H Hodge structure $H = H_{\mathbb{Q}}$ (or $H_{\mathbb{Z}}$), $W_\cdot H_{\mathbb{Q}}$ weight

$F^\cdot H_{\mathbb{C}}$ H pure of wt n $W_n H = H$ $W_{n-1} H = (0)$

$H_{\mathbb{C}} = \bigoplus_{p=0}^{n} H^{p,n-p}$ $H^{p,n-p} = F^p H_{\mathbb{C}} \cap \bar{F}^{n-p} H_{\mathbb{C}}$

Note $F^p = \bigoplus_{q \geq p} H^{q,n-q}$

H mixed $H_{\mathbb{C}} = \bigoplus J^{p,q}$ such $W_n H_{\mathbb{C}} = \bigoplus_{p+q \leq n} J^{p,q}$

$F^p H_{\mathbb{C}} = \bigoplus_{p \geq k} J^{p,q}$ but $\bar{J}^{p,q} \equiv J^{q,p} \bmod \bigoplus_{\substack{p'<p \\ q'<q}} J^{p',q'}$

Reference Cattani, Kaplan, Schmid – Degeneration of HS;
Kato + Collaborators

H MHS $\mathbb{R}$-split if $\bar{J}^{p,q} = J^{q,p}$

$H = \bigoplus J^{p,q} \rightsquigarrow T : H_{\mathbb{C}} \to H_{\mathbb{C}}$ semi-simple T = mult by $p+q$ on $J^{p,q}$

Conversely given T semisimple $\in End(H, W)$ such that
T induces mult by n on $gr^W_n H$.
T splits $W_\cdot H$ $H = \bigoplus gr^W_n H$.

$H = \bigoplus J^{p,q}$ T = mult by $p+q$ on $J^{p,q}$

$\bar{T} : H_{\mathbb{C}} \to H_{\mathbb{C}}$ $T - \bar{T} \in L^{-1,-1}$ $L^{a,b} = \{ f \in End(H_{\mathbb{C}}, W) \mid f(J^{p,q}) \subseteq \bigoplus_{\substack{p' \leq p+a \\ q' \leq q+b}} J^{p',q'} \}$

Not obvious (C-K-S, Prop 2.2).

$\exists !\, Z \in L^{-1,-1}$ such that $\bar{T} = Ad(e^{Z}) T$

Note $\bar{Z} = -Z$ $\delta := \frac{iZ}{2}$ $\delta \in L^{-1,-1}_{\mathbb{R}}$ $\bar{T} = Ad(\exp(-2i\delta)) T$

$\tilde{F} := e^{-i\delta} F$

1. $e^{-i\delta} W = W$

2. $\tilde{F}$ and F induce same H. S's on $gr^W H$
   $(H, W, \tilde{F})$ is a M.H.S

3. $\tilde{T} = Ad(e^{\cancel{+}i\delta}) T = Ad(e^{-i\delta}) \bar{T} \Rightarrow \tilde{T} : H_{\mathbb{R}}$?
   $\tilde{T}$ leaves $\tilde{F}$ invariant, so $\tilde{T}$ gives an $\mathbb{R}$-splitting for $(H, W, \tilde{F})$

$$\begin{array}{ccc} (H, W, F) & \to & \delta : H \to H \\ \downarrow \varphi & & \varphi \downarrow \circlearrowleft \downarrow \varphi \\ (H', W', F') & & H' \xrightarrow{\delta'} H' \end{array}$$

$H_{\mathbb{C}} = \mathbb{C}^{[0,N]}$ = column vectors

$H_{\mathbb{Q}} := \mathbb{Q}$-span of columns of $A(z)$

$$A(z) = \begin{pmatrix} 1 & 0 & - \\ -li_1(z) & 2\pi i & - \\ -li_2(z) & 2\pi i \log z & \cdots \\ \vdots & \vdots & \\ -li_N(z) & 2\pi i \frac{(\log z)^{N-1}}{(N-1)!} & \cdots \end{pmatrix}$$

mixed H.S. $\quad F^p = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} p+1$

$J^{p,p} = F^p \cap W_{2p} \simeq \begin{pmatrix} 0 \\ 0 \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} - p$

$T =$ multipl. by $2p$ on $J^{p,p}$

$A(z) T A(z)^{-1}$ $\quad$ $\mathbb{Q}$-basis

$$D = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 & \\ & & & \ddots \end{pmatrix}$$

$\bar{T} = Ad(e^{i\delta}) T$

$\bar{A} T \bar{A}^{-1} = e^{-i\delta} A \bar{D} \bar{A}^{-1} e^{+i\delta}$

Solve for δ $\quad \bar{A} = e^{-i\delta} A \cdot e^{i\delta} = \cancel{A\bar{A}^{-1}} A D \bar{A}^{-1} \leftarrow$ unipotent

First column of δ

$$D_k(z) = \begin{cases} \sum_{1 \le \ell \le k} b_\ell \frac{\log|z|^{2k}}{\ell!} \operatorname{Im}(li_{k-\ell}(z)) & k \equiv 0 \ (2) \\ \sum b_k \frac{\log|z|^{2\ell}}{\ell!} \operatorname{Re}(li_{k-\ell}(z)) & k \equiv 1 \ (2). \end{cases}$$

$$\begin{pmatrix} 0 \\ D_1 \\ D_2 \\ \vdots \end{pmatrix} \qquad D_0 = 0$$

b_i: Bernoulli no's $\quad 1, -1/2, 1/6, 0, -1/30, 0$

Ex. $D_2 = \operatorname{Im} Li_2(z) - \log|z| \arg(1-z)$

$D_2 : \mathbb{P}^1 - \{0,1,\infty\} \to \mathbb{R}$ $\qquad \arg(1-z) = \operatorname{Im}(\log(1-z))$

Compare + contrast

D_2 with Rogers dilogarithm

$0 < z < 1 \qquad R_2(z) = Li_2(z) + \frac{1}{2}\log z \log(1-z)$

$\mathcal{O}_U$ = sheaf of polyfunctions on $U = \mathbb{P}^1 - \{0,1,\infty\}$

$H = \mathcal{O}^{[0,N]} \qquad \nabla_{\frac{d}{dz}} : \mathcal{O}^{[0,N]} \to \mathcal{O}^{[0,N]}$

Connection

$\nabla_{\frac{d}{dz}} : d - e_0 \frac{dz}{z} - e_1 \frac{dz}{z-1}$

$$\begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & N \end{array} \\ e_0 = \begin{pmatrix} 0 & 0 & 0 & & \\ \vdots & 0 & 0 & & \\ & 1 & 0 & & 0 \\ & & 1 & & \\ 0 & & & \ddots & 0 \\ & 0 & & 1 & 0 \end{pmatrix} \end{array} \qquad e_1 = \begin{pmatrix} 0 & & \\ 1 & & \\ 0 & 0 & \\ \vdots & & \end{pmatrix}$$

$A(z)\, \mathcal{O}^{[0,N]}$ = Horizontal sections

Variation of H.S. over $\mathbb{P}^1 - \{0,1,\infty\}$

vector bundle $\mathcal{O}^{[0,N]}$ connection ∇

~~horizontal~~ columns of $A(z)$ are Horizontal

Griffiths transversality

$\nabla F^p \subseteq F^{p-1}$

$$\nabla F^p = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} N-p \;=\; \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ 0 \end{pmatrix} \Big\} N-p+1$$

F. Brown Real periods

Think of polylogs as extensions

$K_z \hookrightarrow \begin{pmatrix} 1 & 0 \\ \log z & 2\pi i \end{pmatrix}$ Variation of H.S. over $\mathbb{P}^1 - \{0,\infty\} = \mathbb{G}_m$

$0 \to \mathbb{Q}(1)_{\mathbb{G}_m} \to K_z \to \mathbb{Q}(0)_{\mathbb{G}_m} \to 0$ $\qquad$ split at $z=1$

$$0 \to (\mathrm{Sym}^{N-1} K_Z)(1) \to \mathcal{P}^{(N)} \to \mathbb{Q}(0) \to 0$$

the point is $\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0)_U, \mathcal{G}(1)) = \mathbb{Q}_0 \oplus \mathbb{Q}_1$

$U = \mathbb{P}^1 - \{q, 1, \infty\}$

$Q \neq 1$

$\mathcal{G} = \varprojlim \mathrm{Sym}^N K_Z$

$\mathcal{G}(1) = \mathcal{G} \otimes \mathcal{O}(1)_U$

$$\mathrm{Ext}^1_{HS}(\mathbb{Q}(0), \underbrace{\mathrm{Hom}_{LS}(\mathbb{Q}(0)_U, \mathcal{G}(1))}_{(0)})$$

$\cancel{\mathrm{Ext}^1_{HS}(\mathbb{Q}, \mathrm{Hom}_{LS}(\mathbb{Q}(0)_U, \mathcal{G}(1))}$

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{G}(1) & \to & \mathcal{P}^N & \to & \mathbb{Q}(0)_U & \to & 0 \\ & & \downarrow & & & & \| & & \\ 0 & \to & \mathrm{Sym}^{N-1} K_Z(1) & \to & \mathcal{P}^{(N)} & \to & \mathbb{Q}(0)_U & \to & 0 \end{array}$$

$\mathrm{Hom}_{HS}(\mathbb{Q}(0), H^1(U, \mathcal{G}(1))) = \mathbb{Q}^{\oplus 2}$

$\mathcal{G}(1) = \text{"}\varprojlim\text{"} \mathrm{Sym}^N K_Z$.

$\mathcal{G} \cong$ an ext. of $\mathcal{G}(1)$ by $\mathbb{Q}(0)_U$

X reasonable top. space

Π sheaf on X $\Pi_x = \pi_1(X, x)$
of groups

$\mathbb{P}$

$\Pi / [[\Pi, \Pi], [\Pi, \Pi]]$

$$0 \to [\Pi, \Pi]^{ab} \to \mathbb{P} \to \pi_1(X)^{ab}_X \to 0$$

$\mathrm{Lie}(\mathbb{P}) = X$-polylog. sheaf

$$0 \to \mathcal{G}(1) \to \mathcal{P}^{\log} \to \mathbb{Q}(0)^2_U \to 0$$

$\|$

$\mathrm{Lie}(\pi_1(U)_U)$