SOME REMARKS ON SMOOTH SCHUBERT VARIETIES

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1. A Remarkable Formula

Let G be a semi-simple linear algebraic group over $\mathbb{C}$, B a fixed Borel subgroup of G (i.e. a maximal connected solvable subgroup), T a maximal torus in B, and let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ be their respective Lie algebras. Suppose $\Phi \subset \mathfrak{t}^*$ denotes the set of roots for the pair (G, T) and Φ^+ the positive roots associated to B. Letting $\mathfrak{t}$ act on $\mathfrak{g}$ via the adjoint representation, we have Cartan decompositions

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha, \;\; \mathfrak{b} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ denotes the weight space for the root α. There is a remarkable formula, due to Arnold Shapiro (and treated in various contexts by Kostant, Macdonald and Steinberg), which says

$$\prod_{\alpha \in \Phi^+} \frac{1 - t^{2\mathrm{ht}(\alpha)+2}}{1 - t^{2\mathrm{ht}(\alpha)}} = \prod_{i=1}^{\ell} \frac{(1 - t^{2d_i})}{(1 - t^2)} \tag{1}$$

$$= \prod_{i=1}^{\ell} (1 + t^2 + \cdots + t^{2m_i}). \tag{2}$$

Here $d_1, \ldots, d_\ell$ are the degrees of the fundamental generators of the ring of invariants $\mathbb{C}[\mathfrak{t}]^W$ for the Weyl group $W = N_G(T)/T$, and $m_1, \ldots, m_\ell$ are the exponents of (G, T), which differ from the fundamental degrees by 1. The height function $\mathrm{ht} : \Phi^+ \to \mathbb{Z}_+$ is defined by putting $\mathrm{ht}(\alpha) = \sum k_i$ provided $\alpha = \sum k_i \alpha_i$, where $\alpha_1, \ldots, \alpha_\ell$ are the simple roots for Φ^+ determined by B.

2. The Flag Variety G/B and its Schubert Varieties

The homogeneous space G/B is a projective G-variety called *the flag variety of G*. A fundamental result in the theory of algebraic groups says

$$gB \mapsto gBg^{-1}$$

is a bijection between G/B and the set of all Borel subgroups of G. By the Bruhat decomposition $G = BWB$, B acts on G/B with only finitely many orbits BwB/B. The Zariski closure $X_w = \overline{BwB/B}$ is called the *Schubert variety associated to w.*

Example. If $G = GL(n, \mathbb{C})$, then we may take B to be the upper triangular matrices and T the diagonal matrices in B. A nonsingular matrix defines a complete flag via the spans of its columns, and the isotropy group of the identity is B. Thus G/B parameterizes the set of complete flags in $\mathbb{C}^n$.

Let $\mathfrak{h} \subset \mathfrak{t}^*$ denote the real span of Φ. If $\alpha \in \Phi$, let r_α denote the reflection of $\mathfrak{h}$ through the hyperplane $\alpha = 0$ (with respect to a W-invariant inner product). Then $r_\alpha \in W$, and the set S of simple reflections (associated to the simple roots) . (W, S) defines a Coxeter system, so W has a length function $\ell : W \to \mathbb{Z}_+$ and a partial order $\leq$ known as the Bruhat order. The following fundamental properties of Schubert varieties are due to Chevalley.

• For any $w \in W$, the B orbit BwB/B is an affine space of (complex) dimension $\ell(w)$. Consequently, the decomposition

$$G/B = \bigcup_{w \in W} BwB/B$$

is an affine paving of G/B.

• For all $x, w \in W$, $BxB/B \subset \overline{BwB/B}$ iff $x \leq w$. Thus,

$$X_w = \bigcup_{x \leq w} BxB/B.$$

• Let $P_w(t)$ denote the Poincaré polynomial $\sum_{i=0}^{2\ell(w)} b_i(X_w)t^i$. Then

$$P_w(t) = \sum_{x \leq w} t^{2\ell(x)}.$$

Consequently, the odd Betti numbers of a Schubert variety vanish.

3. The tangent space $T_{eB}(X_w)$ of a smooth Schubert variety

A Schubert variety which is nonsingular as an algebraic variety is said to be smooth. Suppose $w \in W$, $\alpha > 0$ and $r_\alpha \leq w$. Then it follows that $\mathfrak{g}_{-\alpha} \subset T_{eB}(X_w)$. In general, let

$$\Phi(w) = \{\alpha > 0 \mid r_\alpha \leq w\}.$$

By Deodhar's Inequality [PSPM1994], $|\Phi(w)| \geq \ell(w)$ for all $w \in W$ with equality if X_w is smooth.

• If X_w is smooth, then

$$T_{eB}(X_w) = \sum_{\alpha \in \Phi(w)} \mathfrak{g}_{-\alpha},$$

but not conversely.

The relevance of the above identity is this: Since the connected solvable group B acts on (any) X_w with the unique fixed point $eB = B/B$, it follows from the Borel Fixed Point Theorem that X_w is smooth iff the Zariski tangent space $T_{eB}(X_w)$ has dimension $\ell(w)$. Note that $T_{eB}(X_w)$ is always B-module, hence also a $\mathfrak{b}$-module. This greatly restricts what form $\Phi(w)$ can take and suggests the following question.

Question: Suppose Ψ is a set of positive roots and $\mathfrak{f} = \sum_{\alpha \in \Psi} \mathfrak{g}_{-\alpha}$ is a B-submodule of $\mathfrak{g}/\mathfrak{b}$. That is, for any $\alpha \in \Psi$, if $\beta, \alpha - \beta > 0$, then $\alpha - \beta \in \Psi$. Does this imply $\mathfrak{f} = T_{eB}(X_w)$ for a smooth Schubert variety X_w with $|\Psi| = \ell(w)$?

4. The Poincaré Polynomial of a smooth Schubert variety

One can (sort of) generalize the remarkable formula (1) and (2) for a smooth Schubert variety as follows. Put $h_i = |\{\alpha \in \Phi(w) \mid \mathrm{ht}(\alpha) = i\}|$. Clearly, $h_1 > 0$.

• If X_w is smooth, then $h_i \geq h_{i+1}$ for all $i > 0$.

The above inequality fails in the singular case, as the following example shows.

Example Suppose α and β denote respectively the short and long simple roots of type B_2, and let $w = r_\alpha r_\beta r_\alpha$. Then $\Phi(w) = \{\alpha, \beta, 2\alpha + \beta\}$. Thus, $h_1 = 2,\ h_2 = 0$, and $h_3 = 1$.

Suppose X_w is smooth and $h_k > 0$ but $h_{k+1} = 0$. Let $d_i = h_i - h_{i+1}$ and $d_k = h_k$. Then,

$$P_w(t) = \prod_{\alpha \in \Phi(w)} \frac{1 - t^{2\mathrm{ht}(\alpha)+2}}{1 - t^{2\mathrm{ht}(\alpha)}} \tag{3}$$

$$= \prod_{1 \leq i \leq k} (1 + t^2 + \cdots + t^{2i})^{d_i}. \tag{4}$$

Note that in the case $X_w = G/B$, (4) explicitly gives the exponents of (G, T) along with their multiplicities. It also suggests possible exponents for smooth Schubert varieties, but doesn't give any geometric interpretation of these exponents.

Let us briefly (and somewhat vaguely) explain how this formula comes about. More details can be found in [MMJ2012]. Suppose X is a smooth projective variety of dimension n over $\mathbb{C}$ which admits an action of the upper triangular subgroup $\mathfrak{B}$ of $SL_2(\mathbb{C})$ such that the unipotent radical $\mathfrak{B}_u$ has unique fixed point, say o. The maximal torus $\mathfrak{T} \subset \mathfrak{B}$ on the diagonal defines a $\mathbb{G}_m$-action on X whose big open cell U is an affine cell about o which is $\mathfrak{T}$-isomorphic with $T_o(X)$, o going to the origin. This induces natural $\mathfrak{T}$-homogeneous coordinates $u_1, \dots, u_n$ on U. If V denotes the algebraic vector field on X induced by $\mathfrak{B}_u$, then the functions $a_i = V(u_i)$ define a regular homogeneous sequence (supported at o). Thus the ideal $(a_1, \dots, a_n)$ defines a punctual scheme Z supported at o. The key fact is that there exists a graded $\mathbb{C}$-algebra isomorphism

$$\mathbb{C}[Z] = \mathbb{C}[u_1, \dots, u_n]/(a_1, \dots, a_n) \cong H^*(X, \mathbb{C}).$$

By elementary commutative algebra, the Poincaré polynomial of X satisfies

$$P_X(t) = \prod_{i=1}^{n} \frac{1 - t^{\deg(a_i)}}{1 - t^{\deg(u_i)}}.$$

Using the fact that an arbitrary X_w admits such an action, the formula (3) for $P_w(t)$ in the smooth case follows from an elementary calculation. The reason that any X_w admits an action of $\mathfrak{B}$ as above is more interesting. It involves a choice of principal nilpotent e in $\mathfrak{b}$. Let $e_\alpha \in \mathfrak{g}_\alpha$ be nonzero, and put

$$e = \sum_{i=1}^{\ell} e_{\alpha_i}.$$

(Recall, the α_i are the simple roots.) Thus we obtain a G_a-action on X_w from the one parameter group $s \to \exp(se)$, which induces an action of $\mathfrak{B}$ by standard results. The reason $\mathfrak{B}_u$ has eB as its unique fixed point is that the unique Borel subalgebra containing e is $\mathfrak{b}$.

Formula (3) suggests the following

Question: Suppose (3) holds for X_w. Does this imply X_w is smooth?

Unfortunately, there is a counter example in type G_2. However, this is the only counter example the author knows of.

5. The Poincaré Polynomial and Rational Smoothness

There is a well known weakening of the notion of smoothness, called rational smoothness, which was studied in depth by Kazhdan and Lusztig in the context of representation theory. The definition is too technical to give here, but we can state a list of conditions (some elementary) equivalent to rational smoothness.

Theorem (see [PSPM1994]) *The following are equivalent:*

(a) *X_w is rationally smooth;*

(b) *if $x \leq w$, then the Kazhdan-Lusztig polynomial $P_{x,w}(q) = 1$;*

(c) *$P_w(t)$ is palindromic; and*

(d) *every vertex of the Bruhat graph of X_w is on exactly $\ell(w)$ edges.*

The equivalence of (a) and (b) is a deep result due to Kazhdan and Lusztig. The equivalence of (b), (c) and (d) is due to the speaker and Dale Peterson. The Bruhat graph of X_w is the graph whose vertices are the xB, where $x \leq w$. Two vertices xB, yB are joined by an edge iff they lie on a T-stable curve in X_w. Since the fixed point set $(X_w)^T$ coincides with $\{xB \mid x \leq w\}$, the Bruhat graph coincides with what is often called the momentum graph of the pair (X_w, T). Every T-stable curve C in G/B contains exactly two fixed points, which have the form xB and $r_\alpha xB$ for a unique $\alpha > 0$. If $x^{-1}(\alpha) < 0$, then C is the closure of $x_\alpha(\mathbb{C})xB$, where $x_\alpha : \mathbb{C} \to B$ is the root subgroup associated to α. If $x, r_\alpha x \leq w$, then $C \subset X_w$.

Since smooth Schubert varieties are obviously rationally smooth, one can ask which rationally smooth Schubert varieties are smooth. This is partially answered by a beautiful result of Peterson

Theorem *If G is of type ADE, then every rationally smooth Schubert variety in G/B is in fact smooth.*

The proof uses a notion we will briefly explain below, namely Peterson translates. More generally, we have the following recent result which brings into context the question of when $\Phi(w)$ defines a $\mathfrak{b}$-module.

• If G doesn't contain any G_2-factors and X_w is rationally smooth, then X_w is smooth iff $\sum_{\alpha\in\Phi(w)} \mathfrak{g}_{-\alpha}$ is a $\mathfrak{b}$-submodule of $T_{eB}(X_w)$.

We refer to [TG2011] for the proof. The following amusing corollary is a simple consequence.

- For any w, X_w is smooth iff $X_{w^{-1}}$ is smooth.

6. The Inversion Arrangement and a result of Oh, Postnikov and Yoo

Recall that the *inversion set of $w \in W$* is by definition

$$I(w) = \{\alpha > 0 \mid w(\alpha) < 0\}.$$

Note that this means $I(w^{-1})$ is precisely the set of T-weights for $T_{wB}(X_w)$. Note also that wB is always a smooth point of X_w. Recently, Oh, Postnikov and Yoo [JCT2010] proved the following surprising result for rationally smooth (hence smooth) Schubert varieties in type A. This was later extended by Oh and Yoo to rationally smooth Schubert varieties in classical types.

Theorem *Assume $G = SL(n, \mathbb{C})$, and consider the hyperplane arrangement $\mathcal{A}$ in $\mathfrak{h}$ (see Section 1) defined by the hyperplanes $\alpha = 0$ for all $\alpha \in I(w)$. Let $R_w(t)$ denote the wall crossing polynomial for this arrangement. Then a Schubert variety X_w is (rationally) smooth iff $R_w(t) = P_w(t)$.*

The *wall crossing polynomial* is defined as follows. Fix a connected component C_0 of the complement $\mathfrak{h} \backslash \mathcal{A}$, and for any other component C, let $n(C, C_0)$ denote the number of walls one needs to cross to pass from C_0 to C. Then

$$R_w(t) = \sum_C t^{2n(C,C_0)}.$$

Although this result is useless in practice, it establishes a beautiful link between global smoothness (or rational smoothness) and the inversion arrangement.

Question: Since $I(w^{-1})$ consists of all T-weights of $T_{wB}(X_w)$ and $R_w(t) = R_{w^{-1}}(t)$, is there a criterion to determine when X_w is smooth just from $I(w)$?

7. Peterson translates and the inversion set

In this section, we will describe an explicit map from $I(w)$ to $\Phi(w)$ in the smooth setting. This map is defined for all $I(w)$ but it turns out in general to be multi-valued in the singular case. Fix X_w and assume $x, r_\alpha x \leq w$. When $\alpha > 0$ and $x^{-1}(\alpha) < 0$, then $r_\alpha x < x$. Recall that the one parameter group $x_\alpha : \mathbb{C} \to B$ gives the T-stable curve $C(x, \alpha) = \overline{x_\alpha(\mathbb{C})xB}$ in X_w whose fixed point set is $\{xB, r_\alpha xB\}$. The limit

$$\lim_{s \to \infty} dx_\alpha(s) T_{xB}(X_w)$$

exists as a T-stable subspace of $T_{r_\alpha xB}(X_w)$ called the *Peterson translate of $T_{xB}(X_w)$ along $C(x,\alpha)$*. If X_w is smooth at xB and $r_\alpha xB$, then we can view this map as being a bijection of the set of T-weights for $T_{xB}(X_w)$ to those of $T_{r_\alpha xB}(X_w)$. We note that although the above definition works in the complex case, a more subtle approach works for any algebraically closed field, independent of the characteristic. See [INVENT2003] for more details.

Irregardless of whether or not X_w is smooth, one can use Peterson translation along any path in the Bruhat graph of X_w from wB to eB by applying Peterson translation to the successive translates. This gives a map depending on the path $\mathcal{P}$ from wB to eB taking the T-weights at wB to those at eB, but in general, this map fails to be onto the weights of $T_{eB}(X_w)$, and in general it depends on the path. The following result seems to be of some importance.

• X_w is smooth iff Peterson translation from wB to eB is independent of path in the Bruhat graph of X_w.

The main question is thus how to determine when Peterson translation is independent of path. A secondary question is how to use Peterson translation to determine when X_w is rationally smooth. Unfortunately, the natural approach doesn't seem to work: Peterson translation doesn't leave the wall crossing polynomial invariant. But how it changes isn't clear yet. Some conjectures have been made by Ed Richmond and William Slofstra, and it is hoped that there will be further developments.

8. References

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