Spencer Bloch — Periods in alg. geom.

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Lecture 1

Periods: $2\pi i = \int_{S^1} \frac{dz}{z}$ — example of a period

I. Polylogarithms (Deligne–Beilinson, AMS Proc. Symp. Pure Math 55)

II. Elliptic polylogarithms (Beilinson, Levin)

III. Motivic Cohomology

IV. Applications: A) Nahm's conjecture
B) Amplitudes of regulators

V. Work of F. Brown on mixed Tate motives $/\mathbb{Z}$

Polylogarithms $Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}$

$$\begin{pmatrix} 1 & 0 & 0 & & \\ -Li_1(z) & 2\pi i & 0 & & 0 \\ -Li_2(z) & 2\pi i \log(z) & (2\pi i)^2 & & \\ \vdots & \vdots & \vdots & & \\ -Li_n(z) & \frac{2\pi i (\log z)^{n-1}}{(n-1)!} & \frac{(2\pi i)^2 (\log z)^{n-2}}{(n-2)!} & \dots & (2\pi i)^n \end{pmatrix} = A(z)$$

$\mathbb{P}^1 \setminus \{0,1,\infty\} =: U$;

$A(z)$ defines a variation of Hodge structures over U

$H_{\mathbb{Q}}$ — f.d. $\mathbb{Q}$-vector space

Weight filtration: $W_\cdot H_{\mathbb{Q}}$ $\quad W_n H \subset W_{n+1} H \subset \dots$

Hodge filtration: $F^\cdot H_{\mathbb{C}}$, $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C}$; $\quad F^n H_{\mathbb{C}} \supset F^{n+1} H_{\mathbb{C}} \supset \dots$

Pure case: $\exists n$: $W_n H_{\mathbb{Q}} = H_{\mathbb{Q}}$, $W_{n-1} H_{\mathbb{Q}} = 0$

Pure Hodge str-re of weight n:

consider $\bar{F} H_{\mathbb{C}}$ - conjugate filtration, then

for any d $\quad H_{\mathbb{C}} = F^d H_{\mathbb{C}} \oplus \bar{F}^{n-d+1} H_{\mathbb{C}}$

Ex. $H_{\mathbb{Q}} = H^1_{Betti}(C, \mathbb{Q})$ - pure HS of weight 1

$F^1 H_{\mathbb{C}} = \Gamma(C, \Omega^1_C) = H^{1,0}$

$\bar{F} H_{\mathbb{C}} = H^{0,1} \qquad H^1(C, \mathbb{Q}) = H^{1,0} \oplus H^{0,1}$

$H_{\mathbb{Q}}$ - mixed HS, if: $gr^W H_{\mathbb{Q}} = \oplus W_n H_{\mathbb{Q}} / W_{n-1} H_{\mathbb{Q}}$

$F^{\cdot} H_{\mathbb{C}}$ induces filtr. on $gr^W H_{\mathbb{Q}}$;

require that $gr^W_n H_{\mathbb{Q}}$ with the induced filtr is pure HS of wt n.

Ex. Tate HS $\mathbb{Q}(n)$, $n \in \mathbb{Z}$

$\mathbb{Q}(n)_{\mathbb{Q}}$ - 1-dim $\mathbb{Q}$-vector space, weight $= -2n$

$F^{\cdot} \mathbb{Q}(n)_{\mathbb{C}}$: $\qquad F^{-n} \mathbb{Q}(n)_{\mathbb{C}} = \mathbb{Q}(n)_{\mathbb{C}}$, $F^{-n+1} = 0$

$H_{\mathbb{Q}}$ - MHS

Def $H_{\mathbb{Q}}$ is mixed Tate HS, if

$$gr^W_n H_{\mathbb{Q}} = \begin{cases} 0, & n \equiv 1 (2) \\ \oplus \mathbb{Q}(\frac{-n}{2}), & n \text{ even} \end{cases}$$

$H_{\mathbb{Q}}$ – HS

Def $H_{\mathbb{Q}}$ is a HS with de Rham structure, if
H_{DR} – $\mathbb{Q}$-vectorspace + $H_{DR} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{\mathbb{C}}$

$\mathbb{Q}(n)_{DR} = \mathbb{Q}$

$\mathbb{Q}(n)_{\mathbb{Q}} = (2\pi i)^n \mathbb{Q}(n)_{DR}$

Betti

$\int_{S^1} \frac{dz}{z} = 2\pi i$ $\qquad \frac{dz}{z} \in H^1_{DR}(G_m, \mathbb{Q})$, $G_m = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}$

circle around zero $\qquad H^1_{DR}(G_m, \mathbb{Q}) = \mathbb{Q} \cdot \frac{dz}{z}$, $H_1(G_m, \mathbb{Q}) = \mathbb{Q} \cdot S^1$

$H^1(G_m, \mathbb{Q}) = H_{\mathbb{Q}}$ wt 2

$\mathbb{Q}(-1)_{\mathbb{Q}} = \mathbb{Q} \cdot \eta \qquad \langle S^1, \eta \rangle = 1$

$\langle S^1, \frac{dz}{z} \rangle = 2\pi i \Rightarrow \frac{dz}{z} = 2\pi i \, \eta$

$\mathbb{Q}(-1)_{DR} = 2\pi i \, \mathbb{Q}(-1)_{\mathbb{Q}}$

Tensor str.: $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$, $\mathbb{Q}(-1) = \mathbb{Q}(1)^{\vee}$

Remark: $X/_{\mathbb{C}}$ compact, smooth $\Rightarrow$ $H^p_{DR}(X, \mathbb{Q})$ has pure wt p,
X – smooth $\Rightarrow$ $H^p(X, \mathbb{Q})$ has wt $\geq p$
X – compact, singular $\Rightarrow$ $H^p(X, \mathbb{Q})$ has wt $\leq p$

$H_{\mathbb{Q}}$ mixed Tate HS

$W_{2p} H_{\mathbb{C}} \subset H_{\mathbb{C}}$

$F^{p+1} H_{\mathbb{C}} \qquad \mathbb{C} \otimes (W_{2p}/W_{2p-2}) = \bigoplus \mathbb{Q}(-p)_{\mathbb{C}} = \bigoplus \mathbb{Q}(-p)_{\mathbb{C}}^{p,p}$

$\Rightarrow F^{p+1} H_{\mathbb{C}} \cap W_{2p} H_{\mathbb{C}} = (0)$

<u>Exercise</u> Show: $H_{\mathbb{C}}$ (MTHS) can be written as

$$\bigoplus W_{2p} H_{\mathbb{C}} \cap F^p H_{\mathbb{C}}$$

<u>Ex</u> $\begin{pmatrix} 1 & 0 \\ \log t & 2\pi i \end{pmatrix}$

Consider the exp. sequence:

$$\begin{tikzcd}
0 \arrow[r] & 2\pi i \mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}^\times \arrow[r] & 0 \\
0 \arrow[r] & 2\pi i \mathbb{Z} \arrow[r] \arrow[u, equal] & H_{\mathbb{Z}} \arrow[r, "e_0 \mapsto 1"] \arrow[u, "\psi"'] & \mathbb{Z} \arrow[u, "1 \mapsto t"'] &
\end{tikzcd}$$

$H_{\mathbb{Z}} = \mathbb{Z} e_0 \oplus \mathbb{Z} e_1$

- $\psi(e_1) = 2\pi i \qquad \psi(e_0) = \log t$

$H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q} \qquad H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$

$F^0 H_{\mathbb{C}} = \ker(\psi_{\mathbb{C}})$

$\parallel$

$\mathbb{C}\left(\frac{e_1}{2\pi i} - \frac{e_0}{\log t}\right) = \mathbb{C}(\log t \, e_1 - 2\pi i \, e_0)$

$W_{-2} H_{\mathbb{Q}} = \mathbb{Q} \cdot e_1 \qquad F^i H_{\mathbb{C}} = \begin{cases} 0, & i > 1 \\ H_{\mathbb{C}} & i < 0 \end{cases}$

given a matrix:

$H_{\mathbb{Q}} = \mathbb{Q}$-span of columns

In the example: $\mathbb{Q}\begin{pmatrix}1\\ \log t\end{pmatrix} \oplus \mathbb{Q}\begin{pmatrix}0\\ 2\pi i\end{pmatrix}$

$F^0 H_{\mathbb{C}} = \mathbb{C}\begin{pmatrix}1\\ 0\end{pmatrix}$

H mixed Tate HS

Assume $gr^W_{2p} H_{\mathbb{Q}}$ is of dim ≤ 1 $\forall p$

$$\begin{pmatrix} 1 & 0 & & 0 \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$W_{2p} H_{\mathbb{Q}}$ spanned by columns starting from the right

$F^{\cdot} H_{\mathbb{C}} = \begin{pmatrix} * \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\qquad \bigoplus_p W_{2p} \cap F^? = H_{\mathbb{C}}$

$A(z)$ describes a $\overbrace{\text{Tate}}^{\text{mixed}}$ HS

$\begin{pmatrix} 1 & 0 \\ \log z & 2\pi i \end{pmatrix}$ K_z 2 dim'l

Hodge str-res = $\overbrace{\text{abelian category}}^{\mathbb{Q}\text{-linear Tannakian}}$

$0 \to \mathbb{Q}(1) \to K_z \to \mathbb{Q}(0) \to 0$

Exercise: $\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{C}^\times \otimes \mathbb{Q}$

$$\underbrace{\left(\mathrm{Sym}^{n-1} K_z\right) \otimes \mathbb{Q}(1)}_{\text{MTHS}} \longleftrightarrow \text{matrix} \quad \begin{pmatrix} 2\pi i & & & \\ 2\pi i \log(z) & (2\pi i)^2 & & 0 \\ \vdots & & & \\ 2\pi i \frac{(\log z)^{n-1}}{(n-1)!} & \frac{(2\pi i)^2 (\log z)^{n-2}}{(n-2)!} & \cdots & (2\pi i)^n \end{pmatrix}$$

Exercise

$$\mathcal{P}^{(n)} \text{ MTHS} \longleftrightarrow \text{columns of } A(z)$$

$$0 \to \left(\mathrm{Sym}^{n-1} K_z\right)(1) \to \mathcal{P}^n \to \mathbb{Q}(0) \to 0$$

$\mathcal{P}^{(n)}$ — n-th polylog. HS

Vary z. local system $\mathcal{P}^{(n)}$ on $\mathbb{P}^1 \setminus \{0,1,\infty\} = U$ whose fibers have MTHS

$W_\cdot \mathcal{P}^{(n)}$ rational $\quad W_{2p} \mathcal{P}^{(n)}$ sub-local systems

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Associated to any mixed HS is a "nearby" $\mathbb{R}$-split HS

Ex $\begin{pmatrix} 1 & 0 \\ \log z & 2\pi i \end{pmatrix}$ $\quad \mathrm{Re} \log z = \log |z|$

H - mixed HS

$F^\cdot H_\mathbb{C} \quad gr^W_n H_\mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$

No evident bigrading of $H_\mathbb{C}$ itself.

In fact, there is: $\quad H_\mathbb{C} = \bigoplus I^{p,q}$, but $I^{p,q} \neq \overline{I^{q,p}}$

$I^{p,q} \equiv \overline{I^{q,p}} \bmod \bigoplus_{\substack{p'<p \\ q'<q}} I^{p',q'}$

Say H is $\mathbb{R}$-split, if $I^{p,q} = \overline{I^{q,p}}$

In general,

$$I^{p,q} = F^p \cap W_{p+q} \cap (\overline{F^q} \cap W_{p+q} + \overline{U}^{q-1}_{p+q-2}), \quad \text{where}$$

$$U^a_b = \sum_{j \geq 0} F^{a-j} \cap W_{b-j}$$

If H is MTHS

$$I^{p,p} = F^p \cap W_{2p}, \qquad I^{p,q} = 0 \quad p \neq q$$