Surfaces containing several circles through each point

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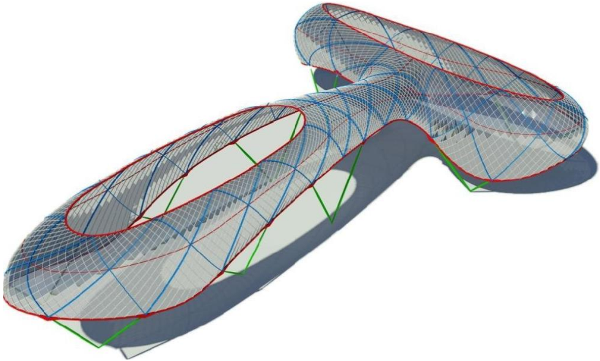
A.G. Khovaskii Conference 4-9.06.2012

Rationalization of an architectural design:



J. Wallner and H. Pottmann. Geometric computing for freeform architecture. J. Math. Industry 1 (2011), #4,1–19.

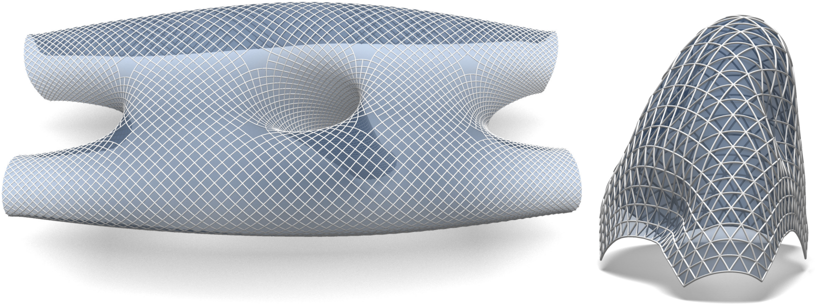
Rationalization of an architectural design:

Building in progress, 2011.

Rationalization of an architectural design:

Yas Viceroy hotel in Abu-Dhabi, 2012.

Circular arc structures (Bo et al., 2011):



- edges are circular arcs;
- nodes have tangent planes;
- nodes are congruent to each other.

Problem. Find all surfaces containing ≥ 2 circles through each point.

Doubly ruled surfaces

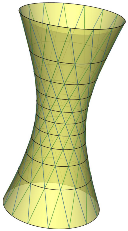
A surface is *doubly ruled*/*doubly circular*, if it contains 2 *segments*/*arcs* through each point.

Classical Theorem. A doubly ruled surface in $\mathbb{R}^3$ is either

- a one-sheeted hyperboloid or
- a hyperbolic paraboloid or
- a plane.

Theorem (Nilov and S., 2011). An analytic ruled circular surface in $\mathbb{R}^3$ is either

- a one-sheeted hyperboloid or
- a quadratic cone or
- an elliptic cylinder or
- a plane.



Example. Not true with $\mathbb{R}^3$ replaced by $\mathbb{C}^3$:

$$(x^2+y^2+z^2)^2+(x+iy)^2-z^2=0.$$

Example. *Darboux cyclide*: an image of a doubly ruled surface under a map $\mathbb{R}^3 \to \mathbb{R}^3$ taking all lines to circles.

Theorem. (A.G. Khovanskii, '80) A map $\mathbb{R}^2 \to S^2$ taking all lines to circles is a composition of the inclusion $\mathbb{R}^2 \to \mathbb{R}^3$ and a central projection $\mathbb{R}^3 \to S^2$.

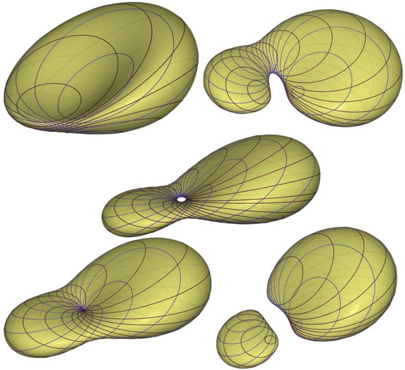
- *true* also in dimension 3 (F. Izadi, '01);
- *not true* in dimension 4 (V. Timorin, '04).

A *Darboux cyclide* is given by the equation

$$a(x^2+y^2+z^2)^2+(x^2+y^2+z^2)(bx+cy+dz)+Q(x,y,z)=0,$$

where $a, b, c, d \in \mathbb{R}$

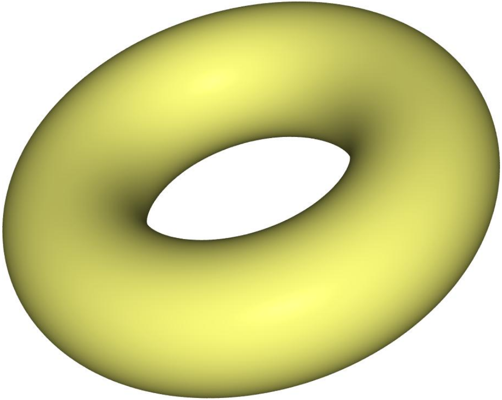
and $\deg Q(x,y,z) \le 2$.



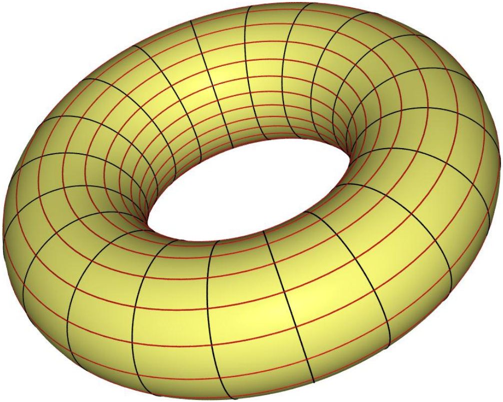
Examples:

- quadrics
- tori
- Dupin cyclides.

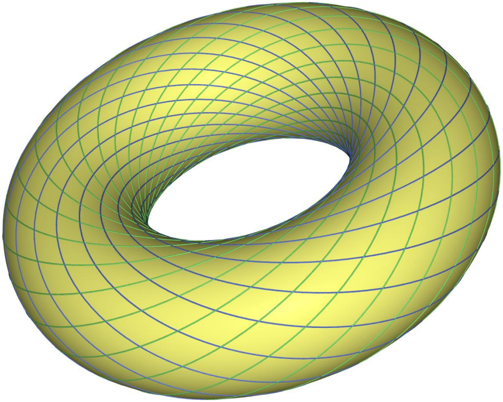
How many families of circles exist on a torus?



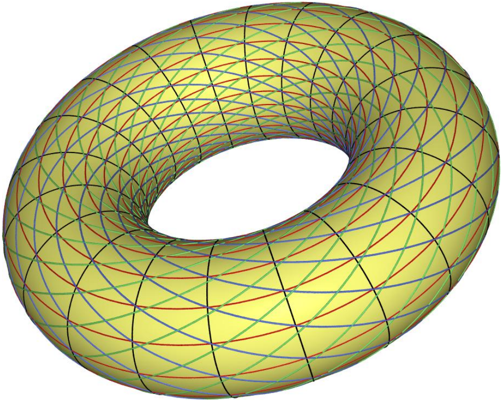
Two obvious families: *meridians and parallels*



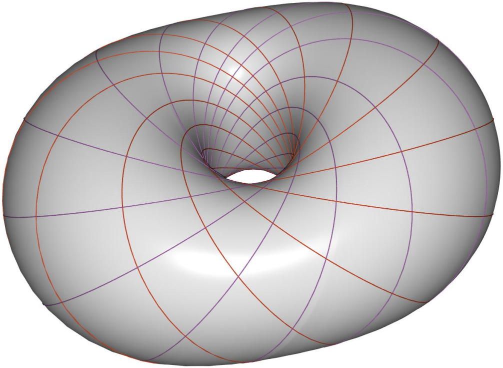
Two more circle families: *Villarceau circles*



A torus is carrying 4 families of circles.



Darboux cyclides contain up to 6 real circles through each point (R. Blum, 1980):



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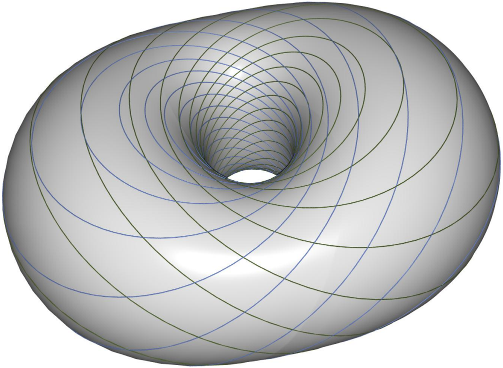

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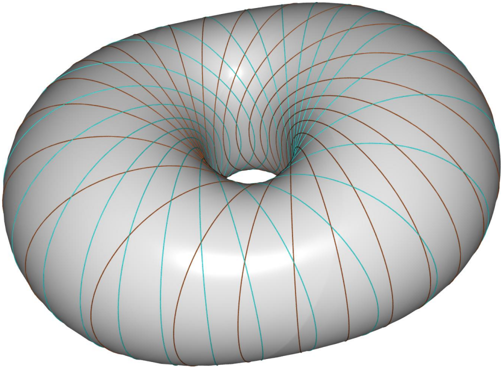

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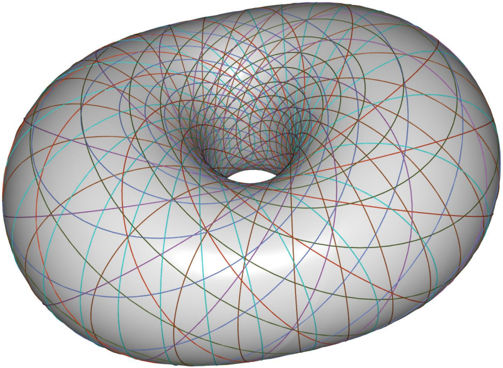
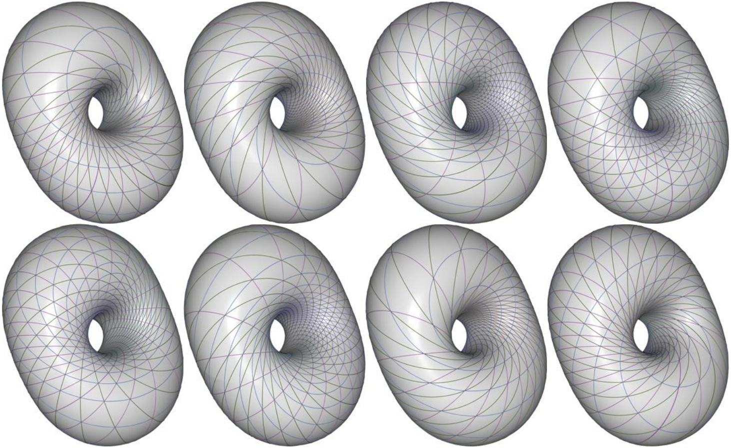
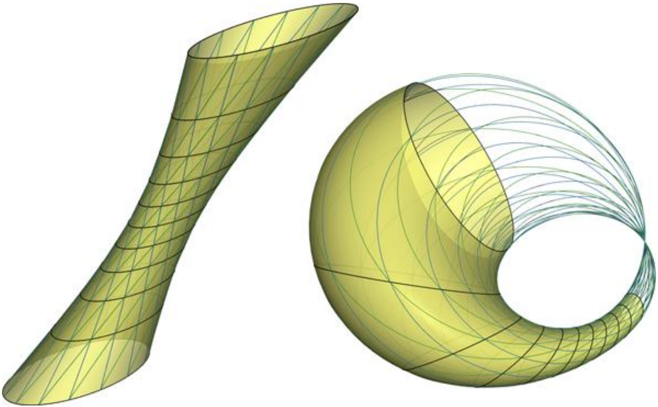


Theorem (Pottmann–Shi–S., 2011). Three families of circles on a nontrivial irreducible Darboux cyclide form a 3-web unless one takes two nonspecial paired families and another family which has a paired one. Thus we have 5 types of 3-webs from circles on a Darboux cyclide.

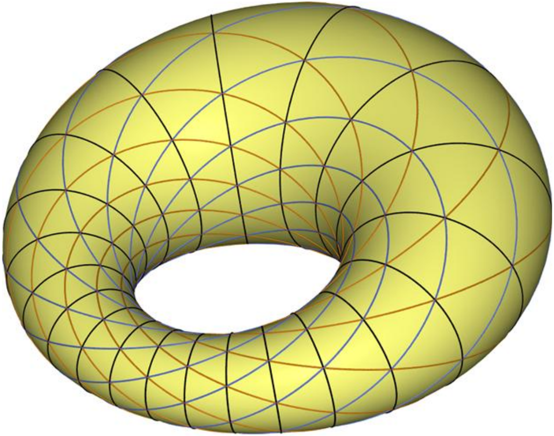
- 3 nonsingle families such that no two of them are paired families



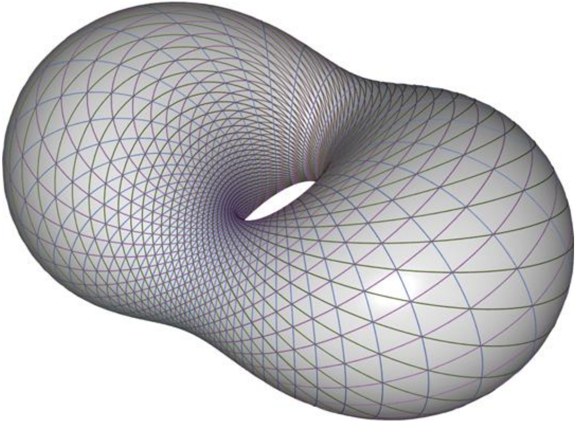
- 2 special paired families and another family which has a paired one



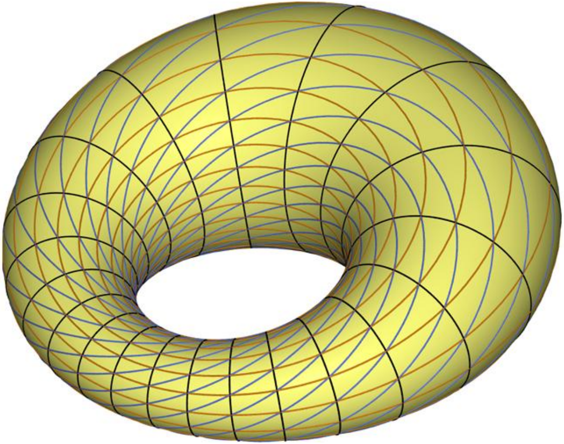
- a single family and 2 paired families



- a single family and 2 nonsingle families, which are not two paired families



- 2 single families and another family which has a paired one

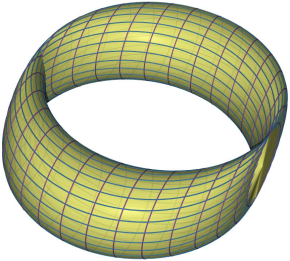


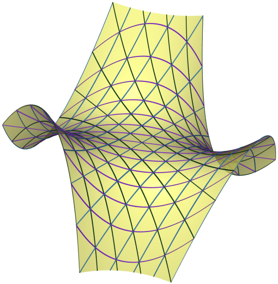
Theorem. A smooth surface of genus at most 1 containing 7 / 4 circles through each point is a *sphere* / *Darboux cyclide* (N. Takeuchi, 1995 / F. Nilov–M.S., 2011).

Theorem. A smooth surface containing 2 *cospherical*/*orthogonal* circles through each point is a *Darboux cyclide*/*Dupin cyclide* (J. Coolidge, 1906/ T. Ivey, 1995).

Example. Doubly circular $\not\Rightarrow$ Darboux cyclide: $(x^2+y^2+z^2+3)^2-4y^2z^2-16x^2-12y^2=0.$

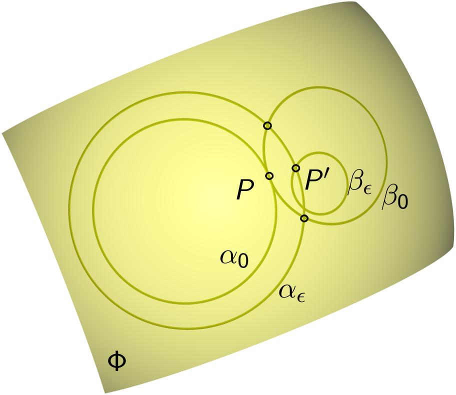
Example. Triply isotropic circular $\not\Rightarrow$ isotropic cyclide: $z=xy(x+y)$.





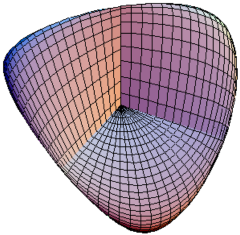
Lemma. From any 7 smooth closed curves intersecting pairwise in finitely many points in a closed surface of genus ≤ 1 one can choose at least 3 curves intersecting pairwise in an *even* number of points (counted with multiplicities).

Lemma. Two circles passing through a generic point of a doubly circular surface are transversal.



A surface is *conical*, if it contains a *conic section* through each point.

Classification results:



- ruled conical surfaces (H. Brauner, 1969);
- multiply conical surfaces (J. Schicho, 2000).

Theorem (S., 2011). An analytic doubly circular surface in $\mathbb{C}^3$ can be parametrized as

$$x(s,s',t,t'):y(s,s',t,t'):z(s,s',t,t'):w(s,s',t,t'),$$

where x, y, z, w are bihomogeneous biquadratic polynomials s.t. $w \mid x^2+y^2+z^2$.

Theorem (S., 2011). An analytic doubly circular surface in 3-sphere can be parametrized as

$$x : y : z : w : v,$$

where x, y, z, w, v are bihomogeneous biquadratic polynomials such that

$$x^2 + y^2 + z^2 + w^2 = v^2.$$

An *isotropic circle* in $\mathbb{R}^3$ is either an ellipse whose projection to Oxy is a circle or a parabola with the axis parallel to Oz.

Problem. Find all 3-webs from isotropic circles on surfaces in 3-space.

Motivation:

- isotropic version is easier than Euclidean;
- it is also interesting for architecture.

References

- H. Pottmann, L. Shi, M. Skopenkov, Darboux cyclides and webs from circles, Computer Aided Geom. Design **29:1** (2012), 77–97. Available at: http://arxiv.org/abs/1106.1354.
- F. Nilov, M. Skopenkov, A surface containing a line and a circle through each point is a quadric, Geom. Dedicata, to appear. Available at: http://arxiv.org/abs/1110.2338.

Acknowledgements

THANKS!