Toric singularities of surfaces and lattice trigonometry

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Contents

Part I

I. Geometric approach to continued fractions. Basics of integer trigonometry.

Continued fractions for 7/5

$$\frac{7}{5} =$$

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$$\frac{7}{5} = 1 + \frac{2}{5}$$

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Continued fractions for 7/5

$$\frac{7}{5}=1+\frac{1}{2+\frac{1}{2}}=1+\frac{1}{2+\frac{1}{1+\frac{1}{1}}}$$

Ordinary continued fractions

The expression (finite or infinite)

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots) \ldots))$$

is an *ordinary continued fraction* if $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}_+$ for $k > 0$. Denote it $[a_0 : a_1; \ldots]$ (or $[a_0 : a_1; \ldots; a_n]$).

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Ordinary continued fraction is *odd* (*even*) if it has odd (even) number of elements.

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1+1/1}}$$

$$\frac{7}{5} = [1 : 2; 2] = [1 : 2; 1; 1]$$

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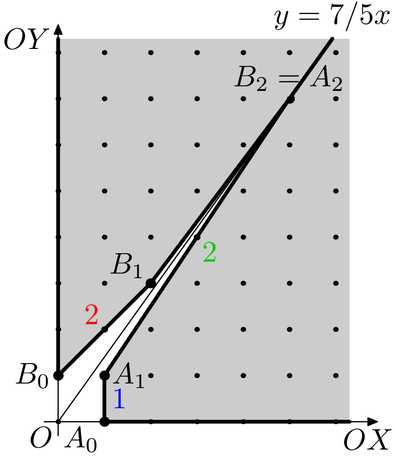
Proposition

Any rational number has a unique odd and even ordinary continued fractions.
Any irrational number has a unique infinite ordinary continued fraction

Geometry of continued fractions



$a_0 = \mathrm{l}\ell(A_0A_1) = 1;$

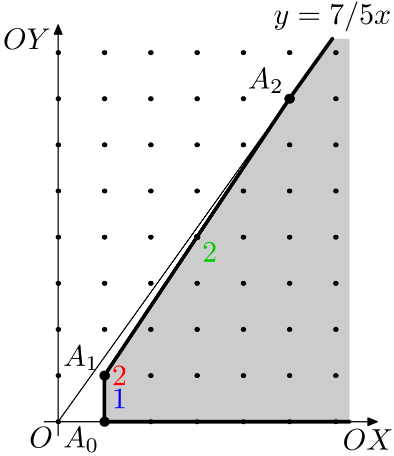
$a_1 = \mathrm{l}\ell(B_0B_1) = 2;$

$a_2 = \mathrm{l}\ell(A_1A_2) = 2.$

$$7/5 = [1; 2 : 2].$$

$\mathrm{l}\ell(AB)$ — the number of primitive vectors in AB.

Geometry of continued fractions



$$a_0 = \mathrm{l}\ell(A_0A_1) = 1;$$

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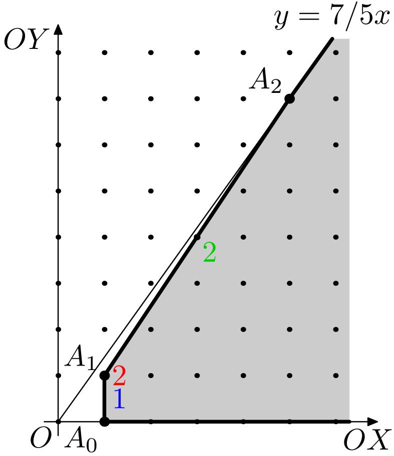
$$\mathrm{lsin}(ABC) = \frac{S(ABC)}{\mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)}$$ (integer sin-formula).

Geometry of continued fractions



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$(a_0, \ldots, a_{2n})$ — lattice length-sine sequence (LLS-sequence).

Integer geometry

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Objects: Integer segments, integer angles, integer polygons.

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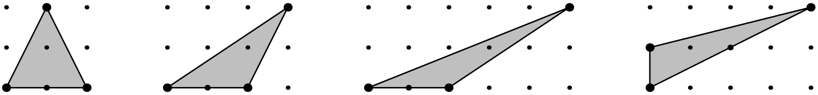
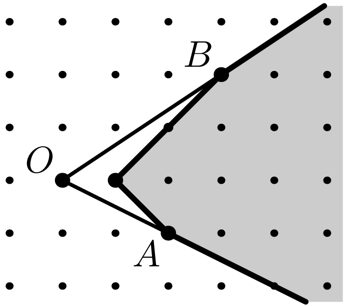
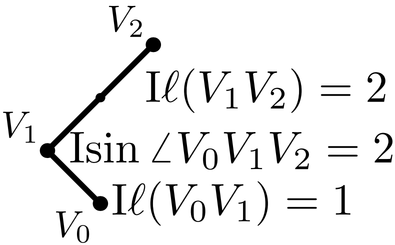
Transformations: Integer lattice preserving affine transformations in the plane.

$$(Aff(2,\mathbb{Z}) = GL(2,\mathbb{Z}) \rtimes \mathbb{Z}^2).$$



Integer trigonometry (O.K. '08)





LLS-sequence for an arbitrary angle

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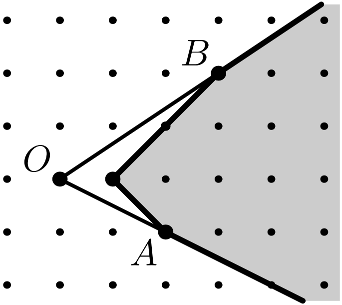

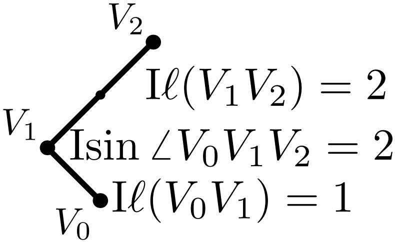

Theorem

LLS-sequence is a complete invariant of integer angles in integer geometry.

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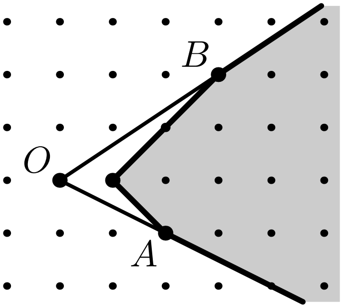
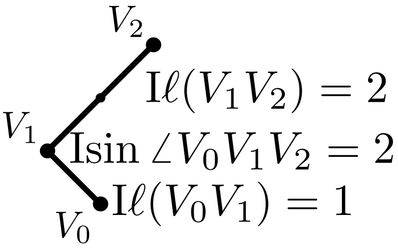
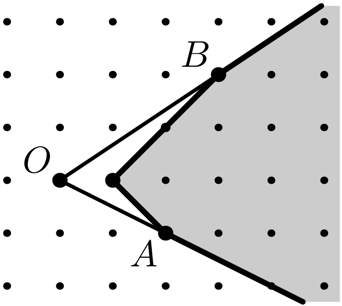


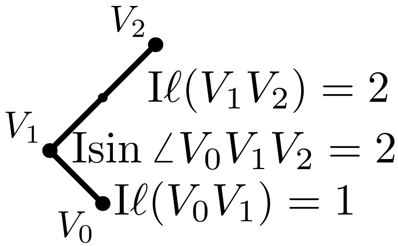


Definition

Let $(a_0, \ldots, a_{2n})$ be the LLS-sequence of α, then
$\mathrm{ltan}\,\alpha = [a_0 : \ldots : a_{2n}]$.

Integer trigonometry (O.K. '08)





$$\mathrm{ltan}\, AOB = [1 : 2; 2] = \frac{7}{5} \quad \Longrightarrow \quad \begin{cases} \mathrm{lsin}\, AOB = 7 \\ \mathrm{lcos}\, AOB = 5 \end{cases}$$

$\alpha + \beta + \gamma = \pi$

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Theorem

Euclidean: a). *There exists a triangle with angles* (α,β,γ) *iff* (α *is acute*)

i) $\tan(\alpha+\beta+\gamma)=0$;

ii) $\tan(\alpha+\beta)\notin[0,\tan\alpha]$.

b). *Two triangles with the same sequences of tangents for angles are homothetic.*

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Integer: a). *There exists an integer triangle with integer angles* (α, β, γ) *iff* (*for some* α *of the triple*)
i) $]\,\mathrm{ltan}\,\alpha, -1, \mathrm{ltan}\,\beta, -1, \mathrm{ltan}\,\gamma[\, = 0$;
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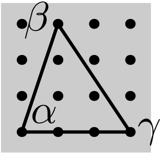
Example



$\mathrm{ltan}\,\alpha = 3 = [3];$
$\mathrm{ltan}\,\beta = 9/7 = [1;3:2];$
$\mathrm{ltan}\,\gamma = 3/2 = [1;1:1].$

i) $[3;-1:1:3:2:-1:1:1:1]=0;$
ii) $[3;-1:1:3:2] = -3/2 \notin [0,3].$

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Part II

II. Global relations on singularities of toric surfaces.

Complex projective surfaces

Complex projective surfaces

Consider an integer convex polygon

$$P = A_0, A_1, \ldots, A_n.$$

Let $A_{n+1}, \ldots, A_m$ be it's inner integer points. Let $A_i = (x_i, y_i)$.

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$$\Omega = \Big\{ \big(t_1^{x_1} t_2^{y_1} t_3^{-x_1-y_1} : \ldots : t_1^{x_m} t_2^{y_m} t_3^{-x_m-y_m} \big) \big| \\ t_1, t_2, t_3 \in \mathbb{C} \setminus \{0\} \Big\}.$$

The set $X_P = \overline{\Omega}$ is the *complex toric variety for P*.

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For $P =$  we have $X_P = \mathbb{C}P^2$.

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For $P =$  we have $X_P = \mathbb{C}P^2$.

For $P =$  X_P is the conic $x_1 x_3 = x_0 x_4$ in $\mathbb{C}P^3$.

Structure of singular sets

Denote $\tilde{A}_i = (0{:}\ldots{:}0{:}1{:}0{:}\ldots{:}0)$.

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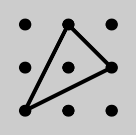
For $P =$ 

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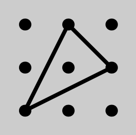
For $P =$ 

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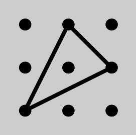
For $P =$ 

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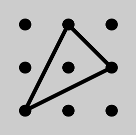

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In appropriate affine charts all singularities are: $xy = z^3$.

Global relation on toric singularities

Corollary

A toric surface singularity of Euler characteristic 3 admits singularities of type $(\operatorname{ltan}\alpha_i, \operatorname{ltan}\alpha_i^t)$ *for* $i = 1, 2, 3$

$$\Longleftrightarrow$$

for some $c_i \in \{\operatorname{ltan}\alpha_i, \operatorname{ltan}\alpha_i^t\}$ *and a permutation* σ *it holds:*
i) $]c_{\sigma(1)}, -1, c_{\sigma(2)}[\notin [0, c_{\sigma(1)}]$*;*
ii) $]c_{\sigma(1)}, -1, c_{\sigma(2)}, -1, c_{\sigma(3)}[= 0.$

Global relation on toric singularities

Corollary

A toric surface singularity of Euler characteristic n admits singularities of type $(\mathrm{ltan}\,\alpha_i, \mathrm{ltan}\,\alpha_i^t)$ for $i = 1, \ldots, n$.

$$\Longrightarrow$$

For $c_i \in \{\mathrm{ltan}\,\alpha_i, \mathrm{ltan}\,\alpha_i^t\}$ there exists a set of integers $\{m_0, \ldots, m_{n-1}\}$ such that:

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Problem

Find a good criterion for "$\Longleftarrow$*" in this case.*

A geometric tool used in the proof

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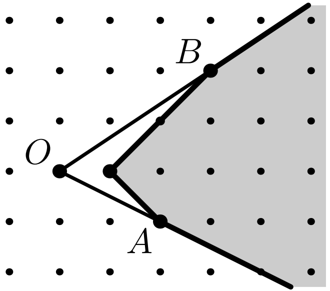
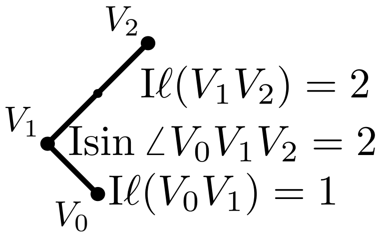
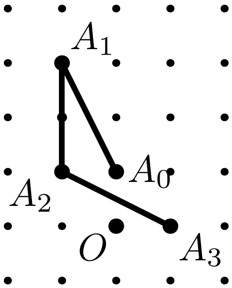




Is it possible to extend the LLS-sequence to arbitrary broken lines?

A geometric tool used in the proof



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Yes.

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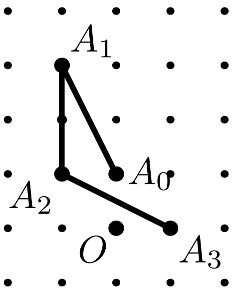

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$$a_{2k} = |OA_k \times OA_{k+1}|, \quad k = 0, \ldots, n;$$

($|v \times w|$ — the oriented area of the parallelogram spanned by v and w)

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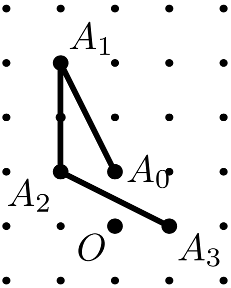

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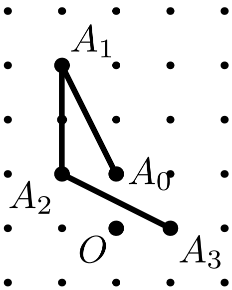

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The sequence $(a_0, \ldots, a_{2n})$ is called the *LLS-sequence*.

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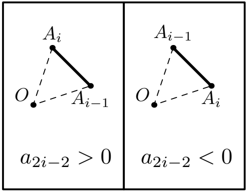

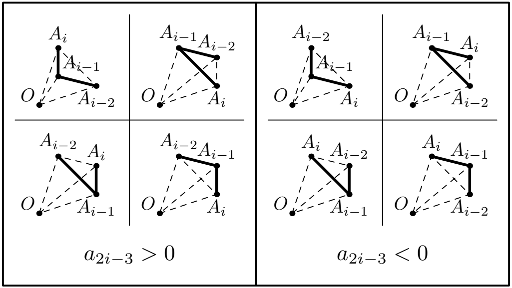

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($|v \times w|$ — the oriented area of the parallelogram spanned by v and w)

Generalized geometry of continued fractions

Theorem

Consider a broken line $A_0 \ldots A_n$ with LLS-sequence $(a_0, a_1, \ldots, a_{2n})$. Let $A_0 = (1,0)$, $A_1 = (1, a_0)$, and $A_n = (x, y)$. Then

$$\frac{y}{x} = [a_0 : a_1; \ldots; a_{2n}].$$

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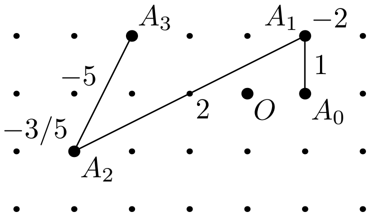

$$[1; -2 : 2 : -3/5 : -5] = \frac{-1}{2}$$

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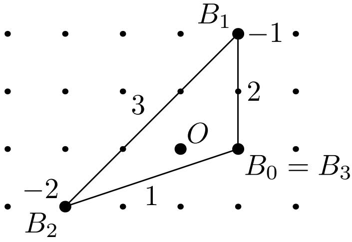

$$[2; -1 : 3 : -2 : 1] = \frac{0}{1}$$

Part III

III. Continued fractions and the second Kepler law.

Smoothing of continued fractions

Recall

Definition

$$a_{2k} = |OA_k \times OA_{k+1}|, \quad k = 0, \ldots, n;$$
$$a_{2k-1} = \frac{|A_kA_{k-1} \times A_kA_{k+1}|}{a_{2k-2}a_{2k}}, \quad k = 1, \ldots, n.$$

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What happens if we consider infinitesimally small edges?

Continued fractions for curves

Definition

(O.K. '11) Let γ be an arc-length parameterized C^2-class curve. Put by definition the *areal density* and the *angular density* at some point t:

$$A(t) = \lim_{\varepsilon \to 0} \frac{|O\gamma(t) \times O\gamma(t+\varepsilon)|}{\varepsilon} = |O\gamma(t) \times \dot{\gamma}(t)|$$

and

$$B(t) = \lim_{\varepsilon \to 0} \frac{|\gamma(t)\gamma(t-\varepsilon) \times \gamma(t)\gamma(t+\varepsilon)|}{\varepsilon |O\gamma(t-\varepsilon) \times O\gamma(t)||O\gamma(t) \times O\gamma(t+\varepsilon)|}.$$

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$$A^2(t)B(t) = \kappa(t).$$

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Proposition

(Areal density and the second Kepler law.) *Suppose that a body moves along the curve γ with velocity $1/A$. Then the sector area velocity of a body is constant and equals 1.*

Examples

Lines. The line $x = a$; O — the origin.

$$A(t) = a \qquad \text{and} \qquad B(t) = 0.$$

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Ellipses and their centers. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a \geq b > 0$; O — the origin (at the symmetry center of the ellipse).

$$A(t) = \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \quad \text{and} \quad B(t) = \frac{1}{ab\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}.$$

Notice:

$$\frac{A(t)}{B(t)} = a^2 b^2 = \text{const}.$$

Examples (planetary motion)

Ellipses and their foci. Consider $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a \geq b > 0$; O — a focus $(-\sqrt{a^2-b^2}, 0)$.

$$A(t)=\frac{ab+b\sqrt{a^2-b^2}\cos t}{\sqrt{a^2\sin^2 t+b^2\cos^2 t}}$$

and

$$B(t)=\frac{a}{b\sqrt{a^2\sin^2 t+b^2\cos^2 t}\left(a+\cos t\sqrt{a^2-b^2}\right)^2}.$$

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The trajectory of the planet is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a \geq b$ (I Kepler law),
Velocity of the planet: $\lambda/A(t)$ (II Kepler law), where

$$\lambda = \pm \frac{T_e}{\int_0^L |1/A(t)|dt} \left(\frac{a}{a_e}\right)^{\frac{3}{2}}.$$

for L — the length of the ellipse for the planet. (III Kepler law)

Examples

Logarithmic spirals. Consider a logarithmic spiral

$$\Big\{\big(ae^{bt}\cos t, ae^{bt}\sin t\big)\Big| t\in\mathbb{R}\Big\}.$$

Then

$$A(t)=\frac{ae^{bt}}{\sqrt{b^2+1}} \quad \text{and} \quad B(t)=\frac{e^{-3bt}\sqrt{b^2+1}}{a^3}.$$

Notice

$$A^3(t)B(t)=\frac{1}{b^2+1}=\text{const.}$$

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Problem

Is there some analog of toric surfaces in the smooth limit? (Develop an approximation theory by toric surfaces.)