Discriminant of system of equations

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Algebra and Geometry,
dedicated to the 65-th anniversary of Askold G. Khovanskii

Multivariate discriminant

$a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \quad \leftrightarrow \quad$ monomial $x^a = x_1^{a_1} \ldots x_n^{a_n}$

$A \subset \mathbb{Z}^n \quad \leftrightarrow \quad \mathbb{C}[A] = \{$linear combinations of x^a, $a \in A\}$

considered as functions on $(\mathbb{C} \setminus 0)^n$

Example

$\mathbb{C}[\text{standard simplex}] = \{\text{linear functions}\}$

$\mathbb{C}[\{0, 1, \ldots, d\}] = \{\text{polynomials of degree } d\}$

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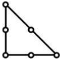
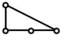
$\Sigma_A \subset \mathbb{C}[A]$ contains $f \quad \Leftrightarrow \quad 0$ is a critical value of $f : (\mathbb{C} \setminus 0)^n \to \mathbb{C}$.

Example

If $A =$ , then the closure of Σ_A consists of

$f(x) = a_{00} + 2a_{10}x + 2a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2$ such that

$$\det \begin{pmatrix} a_{01} & a_{00} & a_{02} \\ a_{00} & a_{10} & a_{20} \\ a_{02} & a_{20} & a_{11} \end{pmatrix} = 0.$$

If $A' =$ , then the closure of $\Sigma_{A'}$ consists of

$f(x) = a_0 + a_1x + a_2x^2 + by$ such that $a_1^2 - 4a_0a_2 = b = 0$.

Dual defect

Definition (Gelfand-Kapranov-Zelevinsky'94)

If the closure of Σ_A if given by one equation, then denote it by $D_A = 0$, otherwise set $D_A \equiv 1$. D_A is the A-**discriminant**.

If $\mathrm{codim}\,\Sigma_A > 1$, then A is **dual defect**. How to classify such A?

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Projectivization of Σ_A is projectively dual to the toric variety X_A.
A is dual defect $\Leftrightarrow$ X_A is dual defect.

Classifying dual defect projective varieties:
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Even if A is dual defect, $\dim \Sigma_A$ is pure. For example (Takeuchi'08), define $d_i = \sum_B (-1)^{\operatorname{codim} B} \left(C^{i-1}_{\dim B - 1} + (-1)^i i \right) \operatorname{Vol}(B) e^B_A$, where B runs over faces of A, and e^B_A is the Euler obstruction of X_A at B.
$0 = d_1 = \ldots = d_r \neq d_{r+1} \;\Rightarrow\; r - 1 = \operatorname{codim} \Sigma_A,\; d_{r+1} = \deg \Sigma_A$.

Discriminant of system of equations

Consider A_0 and A_1 in $\mathbb{Z}^2$. What is the (A_0, A_1)-discriminant?
$\Sigma_{A_0,A_1} \subset \mathbb{C}[A_0] \oplus \mathbb{C}[A_1]$ contains a pair of polynomials (f_0, f_1),

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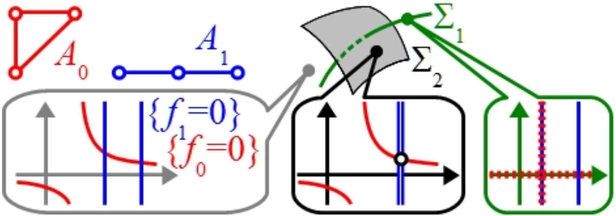
Σ_{A_0,A_1} may contain components of codimension both $= 1$ and > 1.

Example



$f_0(x, y) = a + by + cxy$, $f_1(x, y) = p + qx + rx^2$,
$\Sigma_2 = \{q^2 = 4pr\}$, $\Sigma_1 = \{a = pc^2 + qbc + rb^2 = 0\}$.

Nondegenerate polynomials: definitions

For a polynomial f of n variables and a linear function $v : \mathbb{Z}^n \to \mathbb{Z}$, let f^v be the highest v-homogeneous component of f.

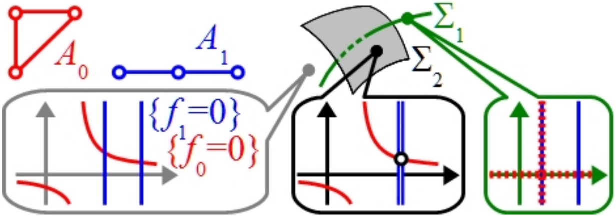
Definition

A tuple $(f_0, \dots, f_k)$ is **degenerate in the sense of Khovanskii**, if:
– 0 is a critical value of $(f_0^v, \dots, f_k^v) : (\mathbb{C} \setminus 0)^n \to \mathbb{C}^{k+1}$ for some v,
– the (non-zero) coefficients of $(f_0, \dots, f_k)$ can be perturbed so that the topological type of $\{f_0 = \dots = f_k = 0\}$ changes,
– the (non-zero) coefficients of $(f_0, \dots, f_k)$ can be perturbed so that the Euler charecteristics of $\{f_0 = \dots = f_k = 0\}$ changes.
– There is a local system L, such that $H\big((\mathbb{C} \setminus 0)^n, L\big) = 0$, but $H(\{f_0 = \dots = f_k = 0\}, L)$ is not only in the middle dimension.

Let $A_0, \dots, A_k$ be finite sets in $\mathbb{Z}^n$.
Let $S \subset \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ be the set of all degenerate tuples.

Nondegenerate polynomials: examples

Example



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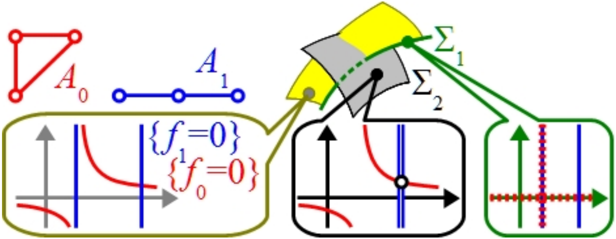

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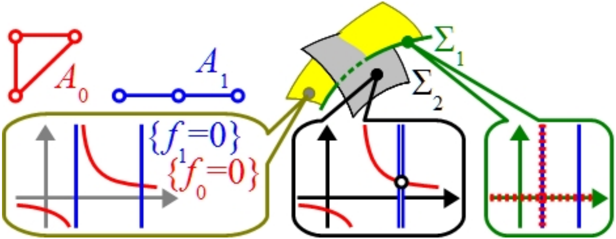

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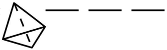
Example

If $(A_0, A_1, A_2, A_3) =$  in $\mathbb{Z}^3$, then $\operatorname{codim} S = 2$:

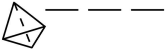
nondegenerate tuple (f_0, f_1, f_2, f_3) has no common roots, and triples (f_1, f_2, f_3) that have a common root are in codimension 2.

Discriminant of system of equations: definition

Definition

$\mathrm{codim}(A_0, \ldots, A_k)$ is the maximum over all $i_1 < \ldots < i_p$ of $p - \dim(\text{convex hull of } A_{i_1} + \ldots + A_{i_p})$.

Example

$\mathrm{codim}($  $) = 2$.

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Example

$\mathrm{codim}($$— — —) = 2.$

$S \subset \mathbb{C}[A_0] \oplus \ldots \mathbb{C}[A_k]$ is the set of all degenerate tuples.

Theorem

If $\mathrm{codim}(A_0, \ldots, A_k) \leqslant 1$*, then* S *is a non-empty hypersurface.*

Let S_i be a component of S, choose $f \notin S$ and a generic $\tilde{f} \in S_i$, then $(-1)^{n-k}(\chi\{\tilde{f} = 0\} - \chi\{f = 0\}) > 0$, denote it by χ_i.

Definition

The equation of the divisor $\sum_i \chi_i S_i$ is called the **Euler discriminant** $E_{A_0,\ldots,A_k}$.

Discriminant of system of equations: examples

Example

If $k = n$, then S is the closure of all tuples $(f_0, \dots, f_n)$ such that $\{f_0 = \dots = f_n = 0\} \neq \varnothing$, and $E_{A_0,\dots,A_n} = R_{A_0,\dots,A_n}$ is the *sparse resultant*.

It can be computed e. g. as $\dfrac{\text{Sylvester-type matrix}}{\text{its certain minor}}$ (D'Andrea'02).

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Example

If $k = 0$, then E_{A_0} is the *principal A-determinant*:
$E_{A_0}(f) = R_{A_0,\dots,A_0}(f, x_1\frac{\partial f}{\partial x_1}, \dots, x_n\frac{\partial f}{\partial x_n})$. $E_{A_0} \neq D_{A_0}$ for $n > 1$!
Its Newton polytope is the **secondary polytope** of A (GKZ'94).

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Proposition (Cayley trick)

$E_{A_0,\dots,A_k}(f_0, \dots, f_k) = \prod_{i_0<\dots<i_p} E(\lambda_{i_0}f_{i_0} + \dots + \lambda_{i_p}f_{i_p})^{(-1)^{n-p}}$
Its Newton polytope is the **mixed secondary polytope** of $A_\bullet$.

Bifurcation set and Euler discriminant

Every algebraic $\pi : M \to \mathbb{C}^m$ admits the maximal open subset $V \subset \mathbb{C}^m$, on which $\pi : \pi^{-1}(V) \to V$ is a fibration.

Definition

$\mathbb{C}^m \setminus V$ is the **bifurcation set** B_π. Fiber $\pi^{-1}(v)$, $v \in V$, is **typical**.

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If $M = \mathbb{C}^m$, then B_π is a hypersurface.

Theorem (Le'84, generalization – Siersma-Tibar, Artal-Luengo-Melle, Nemethi, Parusinski etc.)

Atypical fibers of $\pi : \mathbb{C}^2 \to \mathbb{C}$ differ from typical ones by their Euler characteristics.

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By induction, subdivide $\mathbb{C}^m$ into $\bigsqcup_i V_i$, such that π is a fibration over every V_i. Let is fiber be F_i and V_0 be dense.

Definition

$E_\pi = \sum_{i\,:\,\dim V_i = m-1} (\chi F_i - \chi F_0) \cdot \overline{V}_i$ – **Euler discriminant.**

Base changes in Euler discriminant

Theorem ($k = 1$ – Nemethi'90, $k = n$ – Jelonek'92)

$|E_G| = B_G$ for every map $G = (g_1, \ldots, g_k) : (\mathbb{C} \setminus 0)^n \to \mathbb{C}^k$, such that $(g_1, \ldots, g_k)$ and $(g_1, \ldots, \hat{g}_i, \ldots, g_k)$ for every i are nondegenerate: atypical fibers are parameterized by a hypersurface, and almost all atypical fibers have atypical Euler characteristics.

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Proof: define $\mathcal{J} : \mathbb{C}^k \to \mathbb{C}[A_1] \oplus \ldots \oplus \mathbb{C}[A_k] = \mathbb{C}[A_\bullet]$ as $\mathcal{J}(y_1, \ldots, y_k) = (g_1 - y_1, \ldots, g_k - y_k)$.

$$\begin{array}{ccccc} \text{graph of } G & \to & \{(x, f_1, \ldots, f_k) \mid f_\bullet(x) = 0\} & \subset (\mathbb{C} \setminus 0)^n \times \mathbb{C}[A_\bullet] \\ \downarrow \mathcal{J}^*\pi & & \downarrow \pi & \\ \mathbb{C}^k & \xrightarrow{\mathcal{J}} & \mathbb{C}[A_\bullet] & \end{array}$$

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\end{array}
$$

- $|E_\pi| = B_\pi$ is the Euler discriminant $\{E_{A_0, \ldots, A_k} = 0\}$.
- $\mathcal{J}^* E_\pi = E_{\mathcal{J}^*\pi}$ and $\mathcal{J}^{-1} B_\pi = B_{\mathcal{J}^*\pi}$ by the invariance under base changes.
- $E_{\mathcal{J}^*\pi} = E_G$ and $B_{\mathcal{J}^*\pi} = B_G$ are what we need.

Faces of a tuple of polytopes

Proof of Theorems for $A_0 = \ldots = A_n = A$: singularity theory plus "every face of $\operatorname{conv} A$ is a facet of another face". What if $A_i \neq A_j$?

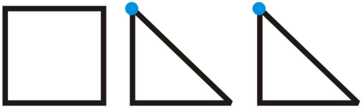
- A tuple B of polytopes is said to be **essential**, if $\operatorname{codim} B > \operatorname{codim} \Gamma$ for every its subtuple Γ
- For linear $v : \mathbb{Z}^n \to \mathbb{Z}$, let A^v be the face of $\operatorname{conv} A$, on which v attains its maximum.
- For every linear $v : \mathbb{Z}^n \to \mathbb{Z}$, the maximal essential subtuple of $A_0^v, \ldots, A_k^v$ is called a **facing** of $(A_0, \ldots, A_k)$.
- A facing Γ is **adjacent** to a facing B, if they are the maximal essential subtuples of $(A_0^v)^u, \ldots, (A_k^v)^u$ and $A_0^v, \ldots, A_k^v$.
- For a facing B of $(A_0, \ldots, A_k)$, define $\dim(B) = \operatorname{codim}(A_0, \ldots, A_k) - \operatorname{codim}(B)$.

Faces of a tuple of polytopes: examples

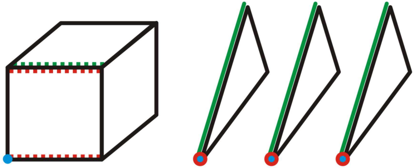
(Facings, adjacency, dim) is the **poset of faces** of $(A_0, \ldots, A_k)$.

Example

Blue is adjacent to **red**.



Blue is adjacent to **black**.



Example

Poset of faces is not a poset:

Blue$<$**red**$<$**green**, **blue**$\not<$**green**

What is the transitive closure of the adjacency relation?

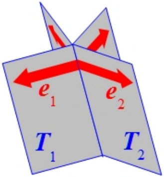
Theorem

Every facing is adjacent to a facing of dimension greater by 1.

Discriminant of system of equations: tropical version

Tropical ring – $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \text{"max"}, +)$, where "max"$(a, b) = \max(a, b)$ for $a \neq b$, and "max"$(a, a) = [-\infty, a]$.

A k-dimensional algebraic set T is a union of k-dimensional polytopes in $\mathbb{T}^n \supset \mathbb{R}^n$, endowed with balanced positive integer tensions: $\sum_i T_i e_i = 0$ on every $(k-1)$-dimensional face.



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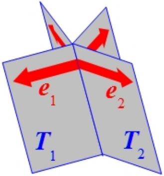

Proposition

If $T \subset \mathbb{T}^n$ is k-dimensional, and $P \subset \mathbb{T}^n$ is an open polytope, then (closure of $T \cap P) \cap \partial P$ is purely $(k-1)$-dimensional.

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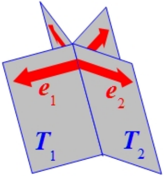

Proposition (Special case of Conjecture)

If $T \subset \mathbb{T}^n$ is k-dimensional, and $P \subset \mathbb{T}^n$ is an open polytope, then (closure of $T \cap P) \cap \partial P$ is purely $(k-1)$-dimensional.

- $x \in T$ is **regular** for $\pi : \mathbb{R}^n \to \mathbb{R}^k$, if a generic fiber of π intersects a neighborhood of x in T at one point.
- T is **regular**, if every its point is regular for some $\mathbb{R}^n \to \mathbb{R}^k$.
- π(all $x \in T$ that are not regular for π) is the **tropical discriminant** $\mathbb{T}D_\pi$ of the projection $\pi : T \to \mathbb{R}^k$.
- Conjecture: if T is regular then $\operatorname{codim} \mathbb{T}D_\pi = 1$.