# Schubert polynomials, pipe dreams, and associahedra

Evgeny Smirnov

Higher School of Economics
Department of Mathematics

Laboratoire J.-V. Poncelet
Moscow, Russia

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# Outline

1. General definitions
   - Flag varieties
   - Schubert varieties and Schubert polynomials
   - Pipe dreams and Fomin–Kirillov theorem
2. Numerology of Schubert polynomials
   - Permutations with many pipe dreams
   - Catalan numbers and Catalan–Hankel determinants
3. Combinatorics of Schubert polynomials
   - Pipe dream complexes
   - Generalizations for other Weyl groups
4. Open questions

# Flag varieties

- $G = \mathrm{GL}_n(\mathbb{C})$
- $B \subset G$ upper-triangular matrices
- $Fl(n) = \{V_0 \subset V_1 \subset \dots \subset V_n \mid \dim V_i = i\} \cong G/B$

**Theorem (Borel, 1953)**

$$\mathbb{Z}[x_1, \dots, x_n]/(x_1 + \dots + x_n, \dots, x_1 \dots x_n) \cong H^*(G/B, \mathbb{Z}).$$

This isomorphism is constructed as follows:

- $\mathcal{V}_1, \dots, \mathcal{V}_n$ tautological vector bundles over $G/B$;
- $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ $(1 \leq i \leq n)$;
- $x_i \mapsto -c_1(\mathcal{L}_i)$;
- The kernel is generated by the symmetric polynomials without constant term.

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# Schubert varieties

- $G/B = \bigsqcup_{w \in S_n} B^- wB/B$ — *Schubert decomposition*;
- $X^w = \overline{B^- wB/B}$, where $B^-$ the opposite Borel subgroup;
- $H^*(G/B, \mathbb{Z}) \cong \bigoplus_{w \in S_n} \mathbb{Z} \cdot [X^w]$ as abelian groups.

**Question**

Are there any "nice" representatives of $[X^w]$ in $\mathbb{Z}[x_1, \dots, x_n]$?

**Answer: Schubert polynomials**

- $w \in S_n \quad \rightsquigarrow \quad \mathfrak{S}_w(x_1, \dots, x_{n-1}) \in \mathbb{Z}[x_1, \dots, x_n]$;
- $\mathfrak{S}_w \mapsto [X^w] \in H^*(G/B, \mathbb{Z})$ under the Borel isomorphism;
- Introduced by J. N. Bernstein, I. M. Gelfand, S. I. Gelfand (1978), A. Lascoux and M.-P. Schützenberger, 1982;
- Combinatorial description: S. Billey and N. Bergeron, S. Fomin and An. Kirillov, 1993–1994.

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# Pipe dreams

Let $w \in S_n$. Consider a triangular table filled by $+$ and $\,\lrcorner\!\ulcorner$, such that:

- the strands intertwine as prescribed by $w$;
- no two strands cross more than once (*reduced* pipe dream).

## Pipe dreams for $w = (1432)$




Pipe dream $P$ $\rightsquigarrow$ monomial $x^{d(P)} = x_1^{d_1} x_2^{d_2} \ldots x_{n-1}^{d_{n-1}}$,
$d_i = \#\{+\text{'s in the } i\text{-th row}\}$

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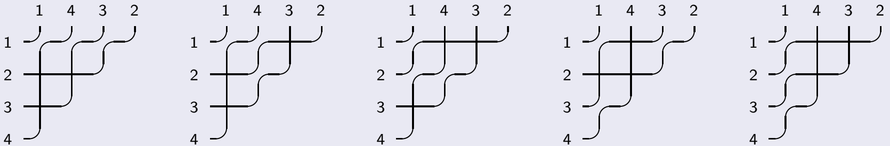


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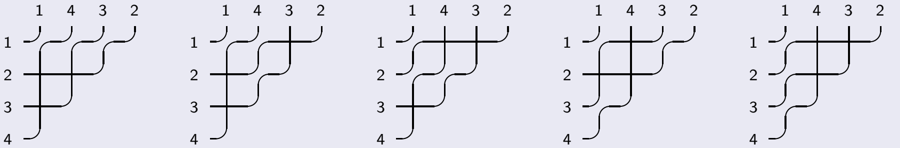


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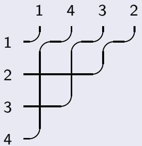

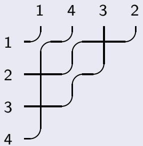

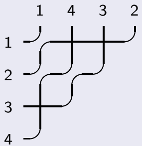

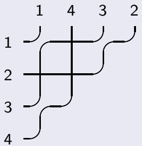

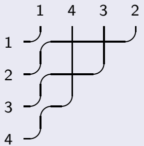


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# Pipe dreams and Schubert polynomials

**Theorem (S. Fomin, An. Kirillov, 1994)**

*Let $w \in S_n$. Then*

$$\mathfrak{S}_w(x_1, \ldots, x_{n-1}) = \sum_{w(P)=w} x^{d(P)},$$

*where the sum is taken over all reduced pipe dreams $P$ corresponding to $w$.*

**Example**

$$\mathfrak{S}_{1432}(x_1, x_2, x_3) = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2.$$

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# Permutations with the maximal number of pipe dreams

## How many pipe dreams can a permutation have?

Find $w \in S_n$, such that $\mathfrak{S}_w(1, \ldots, 1)$ is *maximal*.

## Answers for small $n$

- $n = 3$: $w = (132)$, $\mathfrak{S}_w(1) = 2$;
- $n = 4$: $w = (1432)$, $\mathfrak{S}_w(1) = 5$;
- $n = 5$: $w = (15432)$ and $w = (12543)$, $\mathfrak{S}_w(1) = 14$;
- $n = 6$: $w = (126543)$, $\mathfrak{S}_w(1) = 84$;
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## Definition

$w \in S_n$ is a *Richardson permutation*, if for $(k_1, \ldots, k_r)$, $\sum k_i = n$,

$$w = \begin{pmatrix} 1 & 2 & \ldots k_1 & k_1+1 & \ldots & k_1+k_2 & k_1+k_2+1 & \ldots \\ k_1 & k_1-1 & \ldots 1 & k_1+k_2 & \ldots & k_1+1 & k_1+k_2+k_3 & \ldots \end{pmatrix}.$$

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# Motivation

## Why are we interested in this?

The value $\mathfrak{S}_w(1,\dots,1)$ measures "how singular" is the Schubert variety $X^w$.

## More precisely

- $\mathfrak{S}_w(1,\dots,1)$ equals the degree of the *matrix Schubert variety* $\overline{X^w} \subset M_n$;
- If $w \in S_n$ satisfies the condition

$$\forall 1 \leq i,j \leq n, \quad i+j>n, \qquad \text{either } w^{-1}(i) \leq j \qquad \text{or } w(j) \leq i,$$

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Let $w^0_{k,m} = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & k+m \\ 1 & 2 & \dots & k & k+m & \dots & k+1 \end{pmatrix}$.

**Theorem (Alexander Woo, 2004)**

Let $w = w^0_{1,m}$. Then $\mathfrak{S}_w(1) = Cat(m)$.

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Let $w = w^0_{k,m}$. Then $\mathfrak{S}_w(1)$ is equal to a $(k \times k)$ Catalan–Hankel determinant:

$$\mathfrak{S}_w(1) = det(Cat(m+i+j-2))^k_{i,j=1}.$$

- $\mathfrak{S}_w(1)$ counts the "Dyck plane partitions of height $k$";
- These results have $q$-counterparts, involving Carlitz–Riordan $q$-Catalan numbers.

# Counting pipe dreams of Richardson permutations

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# Pipe dream complex (A. Knutson, E. Miller)

- To each permutation $w \in S_n$ one can associate a shellable CW-complex $PD(w)$;
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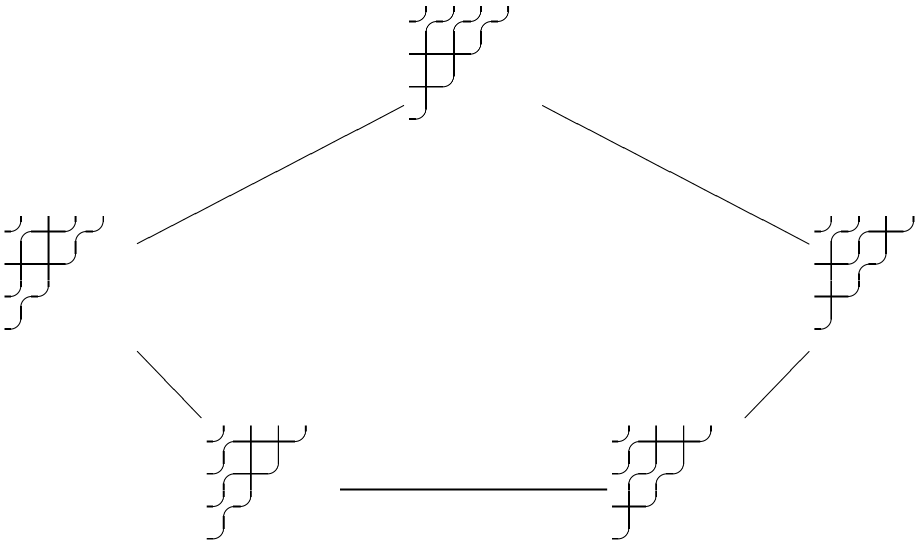

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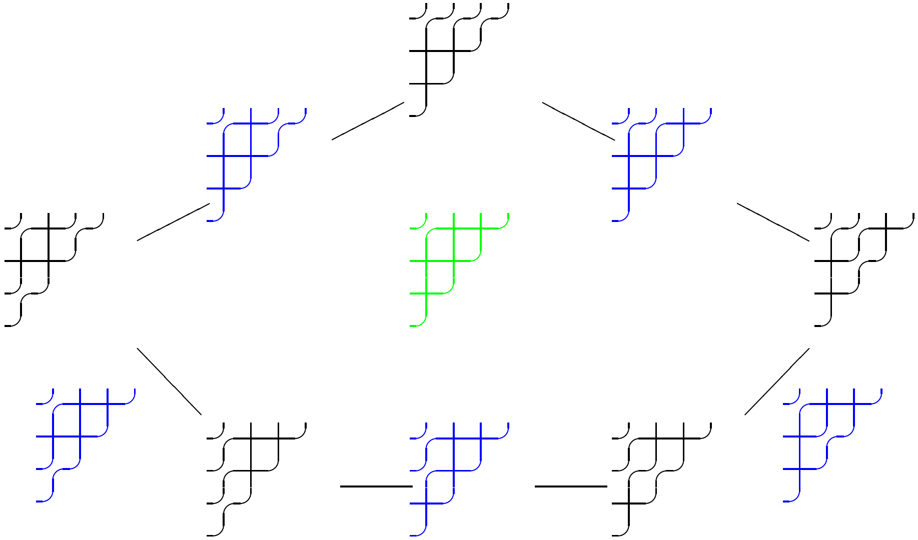
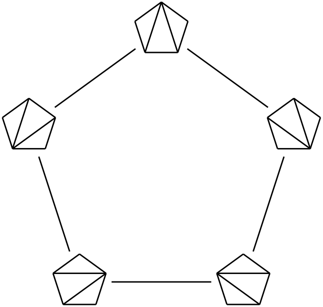
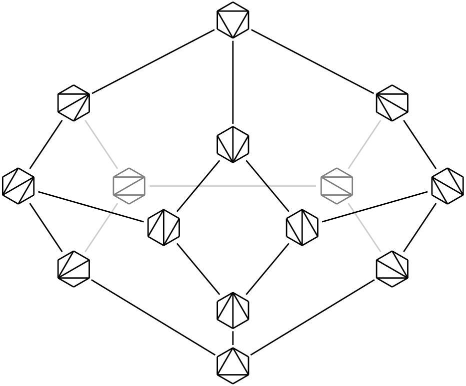


# Associahedra are PD-complexes

**Theorem (probably folklore? also cf. V. Pilaud)**

*Let $w = w^0_{1,n} = (1, n+1, n, \dots, 3, 2) \in S_{n+1}$ be as in Woo's theorem. Then $PD(w)$ is the Stasheff associahedron.*





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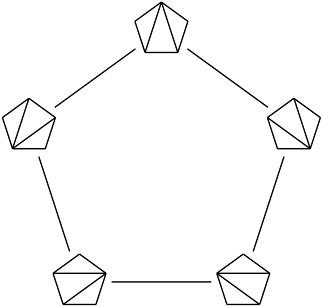

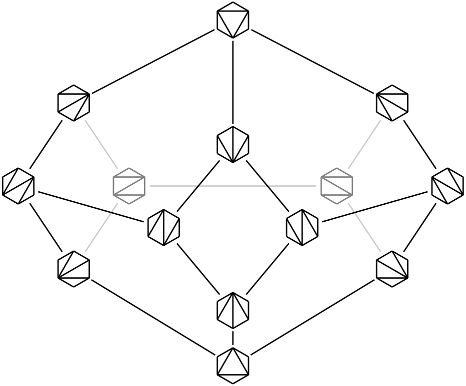

# Zoo of pipe dream complexes

What about $PD(w)$ for other Richardson elements $w$?

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$$C(n, d) = Conv((t_i, t_i^2, \dots, t_i^d))_{i=1}^n \subset \mathbb{R}^d.$$

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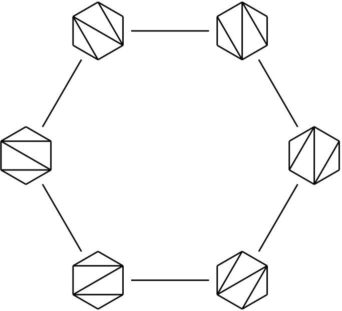
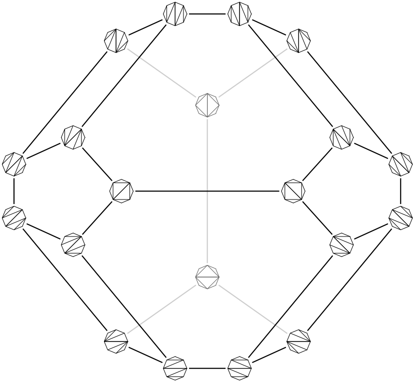
# Cyclohedra are subword complexes

## Theorem

*Let $W$ be of type $C_n$, generated by $s_1, \ldots, s_n$, where $s_1$ corresponds to the longest root $\alpha_1$. Consider a Richardson element $w = (s_1 s_2 \ldots s_{n-1})^{n-1}$. Then $PD(w)$ is a cyclohedron.*





# Questions about $PD(w)$

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С днем рождения!