppose $= 0$

$[Z] = 0 \rightsquigarrow \quad 0 \to H^{2p-1}(X, \mathbb{Q}(p)) \to M_Z \to \mathbb{Q}(0) \to 0$

Cycle class $[Z] = 0$ in H^{2p}, then get $[Z]_H \in \operatorname{Ext}^1_{HS}(\mathbb{Q}(0), H^{2p-1}(X, \mathbb{Q}(p)))$