Lie-operator rings and their schemes **(A. Dosi)**

One of the principal foundations of noncommutative algebraic geometry is to extend the concept of affine schemes to noncommutative rings. As in the commutative case the main motivation for this is to represent a noncommutative ring as the ring of "functions" over its spectrum.

The classical result, which is due to I. M. Gelfand, asserts that a commutative Banach algebra can be realized as the algebra of continuous functions over the space of all its maximal ideals modulo its Jacobson radical.

A commutative ring A is the ring of all global sections $\Gamma\left(\mathsf{Spec}\left(A\right), \mathcal{O}\right)$ of the structure sheaf $\mathcal{O}$ of the ring A over its scheme $\mathsf{Spec}\left(A\right)$ (the space of all prime ideals of A) up to an isomorphism (A. Grothendieck).

The noncommutative ring would generalize here a commutative ring of regular functions on a commutative scheme.

The construction of the relevant schemes, and "functions" over them, for the noncommutative ring requires extraordinary efforts. The problem can not be able to be solved in a unique framework based on a certain special category of objects and morphisms. All these diversity reflects in various methods and constructions picked up in noncommutative geometry.

In the Banach algebra context the indicated direction is closely related to the noncommutative functional calculus problem and noncommutative spectral theory. Lie algebra methods allow us to handle the problem and formulate the relevant restrictions for noncommutative functional calculus to be built up. As shown in many investigations the most reliable case is a family generating a Lie nilpotent algebra. Based on this result from analysis, it is reasonable to expect good behavior of Lie-nilpotent rings in noncommutative algebraic geometry. This proposal has been partially supported in Kapranov's theory of NC-schemes.

Another motivation for the present work is to use a purely operator approach to noncommutative schemes without any sheaf constructions as classically done. The operator realization of many noncommutative algebras is the well known fact.

In the present talk we deal with noncommutative regularity in the general purely algebraic case. We propose a new approach to noncommutative spectra which is based on concrete operator realization of an abstract regularity in a Lie-complete ring.

Let R be a unital (noncommutative) ring. The set of all its nilpotent elements is denoted by $\mathfrak{N}(R)$. Recall that a unital subring $C \subseteq R$ is said to be *an inverse closed subring in* R if each $x \in C$ being a unit in R turns out to be a unit in C too, that is, $x^{-1} \in C$. Any intersection of inverse closed subrings is an inverse closed subring. In particular, a subset $S \subseteq R$ possesses *the inverse closed hull* $R(S)$, which is the smallest inverse closed subring containing S.

A ring R turns out to be a Lie ring denoted by $R_{\mathfrak{lie}}$ equipped with the Lie brackets $[a, b] = ab - ba$, $a, b \in R$. The lower central series $\mathcal{L}^{(k)}$, $k \geq 1$, of a Lie ring $\mathcal{L}$ is defined as the decaying sequence $\mathcal{L}^{(1)} = \mathcal{L}$, $\mathcal{L}^{(k+1)} = \left[\mathcal{L}, \mathcal{L}^{(k)}\right]$, $k \geq 1$, of its Lie-ideals. If $\mathcal{L}^{(n+1)} = \{0\}$ for some $n \geq 1$, we say that $\mathcal{L}$ is a nilpotent Lie ring.

Lie-nilpotent rings. Let A be a ring. The ring A is said to be *a Lie-nilpotent ring* if $A_{\mathfrak{lie}}$ is a nilpotent Lie ring, that is, $A_{\mathfrak{lie}}^{(n+1)} = \{0\}$ for some $n \geq 1$.

Pr. 1. *Let R be a unital ring, $A \subseteq R$ its Lie nilpotent subring, and $R(A)$ the inverse closed hull of A in R. Then $\mathfrak{N}(R(A))$ is a two-sided ideal in $R(A)$ and the ring $R(A)$ is commutative modulo $\mathfrak{N}(R(A))$.*

In particular, if A is a Lie-nilpotent ring then $\mathfrak{N}(A) \lhd A$ and $A/\mathfrak{N}(A)$ is commutative (just put $R = A$). In particular, $I\left(A_{\mathfrak{lie}}^{(2)}\right) \subseteq \mathfrak{N}(A)$. We also put $A_c = A/I\left(A_{\mathfrak{lie}}^{(2)}\right)$ called Kapranov's commutativization of A.

Prime ideals of a Lie-nilpotent ring. A two-sided ideal $\mathfrak{p}$ of a noncommutative ring A is called *a prime ideal* if the inclusion $I_1 I_2 \subseteq \mathfrak{p}$ for two-sided ideals I_1, $I_2 \subseteq A$ implies that $I_1 \subseteq \mathfrak{p}$ or $I_2 \subseteq \mathfrak{p}$. The set of all prime ideals of a ring A is denoted by $\mathsf{Spec}\,(A)$. We define *the nilradical* (McCoy, 1949)

$$\mathfrak{Nil}\,(A) = \cap \{\mathfrak{p} : \mathfrak{p} \in \mathsf{Spec}\,(A)\} \subseteq \mathfrak{N}\,(A)$$

An ideal $\mathfrak{p}$ with multiplicatively closed complement $A \backslash \mathfrak{p}$ is called *a completely prime ideal.*

Pr. 2. *Let A be a Lie-nilpotent ring. Then* $\mathsf{Spec}\,(A)$ *consists of all completely prime ideals, which is identified with* $\mathsf{Spec}\,(A_c)$*, and* $\mathfrak{Nil}\,(A) = \mathfrak{N}\,(A)$*. Thus* $\mathsf{Spec}\,(A)$ *turns out to be T_0-space with respect to the Zariski topology inherited from* $\mathsf{Spec}\,(A_c)$*.*

Lie-complete rings. Let B_ϵ, $\epsilon \in \mathfrak{a}$, be a family of Lie-nilpotent rings (equipped with the discrete topology), $B = \prod_{\epsilon\in\mathfrak{a}} B_\epsilon$ their direct product, and let $A \subseteq B$ be a unital subring. We have a projective (or weak) topology in A such that all projections $\pi_\epsilon : A \longrightarrow B_\epsilon$, $\epsilon \in \mathfrak{a}$, are continuous. We say that A is *a Lie-filtered ring with its weak topology*. Thus $A \subseteq \prod_{\epsilon\in\mathfrak{a}} A_\epsilon$ with $A_\epsilon = \pi_\epsilon(A)$. The ring A is said to be *a Lie-complete ring* if A coincides with its (weak) completion $\widehat{A}$. Put

$$\mathsf{Spf}(A) = \{\text{open prime ideals of } A\}.$$

If A is a Lie-nilpotent ring (equipped with the discrete topology) then $\mathsf{Spf}(A) = \mathsf{Spec}(A)$. The set

$$\mathfrak{Tnil}(A) = \left\{x \in A : \lim_m x^m = 0\right\}$$

is called *the topological nilradical of* A.

Pr. 3. *Let* $A \subseteq \prod_{\epsilon \in \mathfrak{a}} A_\epsilon$ *be a Lie-filtered ring. Then* $\cap \mathsf{Spf}(A) = \mathfrak{Tnil}(A)$, $A/\mathfrak{Tnil}(A)$ *is commutative,* and $\mathsf{Spf}(A) = \{\text{open completely prime ideals of } A\}$. *Thus*

$$\mathsf{Spf}(A) = \cup_{\epsilon \in \mathfrak{a}} \mathsf{Spec}(A_\epsilon) = \mathsf{Spf}(A_c) \subseteq \mathsf{Spec}(A_c),$$

and if A *is a Lie-complete ring then*

$$\overline{\mathfrak{N}(A)} \subseteq \mathfrak{Tnil}(A) \subseteq \mathsf{Rad}(A),$$

where $\mathsf{Rad}(A)$ *is the Jacobson radical of* A, *and* $A_c = A/\overline{I\left(A_{\mathfrak{lie}}^{(2)}\right)}$ is the commutativization.

We define *the formal radical*

$$\sqrt{\mathbf{T}} = \cap \{\mathfrak{p} \in \mathsf{Spf}(A) : \mathbf{T} \subseteq \mathfrak{p}\}$$

in A of an nonempty subset (or tuple) $\mathbf{T} \subseteq A$. In particular, $\sqrt{0} = \mathfrak{Tnil}(A)$. Note that

$$\sqrt{\mathbf{T}} = \left\{x \in A : \lim_{\iota}(x^{n_\iota} - y_\iota) = 0,\ (y_\iota) \subseteq I(\mathbf{T})\right\},$$

where $I(\mathbf{T})$ is the two-sided ideal in A generated by $\mathbf{T}$.

Quantum domains. Fix a unital commutative ring R and let $K \in R$-mod. By *a (quantum) domain on* K we mean a commutative subset $X \subseteq \mathsf{End}_R(K)$ of nonzero projections such that $\sum X = 1_K$ (or $\sum_{e\in X} \mathsf{im}(e) = K$). The commutant (*noncommutative functions on* X)

$$X' = \{T \in \mathsf{End}_R(K) : Te = eT, e \in X\}$$

is an inverse closed unital subring in $\mathsf{End}_R(K)$. For each $S \in X'$ we set (*the noncommutative support*)

$$X_S = \{e \in X : eS \text{ is a unit in } e\,\mathsf{End}_R(K)\,e\}.$$

If $\mathbf{T} \subseteq X'$ then we put $X_{\mathbf{T}} = \cup_{S\in\mathbf{T}} X_S$. For each $\epsilon \subseteq X$ we have the ring homomorphism

$$\pi_\epsilon : X' \to \prod_{e\in\epsilon} e\,\mathsf{End}_R(K)\,e, \pi_\epsilon(T) = (eT)_{e\in\epsilon}.$$

In particular, $\pi_X : X' \hookrightarrow \prod_{e\in X} e\,\mathsf{End}_R(K)\,e$ is a ring embedding. Thus X' is a filtered ring. Assuming each subring $e\,\mathsf{End}_R(K)\,e$ to be equipped with the discrete topology, we obtain the weak (filtered) topology $\mathfrak{w}$ (called *the weak operator topology*) in X' such that all projections π_e, $e \in X$, are continuous.

Operator topologies. $\mathfrak{d}$ is the discrete topology in X'. A covering $\mathfrak{t}$ of X defines a new filtered topology (denoted by $\mathfrak{t}$ as well) in X', which is the weak topology in X' such that all projections π_ϵ, $\epsilon \in \mathfrak{t}$, are continuous (each $\prod_{e\in\epsilon} e\,\mathsf{End}_R(K)\,e$ equipped with the discreet topology). Thus we have the scale

$$\mathfrak{w} \preceq \mathfrak{t} \preceq \mathfrak{d}$$

of the filtered topologies in X' called *the operator topology scale of the domain X*.

If $\mathfrak{t} = \{X\}$ then $\mathfrak{t} = \mathfrak{d}$;

If $\mathfrak{t} = \{\{e\} : e \in X\}$ is the "atomic" covering of X then $\mathfrak{t} = \mathfrak{w}$.

Actually, X' is complete with respect to each topology $\mathfrak{t}$ from the scale.

Lie-operator rings. Let $\mathcal{A} \subseteq X'$ be a unital subring equipped with a topology $\mathfrak{a}$ from the operator topology scale of the domain X.

We say that $\mathcal{A}$ is *a Lie-operator ring in* X' if the range $\pi_\epsilon(\mathcal{A})$ is a Lie-nilpotent subring in $\prod\limits_{e\in\epsilon} e\,\mathsf{End}_R(K)\,e$ for all $\epsilon \in \mathfrak{a}$. In particular, the subring $e\mathcal{A}$ $(= \pi_e(\mathcal{A}))$ is Lie-nilpotent for each $e \in X$.

The Lie-operator ring $\mathcal{A}$ automatically generates so called *support topology* $\mathfrak{s}(\mathcal{A})$ in X, which is based on the key relation

$$X_S \cap X_T = X_{ST}$$

for all $S, T \in \mathcal{A}$. The family $\{X_S : S \in \mathcal{A}\}$ is a topology base of $\mathfrak{s}(\mathcal{A})$ in X, which is not necessarily T_0-space topology.

Noncommutative regularity. *Let $\mathcal{A}$ be a complete Lie-operator ring in X'. It is called an X*-regular ring if

(i) $X_S \subseteq X_{\mathbf{T}}$, $S \in \mathcal{A}$, $\mathbf{T} \subseteq \mathcal{A} \Rightarrow S \in \sqrt{\mathbf{T}}$;

(ii) $S, T \in A \Rightarrow X_{S+T} \subseteq X_S \cup X_T$.

Then $\mathfrak{Tnil}(\mathcal{A}) = \{S \in \mathcal{A} : X_S = \varnothing\}$. Similarly, $X_S = X$ iff S is a unit in $\mathcal{A}$.

For each $e \in X$ the set

$$\mathfrak{p}_e = \{T \in \mathcal{A} : Te \text{ is not a unit in } e \operatorname{End}_R(K) e\}$$

is an open prime ideal of $\mathcal{A}$, $\bigcap\limits_{e \in X} \mathfrak{p}_e = \mathfrak{Tnil}(\mathcal{A})$ and $\bigcup\limits_{e \in X} \mathfrak{p}_e$ consists of all non-unit elements in $\mathcal{A}$. We have a continuous mapping

$$\mathfrak{f} : X \to \operatorname{Spf}(\mathcal{A}), \mathfrak{f}(e) = \mathfrak{p}_e$$

such that $\overline{\{e\}} = \overline{\{f\}}$ in X (with respect to $\mathfrak{s}(\mathcal{A})$) iff $\mathfrak{p}_e = \mathfrak{p}_f$. Thus the Kolmogorov quotient $X_{\mathcal{K}}$ of X is embedded into $\operatorname{Spf}(\mathcal{A})$ up to a homeomorphism.

Pr. *Let $\mathcal{A} \subseteq X'$ be an X-regular ring, Then $X_{\mathcal{K}}$ is a dense subspace in* $\operatorname{Spf}(\mathcal{A})$.

Lie-operator X-rings. As a concrete model of a Lie-complete ring we introduce Lie-operator X-rings:

Let $\mathcal{A} \subseteq X'$ be an X-regular ring with its operator topology $\mathfrak{a}$. We say that $\mathcal{A}$ is a Lie-operator X-ring if for each $\epsilon \in \mathfrak{a}$ we have the following local properties:

(i) $e \notin \epsilon \Rightarrow \exists S \in \mathcal{A}$, $e \in X_S$, $\pi_\epsilon(S) = 0$ *(the separation axiom);*

(ii) $Te = 0$, $T \in \mathcal{A}$, $e \in \epsilon \Rightarrow \exists S \in \mathcal{A}$, $e \in X_S$, with $\mathfrak{w}\text{-lim}_m \, \pi_\epsilon(TS^m) = 0$.

If $\mathfrak{a} = \mathfrak{d}$ on $\mathcal{A}$, then $\mathcal{A}$ is a Lie-nilpotent ring, and we have $Te = 0, T \in \mathcal{A}$, $e \in X \Rightarrow \exists S \in \mathcal{A}, e \in X_S$, $\mathfrak{w}\text{-lim}_m \, TS^m = 0$. The separation axiom (i) is satisfied automatically.

If $\mathfrak{a} = \mathfrak{w}$ on $\mathcal{A}$ then (i) means that for $e \neq f$ in X we have $e \in X_S$, $f \in X_T$ and $Sf = Te = 0$ for some $S, T \in \mathcal{A}$. In particular, X is T_0-space with respect to the support topology $\mathfrak{s}(\mathcal{A})$. In this case the axiom (ii) is satisfied automatically.

The formal spectrum of a Lie-operator X-ring. Now consider the open two-sided ideal

$$\mathfrak{q}_e = \{T \in \mathcal{A} : Te \text{ is nilpotent}\} \subseteq \mathfrak{p}_e.$$

Th.1. *If $\mathcal{A}$ is a Lie-operator X-ring, $\mathfrak{p} \in \mathsf{Spf}\,(\mathcal{A}) \setminus X_{\mathcal{K}}$, then $\mathfrak{q}_e \subseteq \mathfrak{p} \subseteq \mathfrak{p}_e$ for a certain $e \in X$. Thus $X_{\mathcal{K}}$ is a strongly dense subspace in $\mathsf{Spf}\,(\mathcal{A})$, that is, $\mathsf{Spf}\,(\mathcal{A})$ is the Kolmogorov completion of X.*

Cor. *If $X \subseteq \mathcal{A}$ then X is an orthogonal family of projections with $\mathfrak{w}$-$\sum X = 1$, the support topology $\mathfrak{s}\,(\mathcal{A})$ in X is discreet, and $\mathcal{A} = \prod\limits_{e \in X} \mathcal{A}_e$ with $\mathfrak{a} = \mathfrak{w}$. In particular, $\mathsf{Spf}\,(\mathcal{A}) = X$ up to a homeomorphism, and X consists of all open maximal ideals.*

Kolmogorov completion: Let Y be a non-empty T_0-space, $X \subseteq Y$ a subspace.

X is *dense in* Y means $X \cap U \neq \varnothing$ for each non-empty open subset $U \subseteq Y$.

X is T_0-*dense in* Y means $X \cap F \neq \varnothing$ for each non-empty closed subset $F \subseteq Y$.

If X is dense in Y in both senses, we say that X is *a strongly dense subspace in* Y.

Thus strongly dense subspaces reflect our Hausdorff perception of density within T_0-spaces.

We say that Y *is a Kolmogorov completion of a topological space* X if the Kolmogorov quotient $X_{\mathcal{K}}$ of X is embedded (up to a homeomorphism) into Y as a strongly dense subspace. If Y is Hausdorff then T_0-dense subspace of Y is just the whole space Y itself.

The set $\mathbb{Z}$ is an example of a strongly dense subspace in $\mathbb{R}$ with respect to the right-order topology.

Application to joint spectra. Assume that $R = k$ is a field, K a linear space over k, $\mathcal{A} \subseteq X' \subseteq L(K)$ an X-regular ring, and $\mathbf{S} = (S_\iota)_{\iota\in\Omega} \subseteq \mathcal{A}$. The *Harte spectrum of* $\mathbf{S}$ *in* $\mathcal{A}$ is defined as

$$\sigma_{\mathcal{A}}(\mathbf{S}) = \left\{\lambda \in k^{\Omega} : \overline{(\mathbf{S}-\lambda)} \neq (1_K)\right\},$$

where $\overline{(\mathbf{S}-\lambda)}$ is the closure of the left or right ideal in $\mathcal{A}$ generated by the tuple $\mathbf{S}-\lambda$. Then

$$\sigma_{\mathcal{A}}(\mathbf{S}) = \left\{\lambda \in k^{\Omega} : X_{\mathbf{S}-\lambda} \neq X\right\}.$$

Consider the free associative k-algebra $\mathcal{F}_\Omega(\mathbf{x})$ generated by $\mathbf{x} = (x_\iota)_{\iota\in\Omega}$. The mapping $f(\mathbf{x}) \mapsto f(\mathbf{S})$ is *a functional calculus* $\mathcal{F}_\Omega(\mathbf{x}) \to \mathcal{A}$ *for* $\mathbf{S}$. A tuple $\lambda \in k^{\Omega}$ generates itself a functional calculus $\mathcal{F}_\Omega(\mathbf{x}) \to k$, $f(\mathbf{x}) \mapsto f(\lambda)$. If $\mathbf{f}(\mathbf{x}) = (f_\kappa(\mathbf{x}))_{\kappa\in\Xi} \subseteq \mathcal{F}_\Omega(\mathbf{x})$ then we have the Ξ-tuples $\mathbf{f}(\mathbf{S}) = (\mathbf{f}_\kappa(\mathbf{S}))_{\kappa\in\Xi}$ in $\mathcal{A}$ and $\mathbf{f}(\lambda) = (\mathbf{f}_\kappa(\lambda))_{\kappa\in\Xi} \in k^{\Xi}$.

Th 2. *If* $\sigma_{\mathcal{A}}(\mathbf{S}) \neq \varnothing$ *then* $\mathbf{f}(\sigma_{\mathcal{A}}(\mathbf{S})) = \sigma_{\mathcal{A}}(\mathbf{f}(\mathbf{S}))$.

The representation theorem. Finally, we propose the representation theorem for Lie-complete rings being represented as Lie-operator X-rings.

Th 3. *Let $A \subseteq \prod_{\epsilon \in \mathfrak{a}} A_\epsilon$ be a Lie-complete ring. Then A is a Lie-operator X-ring up to a topological ring isomorphism, where $X = \vee_{\epsilon \in \mathfrak{a}} \mathsf{Spec}\,(A_\epsilon)$. Moreover, $\mathfrak{p}_e = e$ for all $e \in X$. In particular, $X_{\mathcal{K}} = \mathsf{Spf}\,(A)$ up to a homeomorphism.*

Thus the Lie-operator X-rings are concrete models of Lie-complete (in particular, Lie-nilpotent) rings. If A is a Lie-nilpotent ring then it is a Lie-operator X-ring equipped with the discrete topology thanks to Th. 3.

Examples. 1) The block-upper triangular matrices. Let $R = k$ be a field, $K = \bigoplus_{e \in X} k^{m_e}$ the direct sum of m_e-dimensional k-linear spaces. We identify the index set X with the family $X \subseteq \mathsf{End}_k(K)$ of orthogonal projections onto indicated finite dimensional subspaces. Note that $\sum X = 1_K$, and each $S \in X'$ has a block diagonal form $S = \bigoplus_{e \in X} S_e$ with $S_e \in \mathsf{End}_k(k^{m_e})$, $e \in X$. In particular, X' possesses the weak (operator) topology $\mathfrak{w}$ with respect to the indicated expansion. Consider the following unital subring

$$\mathcal{A}_{\mathfrak{w}} = \left\{ S \in X' : S_e = \begin{bmatrix} s_e & & * \\ & \ddots & \\ 0 & & s_e \end{bmatrix} \in M_{m_e}(k) \right\}$$

equipped with $\mathfrak{w}$. Obviously, $\mathcal{A}_{\mathfrak{w}}$ is a complete Lie-operator ring. For each $S \in \mathcal{A}_{\mathfrak{w}}$ we have

$$X_S = \{ e \in X : s_e \neq 0 \}.$$

Then $\mathcal{A}_{\mathfrak{w}}$ is a Lie-operator X-ring and $\mathsf{Spf}(\mathcal{A}_{\mathfrak{w}}) = \mathsf{Mf}(\mathcal{A}_{\mathfrak{w}}) = X$ up to a homeomorphism.

2) The glued block-upper triangular matrices. The previous example can be modified in the following way. For each subset $\epsilon \subseteq X$ we put

$$m_\epsilon = \textsf{sup}\left\{m_e : e \in \epsilon\right\}$$

(some of m_e even infinitely many may coincide). Consider a partition $\mathfrak{a}$ of X with the following property $m_\epsilon < \infty$ for each $\epsilon \in \mathfrak{a}$, and let $p_\epsilon = \mathfrak{w}\text{-}\sum_{e\in\epsilon} e \in \mathcal{A}_\mathfrak{w}$ be the relevant projections. The partition $\mathfrak{a}$ associates the following subring

$$\mathcal{A}_\mathfrak{a} = \Big\{S \in \mathcal{A} : s_e = s_f, \forall e, f \in \epsilon, \forall \epsilon \in \mathfrak{a}\Big\} \subseteq \mathcal{A}_\mathfrak{w}.$$

If $\mathfrak{a} = \{\{e\} : e \in X\} = \mathfrak{w}$ is the atomic partition of X then $\mathcal{A}_\mathfrak{a} = \mathcal{A}_\mathfrak{w}$. In the general case, $\mathcal{A}_\mathfrak{a}$ is a Lie-operator X-ring. Note that $\mathfrak{p}_e = \mathfrak{p}_f$ for all $e, f \in \epsilon$ They stick to each other within ϵ. Put $\mathfrak{p}_\epsilon = \mathfrak{p}_e$ whenever $e \in \epsilon$. Then

$$X_\mathcal{K} = \left\{\mathfrak{p}_\epsilon : \epsilon \in \mathfrak{a}\right\},$$

and it is a strongly dense subspace in $\textsf{Spf}\,(\mathcal{A}_\mathfrak{a})$.

3) The local ring $\mathbb{Z}_{(p)}$. Let $p > 1$ be a fixed prime, $K_p = \left\{ \frac{x}{y} \in \mathbb{Q} : x \in p\mathbb{Z}, y \notin p\mathbb{Z} \right\}$, $K = \mathbb{Q} \oplus K_p \in \mathbb{Z}$-mod, and $X = \{e_1, e_2\} \subseteq \mathsf{End}_{\mathbb{Z}}(K)$ the relevant canonical projections. The local ring

$$\mathcal{A} = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : a \in \mathbb{Z}, b \notin p\mathbb{Z} \right\} \subseteq X'$$

acts as the diagonal operators on K. If $z \in \mathbb{Z}_{(p)}$, then $\widehat{z} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$ acts on $\mathbb{Q}$ and K_p as the multiplication operators. Then $X_{\widehat{z}} = \varnothing$, $\{e_1\}$ or X. The support topology $\mathfrak{s}(\mathcal{A})$ is non-Hausdorff T_0-space topology, whose open subsets are $\{\varnothing, \{e_1\}, X\}$. Moreover, $\mathfrak{a} = \mathfrak{d} = \mathfrak{w}$ on $\mathcal{A}$, and $\mathcal{A}$ is a Lie-operator X-ring. Note that

$$\mathfrak{p}_{e_1} = \mathfrak{q}_{e_1}, \mathfrak{p}_{e_2} = \left\{ \frac{a}{b} \in \mathbb{Z}_{(p)} : a \in p\mathbb{Z} \right\}, \mathfrak{q}_{e_2} = \{0\}.$$

By Th 3, $\mathsf{Spf}(\mathcal{A}) = \mathsf{Spec}(\mathcal{A}) = X$ up to a homeomorphism, e_1 is a generic point which responds to $\{0\}$ and e_2 responds to $p\mathbb{Z}$.

4) The algebra of step functions on $[0,1)$. Let K be the algebra of all real-valued functions on the interval $[0,1)$. Then K acts on itself by means of the (left) multiplication operators, it is the so called left regular representation of the algebra K. We restrict this representation to the subring $\mathcal{A} \subseteq K$ of all step-functions like

$$a = a_0\chi_{\left[0,\frac{1}{n}\right)} + \sum_{k=1}^{n-1} a_k\chi_{\left[\frac{1}{n-k+1},\frac{1}{n-k}\right)}, a_k \in \mathbb{R},\ n \in \mathbb{N},$$

where χ_I indicates to the characteristic function of a subset $I \subseteq [0,1)$. The family $\left\{\chi_{\{1/n\}} : n \neq 1\right\}$ (we set $1/0 = 0$) of characteristic functions over the points

$$\Omega = \{0, 1/n : n \neq 1\} \subseteq \mathbb{R}$$

act as the family $X = \{e_n : n \neq 1\}$ of projections in $\mathrm{End}_{\mathbb{R}}(K)$. Undoubtedly, $\mathcal{A} \subseteq X'$. The support topology $\mathfrak{s}(\mathcal{A})$ in X is identified with the restricted topology of the subspace $\Omega \subseteq \mathbb{R}$, which is Hausdorff, compact space topology. Finally, $X_{\mathcal{K}} = X = \Omega = \mathrm{Spec}(\mathcal{A})$ and $\mathfrak{p}_{e_n} = \{a \in \mathcal{A} : a_n = 0\} = \mathfrak{q}_{e_n}$.

5) An example of an X-regular ring which is not Lie-operator X-ring. An X-regular ring may not be a Lie-operator X-ring. Indeed, let $X = \mathsf{Spec}\,(\mathbb{Z})$ be the space of all primes equipped with Zariski topology, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ the finite field related to the prime $p \in \mathbb{Z}$, which is $\mathbb{Z}$-module via pull back along the canonical ring homomorphism $\mathbb{Z} \to \mathbb{Z}_p$, and let $K = \bigoplus_p \mathbb{Z}_p$ be their direct sum. The space X is identified with the set $\{e_p\}$ of all canonical projections in $\mathsf{End}_{\mathbb{Z}}\,(K)$ with respect to the indicated expansion of K. Consider the representation

$$\varphi : \mathbb{Z} \to \mathsf{End}_{\mathbb{Z}}\,(K)\,, \quad \varphi\,(n)\left(\sum x_p\right) = \sum n \cdot x_p.$$

Put $\mathcal{A} = \varphi\,(\mathbb{Z}) \subseteq X'$. Note that

$$\begin{aligned} X_{\varphi(n)} &= \{e_p \in X : \varphi\,(n)\, e_p \text{ is a unit in } e_p\, \mathsf{End}_{\mathbb{Z}}\,(K)\, e_p\} \\ &= \{p \in X : n \notin p\mathbb{Z}\} \\ &= X_n. \end{aligned}$$

Thus the support topology $\mathfrak{s}\,(\mathcal{A})$ is reduced to the original Zariski topology of X, and $\mathcal{A}$ equipped with the discreet topology is an X-regular ring but it is not a Lie-operator X-ring.