3.04.2014

Lecture 2

H - Hodge str-re $H = H_{\mathbb{Q}}$ (or $H_{\mathbb{Z}}$) $W_\bullet H_{\mathbb{Q}}$ weight filtr

$F^\bullet H_{\mathbb{C}}$ - Hodge filtr.

H - pure of weight n, if $W_n H = H$, $W_{n-1} H = 0$,

and $H_{\mathbb{C}} = \bigoplus_{p=0}^{n} H^{p,n-p}$, $H^{p,n-p} = F^p H_{\mathbb{C}} \cap \bar{F}^{n-p} H_{\mathbb{C}}$

Note: $F^p = \bigoplus_{q \geq p} H^{q,n-q}$

H - mixed $\Rightarrow$ $H_{\mathbb{C}} = \bigoplus J^{p,q}$, $W_n H_{\mathbb{C}} = \bigoplus_{p+q \leq n} J^{p,q}$,

$F^k H_{\mathbb{C}} = \bigoplus_{p \geq k} J^{p,q}$,

but $\bar{J}^{p,q} \equiv J^{q,p} \mod \bigoplus_{\substack{p'<p \\ q'<q}} J^{p',q'}$

Cattani, Kaplan, Schmid - Degenerations of HS's

Kato + collaborators

H - MHS is $\mathbb{R}$-split if $\bar{J}^{p,q} = J^{q,p}$

$H_{\mathbb{C}} = \bigoplus J^{p,q} \rightsquigarrow T: H_{\mathbb{C}} \to H_{\mathbb{C}}$ semi-simple

$T =$ multiplication by $p+q$ on $J^{p,q}$

Conversely, given $T \in End^{W}(H, W)$ semisimple,

respect weight filtr

such that T induces mult. by n on $gr_n^W H$.

T splits $W_\bullet H$; $H = \oplus gr_n^W H$

$H = \oplus J^{p,q}$, T = mult by $p+q$ on $J^{p,q}$

$\overline{T}: H_{\mathbb{C}} \to H_{\mathbb{C}}$; $T - \overline{T} \in L^{-1,-1}$

where $L^{a,b} = \{ f \in End(H, W) \mid f(J^{p,q}) \subseteq \bigoplus_{\substack{p' \le p+a \\ q' \le q+b}} J^{p',q'} \}$

Not obvious (C-K-S, Prop. 2.2.):

$T - \overline{T} \in L^{-1,-1} \Rightarrow \exists!\, Z \in L^{-1,-1}$ such that

$\overline{T} = Ad(e^{Z}) T$; Note: $\overline{Z} = -Z$

$\delta = \frac{iZ}{2}$ $\delta \in L^{-1,-1}_{\mathbb{R}}$, $\overline{T} = Ad(\exp(-2i\delta))T$

Define: $\tilde{F} := e^{-i\delta} F$ – new filtration

1) $e^{-i\delta} W = W$

2) $\tilde{F}$ and F induce same HS on $gr_{\cdot}^{W} H$

$(H, W, \tilde{F})$ is a MHS

3) $\tilde{T} = Ad(e^{i\delta}) T = Ad(e^{-i\delta}) \overline{T} \Rightarrow \tilde{T}: H_{\mathbb{R}} \to H_{\mathbb{R}}$

$\tilde{T}$ leaves $\tilde{F}$ invariant, so $\tilde{T}$ gives an $\mathbb{R}$-splitting for $(H, W, \tilde{F})$

$$\begin{array}{ccc} (H, W, F) & \longmapsto & \begin{tikzcd} H \arrow[r, "\delta"] \arrow[d, "\varphi"'] & H \arrow[d, "\varphi"] \\ H' \arrow[r, "\delta'"] & H' \end{tikzcd} \\ \downarrow \varphi & & \\ (H', W', F') & & \end{array}$$

– commutative

$$A(z) = \begin{pmatrix} 1 & 0 & \\ -Li_1(z) & 2\pi i & \\ -Li_2(z) & 2\pi i \log z & \\ \vdots & \vdots & \\ -Li_N(z) & 2\pi i \frac{(\log z)^{N-1}}{(N-1)!} & \cdots \end{pmatrix}$$

$H_{\mathbb{C}} = \mathbb{C}^{[0,N]}$ – column vectors

$H_{\mathbb{Q}} = \mathbb{Q}$-span of columns of $A(z)$

mixed HS $F^p = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \} N-p$

$J^{p,p} = F^p \cap W_{2p}$ $T =$ mult by $2p$ on $J^{p,p}$

$\shortparallel$

$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \} N-p$

$A(z)TA(z)^{-1}$ $\mathbb{Q}$-basis

$\bar{T} = Ad(e^{-i\tilde{\delta}})T$ $\qquad \bar{A}T\bar{A}^{-1} = e^{-i\tilde{\delta}}ATA^{-1}e^{i\tilde{\delta}}$

Solve for $\tilde{\delta}$ $\qquad \bar{A} = e^{-i\tilde{\delta}}A$ $\quad e^{i\tilde{\delta}} = A\bar{A}^{-1}$ – bad

$D = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \\ & & & & \ddots \end{pmatrix}$ $\qquad \bar{A}T\bar{A}^{-1} = e^{-i\tilde{\delta}}ADTD^{-1}A^{-1}e^{i\tilde{\delta}}$

$e^{i\tilde{\delta}} = AD\bar{A}^{-1}$ ← unipotent, can take log

First column of $\tilde{\delta}$:

$$D_k(z) = \begin{cases} \sum_{1\le \ell \le k} b_\ell \frac{\log|z|^{2\ell}}{\ell!} \operatorname{Im}(Li_{k-\ell}(z)), & k \equiv 0(2) \\ \sum b_\ell \frac{\log|z|^{2\ell}}{\ell!} \operatorname{Re}(Li_{k-\ell}(z)), & k \equiv 1(2) \end{cases}$$

$D_0 = 0$ $\qquad b_\ell$ – Bernoulli numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \ldots$

$\begin{pmatrix} 0 \\ D_1 \\ D_2 \\ \vdots \end{pmatrix}$

<u>Ex</u> $D_2 = \operatorname{Im} Li_2(z) - \log|z| \underset{\operatorname{Im}(\log(1-z))}{\underbrace{\arg(1-z)}}$

Compare + contrast D_2 with Rogers dilogarithm

$0<z<1 \qquad R_2(z) = Li_2(z) + \frac{1}{2} \log z \log(1-z)$

$D_n : \mathbb{P}^1 \setminus \{0,1,\infty\} \to \mathbb{R}$ – continuous function

$\mathcal{O}$ = sheaf of poly. functions on $U = \mathbb{P}^1 \setminus \{0,1,\infty\}$

$H = \mathbb{C}^{[0,N]} \qquad H \otimes \mathcal{O} = \mathcal{O}^{[0,N]}$ – trivial vector bundle

connection $\nabla_{\frac{d}{dz}} : \mathcal{E}^{[0,N]} \longrightarrow \mathcal{E}^{[0,N]}$

$$\nabla_{\frac{d}{dz}} : d - e_0 \frac{dz}{z} - e_1 \frac{dz}{z-1}$$

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & \ddots & & \\ \vdots & \vdots & & & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 0 & & \\ 1 & & \\ 0 & 0 & \\ \vdots & & \\ 0 & & \end{pmatrix}$$

$A(z)\,\mathbb{C}^{[0,N]}$ = Horizontal sections

Variation of HS over $\mathbb{P}^1 \setminus \{0,1,\infty\}$

Vector bundle $\mathcal{E}^{[0,N]}$ connection ∇

columns of $A(z)$ are horizontal

Griffiths transversality: $\nabla F^p \subseteq F^{p-1}$

$$N-p \left\{ \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \end{pmatrix} \right. \longmapsto \left. \begin{pmatrix} * \\ \vdots \\ * \\ * \\ 0 \\ \vdots \end{pmatrix} \right\} N-p+1$$

Think of polylogs as extensions

$K_z \longleftrightarrow \begin{pmatrix} 1 & 0 \\ \log z & 2\pi i \end{pmatrix}$ Variation of HS over $\mathbb{P}^1 \setminus \{0,\infty\}$ split at $z=1$

$$0 \to \mathbb{Q}(1)_{\mathbb{G}_m} \to K_z \to \mathbb{Q}(0)_{\mathbb{G}_m} \to 0$$

$$0 \to (\mathrm{Sym}^{N-1} K_z)(1) \to \mathcal{P}^{(N)} \to \mathbb{Q}(0)_{\mathbb{P}^1 \setminus \{0,1,\infty\}} \to 0$$

The point is

$$\mathrm{Ext}^1_{MHSh}(\mathbb{Q}(0)_U, G(1)) = \mathbb{Q}_0 \oplus \mathbb{Q}_1$$

$$G = \varprojlim \mathrm{Sym}^N K_z \ ; \quad G(1) = G \otimes \mathbb{Q}(1)_U$$

$1 \in \mathbb{Q}_1$ gives

$$\begin{tikzcd} 0 \arrow[r] & G(1) \arrow[r] \arrow[d] & \mathcal{P}^\infty \arrow[r] \arrow[d] & \mathbb{Q}(0)_U \arrow[r] & 0 \\ 0 \arrow[r] & (\mathrm{Sym}^{N-1} K_z)(1) \arrow[r] & \mathcal{P}^{(N)} \arrow[r] & \mathbb{Q}(0)_U \arrow[r] & 0 \end{tikzcd}$$

$$\left[\begin{array}{l} \mathrm{Ext}^1_{HS}(\underset{1}{\overset{\mathbb{Q}(0)}{}}, \overbrace{\mathrm{Hom}_{Loc. Sys.}(\mathbb{Q}(0)_U, G(1))}^{0}) \\ \mathrm{Hom}_{HS}(\mathbb{Q}(0), H^1(U, G(1))) = \mathbb{Q}^{\oplus 2} \end{array}\right.$$

$1 \in \mathbb{Q}_0$ gives $G \cong$ extension of $G(1)$ by $\mathbb{Q}(0)_U$

X reasonable topological space

Π – sheaf of groups on X $\quad \Pi_x = \hat{\pi}_1(X, x)$

$$\mathbb{P} = \Pi / [[\Pi,\Pi],[\Pi,\Pi]]$$

$\mathbb{P}$ – sheaf of groups on X

$$0 \to [\pi, \pi]^{ab} \longrightarrow \mathbb{P} \longrightarrow \hat{\pi}_1(X)_x^{ab} \longrightarrow 0$$

$\mathrm{Lie}(\mathbb{P}) = X$-polylog sheaf

$$\left(\begin{array}{l} 0 \to G(1) \to \mathcal{P}^{Big} \to \mathbb{Q}(0)_{\mathcal{U}}^2 \to 0 \\ \qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\quad \mathrm{Lie}(\bar{\pi}_1(\mathcal{U})_u^{ab}) \end{array} \right)$$