

The hyperbolic Ax-Lindemann-Weierstraß conjecture

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Plan of the talk:

- (1) Motivation: Hodge locus and André-Oort conjecture.
- (2) Bi-algebraic geometry and the Ax-Lindemann-Weierstraß conjecture.
- (3) Strategy of the proof.
- (4) Some details.

Motivation: Hodge locus and André-Oort conjecture

- S : smooth quasi-projective variety over \mathbb{C} .
- $\mathcal{H} \rightarrow S$ a polarized \mathbb{Z} VHS.

Example

$\pi : \mathcal{X} \rightarrow S$ a smooth projective family, $\mathcal{H} := (R^i \pi_* \mathbb{Z})_{\text{prim}}$

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$HL(\mathcal{H})$ is a countable union of algebraic subvarieties of S .

Definition:

Special subvariety of (S, \mathcal{H}) := irreducible stratum of $HL(\mathcal{H})$

Special point := special subvariety of dimension 0

Goal:

Understanding the distribution in S of special subvarieties, especially special points, associated to the \mathbb{Z} VHS \mathcal{H} .

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Example

- $S = SL(2, \mathbb{Z}) \backslash \mathbb{H} \stackrel{j}{\simeq} \mathbb{C} = Y_0(1)$
moduli space of \mathbb{C} -elliptic curves, seen as weight 1 polarized \mathbb{Z} HS.
- $\tau \in \mathbb{H} \longleftrightarrow E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$
- Special point:
 τ imaginary quadratic $\longleftrightarrow E_\tau$ has complex multiplication (by $\mathbb{Q}(\tau)$)
- Special points are dense in $Y_0(1) = \mathbb{C}$, even for the usual topology.

Example

- $S = SL(2, \mathbb{Z}) \backslash \mathbb{H} \times SL(2, \mathbb{Z}) \backslash \mathbb{H} \stackrel{j}{\simeq} \mathbb{C} \times \mathbb{C} = Y_0(1) \times Y_0(1)$
moduli space of pairs of \mathbb{C} -elliptic curves.
- Special points in $Y_0(1) \times Y_0(1)$: pairs (x, y) of special points.
- Special curves:
 - $\{x\} \times Y_0(1)$ or $Y_0(1) \times \{x\}$, x special,
 - $\text{Im}(Y_0(N) \rightarrow Y_0(1) \times Y_0(1))$, where $Y_0(N)$ is the moduli space of isogenies $\mathbb{Z}/N\mathbb{Z} \hookrightarrow E_1 \rightarrow E_2$.

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Conjecture: (André, '89)

An irreducible curve of $\mathbb{C} \times \mathbb{C}$ containing infinitely many special points is special.

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Definition:

A connected Shimura variety is a quotient $S = \Gamma \backslash X$ of a symmetric bounded domain X of \mathbb{C}^N by an arithmetic (congruence) subgroup $\Gamma = \mathbf{G}(\mathbb{Z})$, $G = \text{Aut}(X)$.

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Example

- $X = \mathbf{B}_{\mathbb{C}}^n$, $G = PU(n, 1)$.
- $X = D_{p,q}^I = \{Z \in M(p, q, \mathbb{C}) \simeq \mathbb{C}^{pq} : I_q - Z^*Z > 0\}$,
 $G = PU(p, q)$.
- $X = \{Z \in D_{g,g}^I : Z^t = -Z\}$, $G = Sp(g, \mathbb{R})$, $\Gamma = Sp(g, \mathbb{Z})$, $S = \mathcal{A}_g$
moduli space of Abelian varieties of dimension g .

Basic facts:

- Such an $S = \Gamma \backslash X$ has a canonical structure of a quasi-projective variety over \mathbb{C} (Bailey-Borel), even over $\overline{\mathbb{Q}}$ (Shimura-Deligne).
- thanks to its modular definition, S is canonically endowed with a polarized \mathbb{Z} VHS $\mathcal{H} \rightarrow S$.

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- thanks to its modular definition, S is canonically endowed with a polarized \mathbb{Z} VHS $\mathcal{H} \rightarrow S$.

The special subvarieties of $S = \Gamma \backslash S$ are completely understood:

- from the group-theoretical point of view: special subvarieties of S are the irreducible components of Hecke translates of Shimura subvarieties of S .
- from the differential geometric point of view: special subvarieties are totally geodesic subvarieties containing at least one special point.
- Special points are $\overline{\mathbb{Q}}$ -points. They are dense in each special subvariety, in particular in S .

Conjecture: (André-Oort)

Let S be a connected Shimura variety and $Z \subset S$ a closed irreducible algebraic subvariety.

If Z contains a Zariski-dense set of special points then Z is special.

Equivalently: there exists finitely many special subvarieties Y_1, \dots, Y_r of S contained in Z , and maximal for this property.

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This conjecture is similar to the Manin-Mumford conjecture:

Theorem: (Raynaud)

Let A be an Abelian variety and $Z \subset A$ a closed irreducible algebraic subvariety.

If Z contains a Zariski-dense set of torsion points then Z is a translate of an Abelian subvariety by a torsion point.

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Results:

- The AO conjecture has been proven by Klingler-Ullmo-Yafaev under the Generalized Riemann Hypothesis. The proof uses ergodic, algebraic and arithmetic geometry.
- In 2010 Pila proved unconditionnally the AO conjecture in the special case $S = Y_0(1)^N$. His proof relies on the hyperbolic Ax-Lindemann-Weierstraß conjecture in this particular case.

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- Describe the **bi-algebraic** subvarieties, namely the irreducible pairs $(Y, V := \pi(Y))$ with $Y \subset X$ algebraic and $V \subset S$ algebraic.

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- **Ax-Lindemann-Weierstraß statement:** let $Y \subset X$ be an algebraic subvariety. Then any irreducible component of the Zariski closure $\overline{\pi(Y)}^Z$ is bi-algebraic.

Bi-algebraic complex geometry

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- Ax-Lindemann-Weierstraß statement:** let $Y \subset X$ be an algebraic subvariety. Then any irreducible component of the Zariski closure $\overline{\pi(Y)}^Z$ is bi-algebraic.
- Equivalently: consider the diagram

$$\begin{array}{ccccc} Y \subset & \xrightarrow[\text{maximal}]{\text{algebraic, irreducible}} & \pi^{-1}V \subset & \xrightarrow{\quad} & X \\ & & \downarrow & & \downarrow \pi \\ & & V \subset & \xrightarrow{\text{algebraic}} & S. \end{array} \quad \text{Then } Y \text{ is bi-algebraic.}$$

Example (The flat ALW conjecture)

$$\pi = (\exp, \dots, \exp) : \mathbb{C}^n \longrightarrow (\mathbb{C}^*)^n \quad .$$

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Remark:

This is the geometric analog of the classical Lindemann-Weierstraß theorem:

if $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are \mathbb{Q} -linearly independent then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over \mathbb{Q} .

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Zannier noticed that one can obtain a new proof of the Manin-Mumford conjecture using the Abelian ALW conjecture as a crucial ingredient. Pila realized one might use the same kind of techniques for proving the André-Oort conjecture.

The hyperbolic ALW conjecture

Let

$$\pi : X \longrightarrow S = \Gamma \backslash X$$

be the uniformizing map of an arithmetic variety (\sim connected Shimura variety), where $\Gamma = \mathbf{G}(\mathbb{Z})$ is an arithmetic lattice in $G = \text{Aut}(X)$.

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Remark:

Notice that we are not exactly in the setting of bi-algebraic geometry: while S is an algebraic variety, the bounded symmetric domain X is only a semi-algebraic subset of the algebraic variety \mathbb{C}^N .

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Definition:

A subset $Y \subset X$ is an irreducible algebraic subvariety of X if Y is an analytic irreducible component of $D \cap \tilde{Y}$ for $\tilde{Y} \subset \mathbb{C}^N$ an algebraic subvariety.

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The semi-algebraic structure on X is canonical. If you use any other semi-algebraic realization of X (for example the one given by the Borel embedding) you do not change the algebraic subsets of X .

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Proposition: (Ullmo-Yafaev)

The bi-algebraic subvarieties in S are the weakly special (i.e. totally geodesic) ones.

Theorem: (K., Ullmo, Yafaev)

Let $\pi : X \rightarrow S = \Gamma \backslash X$ the uniformizing map of an arithmetic variety. Then the Ax-Lindemann-Weierstraß conjecture holds true for π : let $Y \subset X$ be an algebraic subvariety. Then any irreducible component of $\overline{\pi(Y)}^Z$ is weakly special.

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Corollary:

Using this theorem one can obtain a new proof of the André-Oort conjecture under GRH.

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The André-Oort conjecture holds true unconditionnally for $S = \mathcal{A}_6^n$, for all n .

Strategy of the proof of the main theorem:

We start with

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Theorem (Step 3)

The \mathbb{Q} -group $H_Y := \left(\overline{\mathbf{G}(\mathbb{Z}) \cap \text{Stab}_{\mathbf{G}(\mathbb{R})} Y}^{\text{Zar}/\mathbb{Q}} \right)^0$ is positive dimensional and stabilizes Y .

Then using classical monodromy arguments (Deligne's semi-simplicity theorem) one can conclude that Y is weakly special.

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To prove Step 3, fix a fundamental set $\mathcal{F} \subset X$ for the Γ -action.

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Notice that if $g \in \Sigma(Y)$ then $gY \subset \pi^{-1}V$ is also maximal irreducible algebraic. Suppose you show that $\Sigma(Y)$ contains a positive dimensional semi-algebraic subset W . By maximality of Y the set W has to stabilize Y .

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Hence we are reduced to showing that $\Sigma(Y)$ contains a positive dimensional semi-algebraic subset.

Main idea of Pila-Zannier:

even if $\pi : X \rightarrow S$ is highly transcendental, one can still control its transcendence if it is definable in a “tame topology” in Grothendieck’s sense, i.e. an “o-minimal structure” in the sense of model theory.

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even if $\pi : X \rightarrow S$ is highly transcendental, one can still control its transcendence if it is definable in a “tame topology” in Grothendieck’s sense, i.e. an “o-minimal structure” in the sense of model theory.

Definition:

A structure \mathcal{S} is a collection $\mathcal{S} = (S_n)_{n \in \mathbb{N}}$, where S_n is a set of subsets of \mathbb{R}^n , called definable sets, such that:

- (1) all algebraic subsets of \mathbb{R}^n are in S_n .
- (2) S_n is a boolean subalgebra of powerset of \mathbb{R}^n .
- (3) If $A \in S_n$ and $B \in S_m$ then $A \times B \in S_{n+m}$.
- (4) Let $p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be a linear projection. If $A \in S_{n+1}$ the $p(A) \in S_n$.

The structure \mathcal{S} is said to be o-minimal if the elements of S_1 are precisely the finite unions of points and intervals.

A map $f : A \rightarrow B$ between definable sets is definable if its graph is.

Non-trivial \mathcal{o} -minimal structures do exist:

- \mathbb{R}_{an} (Van den Dries): a function $f : [0, 1]^n \rightarrow \mathbb{R}$ is definable in \mathbb{R}_{an} if it is the restriction of a real analytic function defined on some open neighbourhood of $[0, 1]^n$.
- \mathbb{R}_{exp} (Wilkie): you require $\exp : \mathbb{R} \rightarrow \mathbb{R}$ to be definable.
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Theorem (Pila-Wilkie)

Let $Z \subset \mathbb{R}^m$ be definable in some o -minimal structure.

Let $Z^{\text{alg}} \subset Z$ be the union of all positive-dimensional semi-algebraic subsets of Z . Then:

$$\forall \varepsilon > 0, \quad \exists C_\varepsilon > 0 / \left| \{x \in (Z \setminus Z^{\text{alg}}) \cap \mathbb{Q}^m, \quad H(x) \leq T\} \right| < C_\varepsilon T^\varepsilon .$$

Theorem (Step 1)

There exists a semi-algebraic fundamental set $\mathcal{F} \subset X$ for Γ such that

$$\pi|_{\mathcal{F}} : \mathcal{F} \longrightarrow S$$

is definable in $\mathbb{R}_{\text{an},\text{exp}}$.

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Remarks:

- for S compact: this is obvious, even in \mathbb{R}_{an} .
- for $S = \mathcal{A}_g$: this was proven by Peterzil-Starchenko, using explicit intricate computations with θ -functions. Crucially used by Pila-Tsimerman.
- our proof is general and purely geometric, relying on the structure of toroidal compactifications of S .

Theorem (Step 1)

There exists a semi-algebraic fundamental set $\mathcal{F} \subset X$ for Γ such that

$$\pi|_{\mathcal{F}} : \mathcal{F} \longrightarrow S$$

is definable in $\mathbb{R}_{\text{an},\text{exp}}$.

Remarks:

- for S compact: this is obvious, even in \mathbb{R}_{an} .
- for $S = \mathcal{A}_g$: this was proven by Peterzil-Starchenko, using explicit intricate computations with θ -functions. Crucially used by Pila-Tsimerman.
- our proof is general and purely geometric, relying on the structure of toroidal compactifications of S .

Corollary:

The set $\Sigma(Y) := \{g \in \mathbf{G}(\mathbb{R}) / \dim(gY \cap \pi^{-1}V \cap \mathcal{F}) = \dim Y\}$ is definable in $\mathbb{R}_{\text{an},\text{exp}}$.

To show that $\Sigma(Y)$ contains a positive dimensional semi-algebraic set, we are reduced, using the Pila-Wilkie theorem, to showing that

$$\Sigma(Y) \cap \mathbf{G}(\mathbb{Z}) = \{\gamma \in \mathbf{G}(\mathbb{Z}) / \gamma^{-1}\mathcal{F} \cap Y \neq \emptyset\}$$

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Theorem: (Step 2)

Let $Y \subset X$ be an irreducible algebraic subset of X .

There exists $c_1 > 0$ such that

$$|\{\gamma \in \mathbf{G}(\mathbb{Z}) / Y \cap \gamma\mathcal{F} \neq \emptyset, H(\gamma) \leq T\}| \geq T^{c_1} .$$

About the proof of Step 2 assuming Step 1

Height on $\mathbf{G}(\mathbb{Z})$:

Fix $\mathbf{G} \subset \mathbf{GL}(E)$ a faithful linear representation of \mathbf{G} . Write $X = \mathbf{G}(\mathbb{R})/K$, where K is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. Fix $\|\cdot\|_\infty$ a K -invariant norm on E and denote in the same way the operator norm on $\text{End}(E)$.

Definition:

For $g \in \mathbf{G}(\mathbb{Z})$, we define $H(g) := \max(1, \|g\|_\infty)$.

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$\exists B > 0 / \forall \gamma \in \mathbf{G}(\mathbb{Z}), \forall u \in \gamma\mathcal{F}, \quad H(\gamma) \leq B\|u\|_\infty.$

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$$\text{Taking volumes: } \text{Vol}_C(C(T)) \leq \sum_{\substack{\gamma \in \Gamma, \gamma\mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T}} \text{Vol}_C(\mathcal{F} \cap \gamma^{-1}C) .$$

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$\exists A > 0 / \forall C \subset X$ algebraic curve of degree d , $\text{Vol}_C(C \cap \mathcal{F}) \leq A \cdot d$.

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Hence $C \cap B(x_0, \log T) \subset C(T)$. Thus:

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Let us consider the diagram of holomorphic maps:

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We claim that this composite is definable in $\mathbb{R}_{\text{an}, \text{exp}}$. It follows from the following observations:

- $\exp(2\pi iz) = \exp(-2\pi \text{Im}(z)) \cdot \exp(2\pi i \text{Re}(z))$. The first factor is definable in \mathbb{R}_{exp} . On the other hand $\text{Re}(x)$ is bounded on \mathcal{F} , hence the second factor, on \mathcal{F} , is definable in \mathbb{R}_{an} .
- The j -function $j : D^* \rightarrow \mathbb{C}$ extends to $D \rightarrow \mathbf{P}^1 \mathbb{C}$ hence is definable in \mathbb{R}_{an} .

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This picture extends to any arithmetic variety using toroidal compactifications for S and Siegel fundamental domain in X for Γ .