Periods $\quad 2\pi i = \int_{S^1} \frac{dz}{z}$

I. Polylogarithms (Deligne - Beilinson, AMS Proc. Symp pure math. 55)

II Elliptic polylogarithms (Beilinson, Levin)

III motivic Cohomology

IV Applications - A. Nahm's conjecture

B. Amplitudes + Regulators

V Work of F. Brown on mixed Tate motives $/\mathbb{Z}$

Polylogarithms $\quad Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ polylogarithms

matrix (*)

$$\begin{pmatrix} 1 & 0 & 0 & & \\ -Li_1(z) & 2\pi i & 0 & & \\ -Li_2(z) & 2\pi i \log(z) & (2\pi i)^2 & 0 & \\ \vdots & \vdots & \vdots & & \\ -Li_n(z) & \frac{2\pi i (\log z)^{n-1}}{(n-1)!} & \frac{(2\pi i)^2(\log z)^{n-2}}{(n-2)!} & \cdots & (2\pi i)^n \end{pmatrix} =: A(z)$$

$\mathbb{P}^1 - \{0, 1, \infty\} =: U$

Variation of Hodge structures on U!

$H_{\mathbb{Q}}$ f.-d. $\mathbb{Q}$-vectors space.

weight filtration $W_\cdot H$ $\quad W_n H \subset W_{n+1} H \subset \dots$

Hodge filtration $F^\cdot H_{\mathbb{C}}$ $\quad H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C}$

$F^n H_{\mathbb{C}} \supset F^{n+1} H_{\mathbb{C}} \dots$

Pure case $\exists n$ $\quad W_n H_{\mathbb{Q}} = H_{\mathbb{Q}}$, $W_{n-1} H_{\mathbb{Q}} = 0$

Pure H.S. of wt n if writing $\bar{F} H_{\mathbb{C}}$ conjugate filtr

any d $\quad H_{\mathbb{C}} = F^d H_{\mathbb{C}} \oplus \bar{F}^{n-d+1} H_{\mathbb{C}}$

<u>Example</u> $\quad H_{\mathbb{Q}} = H^1_{\text{Betti}}(C, \mathbb{Q})$

compact Riemann surface. $\qquad g = \text{genus } C$

pure H.S. of wt $\underline{1}$ $\qquad F^1 H_{\mathbb{C}} = \Gamma(C, \Omega^1_C) \cong \mathbb{C}^g$ $\quad = H^{1,0}$

$\bar{F}^1 H = H^{1,0}$ $\qquad H^1(C, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$

$H_{\mathbb{Q}}$ mixed H.S. $\qquad gr^W_\cdot H_{\mathbb{Q}} = \bigoplus W_n H_{\mathbb{Q}} / W_{n-1} H_{\mathbb{Q}}$

$F^\cdot H_{\mathbb{C}}$ induces filtr. on $gr^W_\cdot H_{\mathbb{C}}$

Require $gr^W_n H_{\mathbb{Q}}$ with this filtration should be pure of wt n.

<u>Ex.</u> Tate H.S. $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

$\mathbb{Q}(n)_{\mathbb{Q}}$ 1-dim $\mathbb{Q}$-v.s.

weight $-2n$

$F^\cdot \mathbb{Q}(n)_\mathbb{C}$ $\qquad \mathbb{Q}(n)_\mathbb{C}^{-n,-n}$

$F^{-n}\mathbb{Q}(n)_\mathbb{C} = \mathbb{Q}(n)_\mathbb{C} \qquad F^{-n+1} = (0)$

$H_\mathbb{Q}$ m. H. S.

$H_\mathbb{Q}$ mixed Tate H.S. if $gr_n^W H_\mathbb{Q} = \begin{cases} 0 & n \equiv 1 \ (2) \\ \oplus \mathbb{Q}(-\frac{n}{2}) & n \text{ even} \end{cases}$

$H_\mathbb{Q}$ H.S. + D.R structure.

$H_{DR} \otimes_\mathbb{Q} \mathbb{C} \xrightarrow{\sim} H_\mathbb{C}$

$\mathbb{Q}(n)_{DR} = \mathbb{Q}$

$\mathbb{Q}(n)_\mathbb{Q} = (2\pi i)^n \mathbb{Q}(n)_{DR}$

Betti

$\int_{S^1} \frac{dz}{z} = 2\pi i$

circle around 0

$\frac{dz}{z} \in H^1_{DR}(\mathbb{G}_m, \mathbb{Q})$

$\mathbb{G}_m = \mathbb{P}^1 - \{0, 1, \infty\}$

$H^1_{DR}(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q}\cdot\frac{dz}{z}$

$H^{\mathbb{Z}}_{1,\text{Betti}}(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q}\cdot S^1$

$H^1(\mathbb{G}_m, \mathbb{Q}) = H_\mathbb{Q}$ wt 2

$\quad \|$

$\mathbb{Q}(-1)_\mathbb{Q} = \mathbb{Q}\cdot\gamma \qquad \langle S^1, \gamma\rangle = 1.$

$\langle S^1, \frac{dz}{z}\rangle = 2\pi i \qquad \frac{dz}{z} = 2\pi i \gamma$

$\mathbb{Q}(-1)_{DR} = 2\pi i\, \mathbb{Q}(-1)_\mathbb{Q}$

$\mathbb{Q}(-1)_\mathbb{Q} = (2\pi i)^{-1} \mathbb{Q}(-1)_{DR}$

Tensor structure

$\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$, $\mathbb{Q}(-1) = \mathbb{Q}(1)^{\vee}$

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why $H^1(\mathbb{G}_m, \mathbb{Q})$ wt 2 ?

$X/\mathbb{C}$ smooth, compact $H^p_{\text{Betti}}(X, \mathbb{Q})$ pure wt p

X smooth $H^p(X, \mathbb{Q})$ wt $\geq p$

X compact but singular $H^p(X, \mathbb{Q})$ ~~wt~~ wts $\leq p$.

$H_{\mathbb{Q}}$ mixed Tate H.S.

$W_{2p} H_{\mathbb{C}} \subset H_{\mathbb{C}}$

$F^{p+1} H_{\mathbb{C}}$ $\quad F^{\cdot}$ $\quad \left(W_{2p} / W_{2p-2} \right)_{\mathbb{C}}^{\oplus} = \oplus\, \mathbb{Q}(-p)_{\mathbb{C}}$

$$\oplus\, \mathbb{Q}(-p)^{p,p}_{\mathbb{C}}$$

$\Rightarrow F^{p+1} H_{\mathbb{C}} \cap W_{2p} H_{\mathbb{C}} = (0)$

<u>Exercise</u> Show $H_{\mathbb{C}}$ (m.T.H.S.)

$= \oplus\, W_{2p} H_{\mathbb{C}} \cap F^p H_{\mathbb{C}}$

<u>Ex.</u> $\begin{pmatrix} 1 & 0 \\ \log t & 2\pi i \end{pmatrix}$

$$\begin{tikzcd}
0 \arrow[r] & 2\pi i\mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^* \arrow[r] & 0 \\
0 \arrow[r] & 2\pi i\mathbb{Z} \arrow[u, equal] \arrow[r] & H_{\mathbb{Z}} \arrow[u, "t"] \arrow[r, bend left, "\cancel{\mathbb{Z}\oplus\mathbb{Z}}"] & \mathbb{Z} \arrow[u, "t \mapsfrom 1"'] \arrow[r] & 0
\end{tikzcd}$$

$2\pi i\mathbb{Z} = \mathbb{Z}\cdot 2\pi i\, e_1$ $\quad H_{\mathbb{Z}} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_0$ $\quad e_0 \mapsto 1$

$\psi(e_1) = 2\pi i$ $\qquad$ $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$ $\qquad$ (5)

$\psi(e_0) = \log t$ $\qquad$ $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C}$

$$F^0 H_{\mathbb{C}} = \ker(\psi_{\mathbb{C}})$$

$$\mathbin{/\!/}$$

$$\mathbb{C}\left(\frac{e_1}{2\pi i} - \frac{e_0}{\log t}\right) = \mathbb{C}(\log t\, e_1 - 2\pi i\, e_0)$$

$W_{-2} H_{\mathbb{Q}} = \mathbb{Q} \cdot e_1$

$F^i H_{\mathbb{C}} = \begin{cases} 0 & i > 0 \\ H_{\mathbb{C}} & i < 0 \end{cases}$

$H_{\mathbb{Q}} = \mathbb{Q}$ span of columns of $\begin{pmatrix} 1 & 0 \\ \log t & 2\pi i \end{pmatrix}$

$\mathbb{Q}\begin{pmatrix} 1 \\ \log t \end{pmatrix} \oplus \mathbb{Q}\begin{pmatrix} 0 \\ 2\pi i \end{pmatrix}$ $\qquad$ $F^0 H_{\mathbb{C}} = \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

H mixed Tate H.S.

Assume $gr^W_{2p} H_{\mathbb{Q}}$ dim ≤ 1 $\forall p$

$$\begin{pmatrix} 1 & 0 & & 0 \\ a_{21} & a_{22} & \ddots & \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & & 0 \\ & & & a_{nn} \end{pmatrix}$$

$W_{2p} H_{\mathbb{Q}}$ spanned by columns starting from the right.

$F^p H_{\mathbb{C}} = \begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad$ $\bigoplus_p W_{2p} \cap F^p = H_{\mathbb{C}}$

$$\mathbin{/\!/}$$

$$\begin{pmatrix} 0 \\ * \\ \vdots \\ 0 \end{pmatrix}$$

$A(z)$ describes mixed Tate H.S.

$\begin{pmatrix} 1 & 0 \\ \log z & 2\pi i \end{pmatrix}$ $\qquad$ K_z 2 dim'l.

symbol

Hodge structures – Abelian category

$0 \to \mathbb{Q}(1) \to K_z \to \mathbb{Q}(0) \to 0$] $\mathbb{Q}$-linear Tannakian

<u>Exercise</u> $\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{C}^* \otimes \mathbb{Q}$

$$\underbrace{(\mathrm{Sym}^{n-1} K_z) \otimes \mathbb{Q}(1)}$$

MTHS $\leftrightarrow$ matrix $(*)$

<u>Exercise</u> $\mathcal{P}^{(n)}$ MTHS $\leftrightarrow$ columns of $A(z)$

$$0 \to (\mathrm{Sym}^{n-1} K_z)(1) \to \mathcal{P}^n \to \mathbb{Q}(0) \to 0$$

n-th Polylog. H.S

Vary z local system $\mathcal{P}^{(n)}$ on $\mathbb{P}^1 - \{0,1,\infty\} = \mathbb{C}$
whose fibres have MTHS.
$W_\cdot \mathcal{P}^{(n)}$ rational $W_{2p}\mathcal{P}^{(n)}$ sub-local systems

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Canonically associated to a mixed H.S. is
a "nearby" $\mathbb{R}$-split HS

<u>Ex.</u> $\begin{pmatrix} 1 & 0 \\ \log z & 2\pi i \end{pmatrix}$ $\mathbb{R}$-valued fixed

$\mathrm{Re}\log z = \log|z|$

$\mathbb{R}$-valued function $\leftrightarrow \mathrm{Li}_2(z)$

H mixed H.S.

$F^\cdot H_{\mathbb{C}}$ $\mathrm{gr}^W_n H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$

no evident bigradity on $H_{\mathbb{C}}$ itself

In fact there is $H_{\mathbb{C}} = \bigoplus I^{p,q}$, but $I^{p,q} \neq \overline{I}^{q,p}$

$$I^{p,q} \equiv \overline{I}^{q,p} \bmod \bigoplus_{\substack{p'<p \\ q'<q}} I^{p',q'}$$

Say H is $\mathbb{R}$-split if $I^{p,q} = \overline{I^{q,p}}$

In general, $I^{p,q} = F^p \cap W_{p+q} \cap \left(\overline{F}^q \cap W_{p+q} + \overline{U}^{q-1}_{p+q-2} \right)$

$$U^q_b = \sum_{j \geq 0} F^{q-j} \cap W_{b-j}$$

If H is MTHS $I^{p,p} = F^p \cap W_{2p}$, $I^{p,q} = 0$ $p \neq q$