Symplectic varieties with invariant Lagrangian subvarieties

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Main Thesis

We consider symplectic algebraic varieties equipped with a Hamiltonian reductive group action which contain an invariant Lagrangian subvariety.

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Hamiltonian symplectic varieties with invariant Lagrangian subvarieties behave similar to cotangent bundles.

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Outline

Symplectic geometry

- (M,ω), symplectic manifold (over $\mathbb{K}=\mathbb{R},\mathbb{C}$) or
  smooth algebraic variety (over $\mathbb{K}=\mathbb{C}$),
  ω is a nondegenerate closed 2-form on M;
- ∇f, skew gradient of $f : M \supset U \to \mathbb{K}$,
  $$df(v)=\omega(\nabla f, v),\ \forall v \in TM;$$
- $\{f,g\}=\omega(\nabla f,\nabla g)$, Poisson bracket.

A submanifold / smooth algebraic subvariety $S \subseteq M$ is:

- *isotropic* if $\omega|_{T_pS}=0$, $\forall p \in S$;
- *coisotropic* if $\omega|_{(T_pS)^\angle}=0$, $\forall p \in S$;
- *Lagrangian* = isotropic + coisotropic.

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Hamiltonian actions

Lie/algebraic group action $G \curvearrowright M$ is *Hamiltonian* if:

- it preserves ω;
- $\exists$ *moment map* $\Phi : M \to \mathfrak{g}^*$:
  - Φ is G-equivariant;
  - $\nabla(\Phi^*\xi) = \xi_*$, $\forall \xi \in \mathfrak{g}$,
  - $\langle d_p\Phi(v), \xi \rangle = \omega(\xi p, v)$, $\forall p \in M$, $v \in T_pM$;

  Notation: $\xi_*(p) = \xi p = \frac{d}{dt}\big|_{t=0} \exp(t\xi)p$: velocity vector
- $\{\Phi^*\xi, \Phi^*\eta\} = \Phi^*([\xi, \eta])$, $\forall \xi, \eta \in \mathfrak{g}$.

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Basic example: cotangent bundles

Example

$M = T^*X$, $\omega = \sum_i dx_i \wedge dy_i$,
x_i are local coordinates on X, y_i are dual coordinates in T_x^*X.

$G \curvearrowright X$ induces Hamiltonian action $G \curvearrowright T^*X$,
$\langle \Phi(p), \xi \rangle = \langle p, \xi x \rangle$, $\forall x \in X$, $p \in T_x^*X$, $\xi \in \mathfrak{g}$.

Zero section $S \subset T^*X$ is Lagrangian.

Conormal bundles $N^*(X/Y) = \{p \in T_x^*X \mid x \in Y,\ \langle p, T_xY \rangle = 0\}$ are Lagrangian for any $Y \subseteq X$.

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Structure of a neighborhood of a Lagrangian submanifold

Assume $S \subset M$ Lagrangian.

Darboux–Weinstein Theorem $\Longrightarrow$ $M \simeq T^*S$ (C^∞ symplectomorphism) in a neighborhood of S

G compact Lie group, $G \curvearrowright M$ Hamiltonian, S G-stable $\Longrightarrow$ G-equivariant local symplectomorphism $M \simeq T^*S$ (B. Kostant)

G reductive algebraic group, M Hamiltonian G-variety, $S \subset M$ G-stable Lagrangian subvariety: G-equivariant local symplectomorphism may not exist.

Obstruction: the structure of isotropy representations.

- $G_p \curvearrowright T_p(T^*S) = T_pS \oplus T_p^*S$ splits.
- May happen that T_pS has no G_p-stable complement in T_pM.

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Example: complete conics

Example

$\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}_{3\times 3}(\mathbb{C}))$, space of conics in $\mathbb{P}^2$
$F \subset \mathbb{P}^5$, set of double lines
$X = \mathrm{Bl}_F(\mathbb{P}^5)$, variety of *complete conics*
$G = SL_3(\mathbb{C}) \curvearrowright X \supset Y$, the unique closed orbit

Put $M = T^*X$, $S = N^*(X/Y)$.

$\exists$ unique $y \in Y$ such that $G_y = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$

$p \in S_y$ general point $\implies G_p^\circ = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$

T_pS has no G_p°-stable complement in T_pM.

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Setup

From now on:

M is an irreducible symplectic algebraic variety;
G is a connected reductive algebraic group, $\mathfrak{g} = \operatorname{Lie} G$;
$G \curvearrowright M$ is a Hamiltonian action;
$\Phi : M \to \mathfrak{g}^*$ is the moment map.

Notation: $\mathfrak{g}p = T_p Gp = \{\xi p \mid \xi \in \mathfrak{g}\}$, $\forall p \in M$

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Invariants of a Hamiltonian action

Definition

Corank $\operatorname{cork} M = \operatorname{rk} \omega|_{(\mathfrak{g}p)^\angle}$;
Defect $\operatorname{def} M = \dim \mathfrak{g}p \cap (\mathfrak{g}p)^\angle$ $\qquad$ ($p \in M$ general point).

Properties:

1. $\operatorname{Ker} d_p\Phi = (\mathfrak{g}p)^\angle$;
2. $\operatorname{Im} d_p\Phi = (\mathfrak{g}_p)^\perp$;
3. $\dim \overline{\Phi(M)} = \dim Gp$ (p general) $\Longleftarrow$ (2);
4. $\operatorname{def} M = \dim \overline{\Phi(M)}/G$ $\Longleftarrow$ (3), (1);
5. $\operatorname{cork} M = \dim M - \dim \overline{\Phi(M)} - \dim \overline{\Phi(M)}/G$ $\Longleftarrow$ (6), (3), (4);
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5. $\operatorname{cork} M = \dim M - \dim \overline{\Phi(M)} - \dim \overline{\Phi(M)}/G \Longleftarrow$ (6), (3), (4);
6. $\operatorname{cork} M + \operatorname{def} M = \dim M/G$.

Main result

Let $S \subset M$ be irreducible G-stable Lagrangian subvariety.
$\Phi(S) = \{G\text{-fixed point in } \mathfrak{g}^*\} \Longleftarrow (1)$
May assume: $\Phi(S) = \{0\}$

Theorem

$$\overline{\Phi(M)} = \overline{\Phi(T^*S)}$$

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Ideas of the proof

1. Deformation to the normal bundle

   $\exists$ flat family $\widehat{M} \to \mathbb{A}^1$ with fibers
   $M_c \simeq M$, $\forall c \neq 0$,
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2. Foliation of horospheres

   Horosphere = orbit of a (fixed) maximal unipotent subgroup $U \subset G$

   Suppose S is quasiaffine. Denote:
   $\mathcal{U} \subset T^*S$, conormal bundle to foliation of generic horospheres in S;
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Theorem ([Knop, 1994])

$$\overline{\Phi(\mathcal{U})} = \mathfrak{p}_0^{\perp}, \qquad \overline{G\mathcal{U}} = T^*S, \qquad \overline{\Phi(T^*S)} = G\mathfrak{p}_0^{\perp}.$$

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$T^*S \rightsquigarrow M, \quad \mathcal{U} \rightsquigarrow \mathcal{W}$

Construction of $\mathcal{W}$:

- Choose P_0-invariant functions $F_1, \dots, F_m : M \supset \mathring{M} \to \mathbb{C}$ such that $dF_1|_S, \dots, dF_m|_S$ span $\mathcal{U}$.
- Spread S along the trajectories of $\nabla F_1, \dots, \nabla F_m$.

Proposition

$$\overline{\Phi(\mathcal{W})} = \mathfrak{p}_0^\perp, \qquad \overline{G\mathcal{W}} = M, \qquad \overline{\Phi(M)} = G\mathfrak{p}_0^\perp.$$

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Let now $S \subset M$ be coisotropic G-stable.
Assume:

$$\mathfrak{g}x \subset (T_x S)^{\angle}, \qquad \forall x \in S \tag{$\diamondsuit$}$$

Theorem

If $(\diamondsuit)$ *holds, then*

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Elashvili's conjecture

Let H be an algebraic group, $\mathfrak{h} = \operatorname{Lie} H$.

Definition

Index $\operatorname{ind} \mathfrak{h} = \dim(\mathfrak{h}^*/H) = \dim H_p$, $p \in \mathfrak{h}^*$ general point.

Conjecture (Elashvili, late 90's)

$$\forall x \in \mathfrak{g}^* \simeq \mathfrak{g} : \quad \operatorname{ind} \mathfrak{g}_x = \operatorname{ind} \mathfrak{g}$$

- reduced to nilpotent x;
- verified case by case for classical $\mathfrak{g}$ (Yakimova, 2006) and exceptional $\mathfrak{g}$ (De Graaf, 2008);
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Elashvili's conjecture: towards elementary proof

Natural action $G \times G \curvearrowright G$ (by left/right multiplication) yields Hamiltonian action $G \times G \curvearrowright M = T^*G \simeq G \times \mathfrak{g}^*$.

$M \supset S = (G \times G)x \simeq G \times \mathrm{Ad}^*(G)x$ is coisotropic.

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Equality of dim's of LHS's would imply $\operatorname{ind}\mathfrak{g} = \operatorname{ind}\mathfrak{g}_x$.
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