Injectivity theorems with multiplier ideal sheaves and their applications

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Complex manifolds, dynamics
and birational geometry

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This talk is organized in the following way:

- **Section 1: Introduction to the injectivity thm**
- **Section 2: Preliminaries**
- **Section 3: Proof of the injectivity thm**
- **Section 4: Applications to the minimal model program (Section 4 is a joint work with Gongyo)**
- **Section 5: Proof of the applications (if time permits)**

Section 1 Introduction

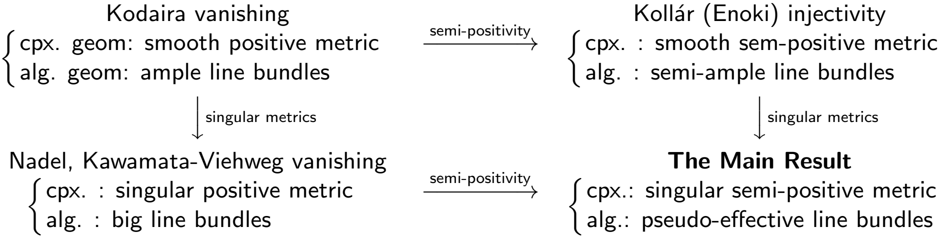
- X : compact Kähler manifold of dimension n.
- F : (holomorphic) line bundle on X

Theorem (The Kodaira vanishing)

When F is strictly positive (ample), then

$$H^q(X, K_X \otimes F) = 0 \quad \textit{for any } q > 0.$$

- By using **singular metrics** with strictly positive curvature, the Kodaira vanishing thm can be generalized to the Nadel vanishing theorem (the Kawamata-Viehweg vanishing thm).
- However the vanishing thm does not hold for “semi-positive” line bundles.
- Then Kollár (Enoki) gave the injectivity thm instead of the vanishing thm for “semi-positive” line bundles.



Kodaira vanishing

cpx. geom: smooth positive metric
alg. geom: ample line bundles

Kollár (Enoki) injectivity

cpx. : smooth sem-positive metric
alg. : semi-ample line bundles

Nadel, Kawamata-Viehweg vanishing

cpx. : singular positive metric
alg. : big line bundles

The Main Result

cpx.: singular semi-positive metric
alg.: pseudo-effective line bundles

- The main result is a generalization of the injectivity thm to pseudo-effective line bundles with singular metrics.
- The main result can be seen as a generalization of both Kollár's injectivity thm and the Nadel vanishing thm.
- The proof is based on combinations of the L^2-method for $\overline{\partial}$-equations and the theory of harmonic integrals.
- As applications, we can obtain the extension thm of (holomorphic) sections from subvarieties to the ambient space.

Theorem (Kollár's injectivity theorem and Enoki's generalization)

- *Assume that X is projective and F is semi-ample.*

Then for any section s of F^m, the multiplication map

$$\Phi_s : H^q(X, K_X \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1})$$

is injective for any q.

- *Enoki obtained the same conclusion under the weaker assumption that X is Kähler and F is semi-positive.*

Let us explain the definition of semi-ample and semi-positive line bundles in the next page.

- ω : Kähler form on X

Definition

(1) *F is said to be* **semi-ample** *if F^m admits sections $\{s_i\}_{i=1}^N$ such that $\cap_{i=1}^N s^{-1}(0) = \phi$.*

(2) *F is said to be* **semi-positive** *if F admits a smooth hermitian metric g with semi-positive curvature $\sqrt{-1}\Theta_g := -dd^c \log g$.*

(3) *F is said to be* **nef** *if for any $\varepsilon > 0$ the line bundle F admits a smooth hermitian metric h_ε with $\sqrt{-1}\Theta_{h_\varepsilon} \geq -\varepsilon\omega$.*

- $\exists$ counterexample to the injectivity thm for nef line bundles.
- However, by taking suitable limit h_0 of h_ε, we can obtain the **singular** metric h_0 with semi-positive curvature for nef line bundles (more widely, pseudo-effective line bundles).
- It's natural to consider the injectivity thm for (pseudo-effective) line bundles equipped with singular metrics.

Theorem (The main result, arXiv:1308.2033v2)

Let h be a singular metric on F with semi-positive curvature. Then for any section s of F^m with $\sup_X |s|_{h^m} < \infty$, the multiplication map

$$\Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$$

is (well-defined and) injective for any q.

- $\mathcal{I}(h)$ is the multiplier ideal sheaf of h.
- The condition of $\sup_X |s|_{h^m} < \infty$ is a natural assumption to make the multiplication map well-defined.
- We can always take the (special) singular metric $h_{\min}$ such that $\sup_X |s|_{h^m_{\min}} < \infty$ for any section s of F^m.

Section 2 Preliminaries

- g : smooth hermitian metric on F

Definition (Singular metrics, Curvature currents)

(1) *For an L^1-function φ on X, the metric $h := ge^{-\varphi}$ on F is called a* **singular metric**. *The function φ is called the weight of h.*
(2) *The* **curvature** $\sqrt{-1}\Theta_h(F)$ *of $h = ge^{-\varphi}$ is defined by*

$$\begin{aligned}\sqrt{-1}\Theta_h(F) &= \sqrt{-1}\Theta_g(F) + dd^c\varphi \\ &= -dd^c \log g + dd^c\varphi.\end{aligned}$$

(3) *The curvature $\sqrt{-1}\Theta_h(F)$ is said to be* **semi-positive** *if $\sqrt{-1}\Theta_h(F) \geq 0$ in the sense of currents, that is to say, $-\log g + \varphi$ is a psh function.*

Definition (Multiplier ideal sheaves)

Let $h = ge^{-\varphi}$ *be a metric on* F *such that* $\sqrt{-1}\Theta_h(F) \geq \gamma$ *for some smooth* $(1,1)$*-form* γ *on* X*.*
Then the **multiplier ideal sheaf** $\mathcal{I}(h)$ *of* h *is defined to be*

$$\mathcal{I}(h)(U) := \{f \in \mathcal{O}_X(U) \mid |f|e^{-\varphi} \in L^2_{\mathrm{loc}}(U).\}$$

for an open set $U \subset X$*.*

Example

Let $\{s_i\}_{i=1}^N$ *be sections of* F^m. *We define* φ *by*

$$\varphi := \frac{1}{2m} \log \sum_{i=1}^{N} |s_i|^2_{g^m}.$$

Then

(1) *The curvature of h is semi-positive.* (*Because it is locally written as* $\sqrt{-1}\Theta_h(F) = (1/m)\ dd^c\ \log \sum_{i=1}^N |s_i| \geq 0.$)

(2) *This type metric is said to have* **algebraic singularities**.

(3) *Take a modification* $\pi : \widetilde{X} \to X$ *such that* $\pi^*\mathcal{I}$ *is an ideal sheaf* $\mathcal{O}(-D)$ *with a simple normal crossing divisor* $D = \sum a_j D_j$. *Then*

$$\mathcal{I}(h) = \pi_*(K_{\widetilde{X}/X} - \sum_j \lfloor a_j/m \rfloor D_j).$$

- In complex geometry, there are metrics that are important but have non-algebraic singularities.

- As an application, we can obtain the following Nadel vanishing thm.

Corollary (Firstly proved in another preprint)

Let F be a big line bundle and $h_{\min}$ be a metric with minimal singularities on F. Then we have

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h_{\min})) = 0 \quad \textit{for } q > 0.$$

- F is said to big if

$$cm^n \leq \dim H^0(X, F^m) \quad \text{as } m \to \infty.$$

- $h_{\min}$ is a singular metric with the mildest singularities among metrics on F with semi-positive curvature.
- $h_{\min}$ admits an analytic Zariski decomposition, namely

$$H^0(X, F^m \otimes \mathcal{I}(h^m_{\min})) \cong H^0(X, F^m).$$

- Unfortunately, $h_{\min}$ does not have algebraic singularities even if F is big.
- It is important to study metrics with minimal singularities.

Corollary (Firstly proved in another preprint)

Let F be a big line bundle and $h_{\min}$ be a metric with minimal singularities on F. Then we have

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h_{\min})) = 0 \quad \textit{for } q > 0.$$

- The above result was an open problem, although we can easily obtain the vanishing thm for $\mathcal{I}_+(h_{\min}) := \mathcal{I}(h_{\min}^{(1+\delta)})$ from the standard Nadel vanishing.
- Recently, Guan-Zhou proved the (strong) openness conjecture, namely $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ for any psh function φ.
- Therefore the above result follows from the standard Nadel vanishing and the openness conjecture.

- Let us prove the corollary.
- Assume that non-zero $\exists \alpha \in H^q(X, K_X \otimes F \otimes \mathcal{I}(h_{\min}))$.
- If $\{s_i\}_{i=1}^N$ are linearly independent sections of F^m, then $\{s_i \alpha\}_{i=1}^N$ are linearly independent in $H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h_{\min}^{m+1}))$ by our injectivity theorem.
- Therefore we have

$$\dim H^0(X, F^m) \leq \dim H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h_{\min}^{m+1})).$$

- By bigness of F, we have

$$cm^n \leq \dim H^0(X, F^m) \quad \text{as } m \to \infty.$$

- On the other hand, we can prove the asymptotic vanishing:

$$\dim H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h_{\min}^{m+1})) = O(m^{n-q}) \quad \text{as } m \to \infty.$$

- It is a contradiction.

Proof of the injectivity thm

Theorem (The main result, arXiv:1308.2033v2)

Let h be a singular metric on F with semi-positive curvature.

Then for any section s on F^m with $\sup_X |s|_{h^m} < \infty$, *the multiplication map*

$$\Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$$

is (well-defined and) injective for any q.

The difficulties of the proof are as follows:

- We must handle a metric with non-algebraic singularities.
- h does **not** have **strictly**-positive curvature.

To overcome these difficulties, we study

- combinations of the theory of harmonic integrals and the L^2-method for $\overline{\partial}$-equations.
- the asymptotics of harmonic forms.

the special case where h is smooth

We first study the case where h is smooth (Enoki's proof).
In this case, we have

$$H^q(X, K_X \otimes F) \cong \mathrm{Ker}\overline{\partial}/\mathrm{Im}\overline{\partial} \cong \mathcal{H}^{n,q}(F)_h,$$

$$\mathcal{H}^{n,q}(F)_h := \{u \mid u \text{ is an } F\text{-valued } (n,q)\text{-form such that } \Delta_h u = 0\}.$$

For an arbitrary harmonic form $u \in \mathcal{H}^{n,q}(F)_h$, we can conclude that $\Delta_{h^{m+1}} su = 0$ by Bochner-Kodaira-Nakano identity and semi-positivity of the curvature of h. Namely, su is also harmonic. Then the multiplication map Φ_s induces the map from $\mathcal{H}^{n,q}(F)_h$ to $\mathcal{H}^{n,q}(F^{m+1})_{h^{m+1}}$, thus the injectivity is obvious.

Step 1: Approximation

- By the approximation of Demailly-Peternell-Schneider, we can obtain metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on F with the following properties:
  - (a) h_ε is smooth on $X \setminus Z$, where Z is a subvariety on X.
  - (b) $h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.
  - (c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$.
  - (d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega$.
- $\widetilde{\omega}$: complete Kähler form on $Y := X \setminus Z$ with the following properties:
  - $\widetilde{\omega} \geq \omega$
  - $\widetilde{\omega} = dd^c \exists\Psi$, Ψ is bounded, on a nbd of any point in Z.
- Take an arbitrary cohomology class $\{u\} \in H^q(K_X \otimes F \otimes \mathcal{I}(h))$ represented by a smooth (n, q)-form u valued in $F \otimes \mathcal{I}(h)$.
- Assume $\{su\} = 0 \in H^q(K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$.
- Our goal is to prove $\{u\} = 0 \in H^q(K_X \otimes F \otimes \mathcal{I}(h))$.
- Namely, our goal is to solve $\overline{\partial}v = u$ in a suitable range since $H^q(K_X \otimes F \otimes \mathcal{I}(h)) \cong \mathrm{Ker}\overline{\partial}/\mathrm{Im}\overline{\partial}$

Step 2: A generalization of Enoki's method

- We can obtain the orthogonal decomposition
$$L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}} = \mathrm{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}(F)_{h_\varepsilon,\widetilde{\omega}} \oplus \mathrm{Im}\overline{\partial}^*_{h_\varepsilon}.$$
- Then u can be decomposed into $\overline{\partial}v_\varepsilon + u_\varepsilon$.
- By generalizing Enoki's method, we can prove
$$\|\Delta_{h_\varepsilon} s u_\varepsilon\|^2_{h_\varepsilon,\widetilde{\omega}} := \int_Y |\Delta_{h_\varepsilon} s u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}} \, \widetilde{\omega}^n \to 0.$$
- Remark that we can not obtain the above convergence from only the curvature condition (d).
The important point is the following inequality:
$$\|u_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h,\omega}.$$

Step 3: Solutions of $\overline{\partial}$-equations with L^2-estimates

- By $\{su\} = 0 \in H^q(K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$, we can take a solution $\gamma_\varepsilon \in L^{n,q-1}_{(2)}(F^{m+1})_{h_\varepsilon^{m+1}}$ of $\overline{\partial}\gamma_\varepsilon = su_\varepsilon$.
- However, for our purpose, we need the L^2-estimate $\|\gamma_\varepsilon\|_{h_\varepsilon, \widetilde{\omega}} \leq C$ for some C (independent of ε).
- To construct such solution, we convert

$$\overline{\partial}\text{-equation } \overline{\partial}\gamma_\varepsilon = su_\varepsilon \text{ of } L^{n,\bullet}(K_F \otimes F)_{h_\varepsilon, \widetilde{\omega}}$$

to

$$\delta\text{-equation } \delta B_\varepsilon = A_\varepsilon \text{ of } C^\bullet(K_F \otimes F \otimes \mathcal{I}(h_\varepsilon))$$

where A_ε is a q-coboundary constructed by su_ε,
by chasing De-Rham Weil isomorphism.

Step 4: Asymptotics of harmonic forms

- From now on, we fix $\varepsilon_0 > 0$.
- By $\|u_\varepsilon\|_{h_{\varepsilon_0}} \le \|u_\varepsilon\|_{h_\varepsilon} \le \|u\|_{h,\omega}$, $\exists$ a subsequence of u_ε which converges to α in $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0}}$ w.r.t. the weak L^2-topology.
- We can prove $\alpha = 0$ by Step 2 and 3 (and some techniques).
- Indeed, the following convergence and (some techniques) imply $\alpha = 0$.

$$\begin{aligned}\|su_\varepsilon\|^2_{h_\varepsilon^{m+1}} &= \langle\langle su_\varepsilon,\ \overline{\partial}\gamma_\varepsilon\rangle\rangle_{h_\varepsilon^{m+1}} \\ &= \langle\langle \overline{\partial}^*_{h_\varepsilon} su_\varepsilon,\ \gamma_\varepsilon\rangle\rangle_{h_\varepsilon^{m+1}} \\ &\le \|\overline{\partial}^*_{h_\varepsilon} su_\varepsilon\|_{h_\varepsilon^{m+1}}\|\gamma_\varepsilon\|_{h_\varepsilon^{m+1}} \to 0.\end{aligned}$$

- Then by $u = u_\varepsilon + \overline{\partial} v_\varepsilon$ and $u_\varepsilon \to 0$,
  we have $\{u\} = 0 \in H^q(K_X \otimes F \otimes \mathcal{I}(h_{\varepsilon_0}))$.
- By $\mathcal{I}(h_{\varepsilon_0}) = \mathcal{I}(h)$,
  we can conclude $\{u\} = 0 \in H^q(K_X \otimes F \otimes \mathcal{I}(h))$.

Section 4 Applications (j.w. Gongyo)

Conjecture (Generalized abundance conjecture)

Let X be a normal projective variety and Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a KLT pair. Then if $K_X + \Delta$ is nef, then it should be semi-ample.

- In general, semi-ample ⇒ semi-positive ⇒ nef.
  The abundance conjecture says that the converse should be true for the (log) canonical bundle $K_X + \Delta$ by the special characteristic of $K_X + \Delta$.
- The abundance conjecture can be decomposed into **the non-vanishing conjecture** and **the extension conjecture**
- We will give some extension theorems by using a version of the injectivity thm.
- For simplicity, I will explain simple cases of them.

The non-vanishing conjecture

Conjecture (A simple case of the non-vanishing conjecture)

Under the same situation as the abundance conjecture, if $K_X + \Delta$ *is nef, then* $m_0(K_X + \Delta)$ *admits a (holomorphic) section for some* $m_0 > 0$.

Theorem (Verbitsky)

Let Δ *be a nef divisor on a hyperKähler manifold* X*. If* Δ *admits a singular metric* h *whose Lelong number* $\nu(h, x) \equiv 0$*, then* $m_0\Delta$ *admits a section for some* $m_0 > 0$.

- The Lelong number is defined by $\nu(h, x) := \nu(\varphi, x) := \liminf_{z \to x} \varphi / \log |z - x|$.
- If h is smooth at x, the Lelong number $\nu(h, x)$ is zero. In particular, if Δ is semi-positive, $m_0\Delta$ admits a section.
- We study the extension conjecture under the same assumption as Verbitsky.

The extension conjecture

Conjecture (A simple case of the extension conjecture)

Let X be a smooth projective variety and $\Delta := S + B$ be $\mathbb{Q}$-divisor with the following assumptions:

- *Δ is simple normal crossing and $0 \leq \Delta \leq 1$.*
- *$\lfloor \Delta \rfloor = S$.*
- *$K_X + \Delta$ is nef.*
- *$\exists t \in H^0(X, m_0(K_X + \Delta))$ with $S \subset \mathrm{Supp\ div(t)}$.*

Then

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$$

is surjective for all sufficiently divisible integers $m \geq 2$.

Remark

When S is a smooth divisor and $\mathrm{Supp\ div(t)} \subset \mathrm{Supp(S + B)}$, Demailly-Hacon-Păun proved this conjecture!!

Theorem (A simple case of the main result of Gongyo-M.)

Under the same situation, if $K_X + \Delta$ admits a singular metric h whose Lelong number $\nu(h, x) = 0$ at $x \in S$, then we have the conclusion of the extension conjecture.

- The proof of DHP is based on a version of the Ohsawa-Takegoshi extension thm.
- Our proof is based on a version of the injectivity thm.
- The advantage of using the injectivity theorem instead of the O-T thm is that we can extend sections even if S has singularities.
- If S is smooth, we can construct a good metric thanks to the techniques by DHP, whose strategy is as follows:

- The restriction of $G := K_X + \Delta$ to S is semi-ample.
- Take sections u_i of $m_0 G|_S$ with $\cap u_i^{-1}(0) = \phi$.
- Extend $s_A \otimes u_i^k$ to a section $U_{i,k}$ of $m_0 kG + A$ on X.
- Consider a metric h_k on $m_0 G + \frac{1}{k} A$ defined by

$$\frac{1}{2k} \log \sum_i |U_{i,k}|^2.$$

- The (suitable) limit $h := \lim h_k$ determines a metric on $m_0 G$, which has good properties along S by the construction.
- When we take the limit, we need the L^2-estimate of the extended section $U_{i,k}$ like the O-T thm.
- The disadvantage is that we do not obtain the L^2-estimate of the extended section.
- Due to this disadvantage, it seems to be difficult to construct a good metric by a method similar to DHP.
- But we have some ideas and we are trying.

Result related to the abundance conjecture

By combining techniques of the minimal model program, we obtain some results related to the abundance conjecture.

Theorem (Gongyo-M.)

Under the same situation as the abundance conjecture, further we assume the following conditions

- *the abundance conjecture holds in dimension* $n-1$.
- *the non-vanishing conjecture*
- $\exists$ *birational morphism* $\varphi : Y \to X$ *such that* Y *is smooth and* $\varphi^*(m_0(K_X + \Delta))$ *admits a singular metric whose Lelong number is identically zero.*

Then $K_X + \Delta$ *is semi-ample.*

Corollary (Gongyo-M.)

Let Δ be a nef divisor on a 4-dimensional projective hyperKähler manifold X. If Δ admits a singular metric h whose Lelong number $\nu(h, x) \equiv 0$. Then Δ is semi-ample.

This corollary follows from the following result:

- The abundance conjecture for 3-dimensional projective varieties.
- The non-vanishing conjecture under the assumption of $\nu(h, x) \equiv 0$ (Verbitsky).
- Our extension thm.

Strictly speaking, we need **a precise case** (not simple case) of our extension theorem for this corollary.

Sketch of the proof

Recall the situation of our extension theorem.

- $(X, \Delta := S + B)$: a log smooth LC pair.
- $\exists t \in H^0(X, m_0(K_X + \Delta))$ with $S \subset \mathrm{Supp}\ \mathrm{div}(\mathrm{t})$.
- $G := K_X + \Delta$ admits a singular metric h whose Lelong number $\nu(h, x) = 0$ at $x \in S$

Our goal is to prove the surjectivity of

$$H^0(X, \mathcal{O}_X(mG)) \to H^0(S, \mathcal{O}_S(mG)).$$

For this goal, we consider the long exact sequence derived from

$$0 \to \mathcal{O}_X(mG - S) \to \mathcal{O}_X(mG) \to \mathcal{O}_S(mG) \to 0$$

twisted by $\mathcal{I}(h^{m-1}h_B)$.

- It is sufficient to prove that the following is injective:

$$H^1(\mathcal{O}_X(mG-S)\otimes\mathcal{I}(h^{m-1}h_B)) \to H^1(\mathcal{O}_X(mG)\otimes\mathcal{I}(h^{m-1}h_B))$$

- Take $a > 0$ with $S \leq aD$, where $D := \mathrm{div}(\mathrm{t})$ and consider

$$\begin{tikzcd} H^q(\mathcal{O}_X(mG - S) \otimes \mathcal{I}(h^{m-1}h_B)) \arrow[r] \arrow[dr] & H^q(\mathcal{O}_X(mG) \otimes \mathcal{I}(h^{m-1}h_B)) \arrow[d] \\ & H^q(\mathcal{O}_X(mG - S + aD) \otimes I(h^{a+m-1}h_B)). \end{tikzcd}$$

- $mG - S = K_X + (m-1)(K_X + \Delta) + B$ and
  $mG - S + aD = K_X + (m-1)(K_X + \Delta) + aD + B$
- By giving a version of the injectivity theorem, we can prove that the above map is injective. Therefore we obtain the surjectivity of
  $H^0(X, \mathcal{O}_X(G) \otimes \mathcal{I}(h^{m-1}h_B)) \to H^0(S, \mathcal{O}_S(G) \otimes \mathcal{I}(h^{m-1}h_B))$.
- By the assumption of $\nu(h, x) \equiv 0$ for $x \in S$, any section on S belongs to $H^0(S, \mathcal{O}_S(G) \otimes \mathcal{I}(h^{m-1}h_B))$.

Thank you
for your attention!!