Initial singularity of flat Lorentzian spacetimes, according to M. Belraouti

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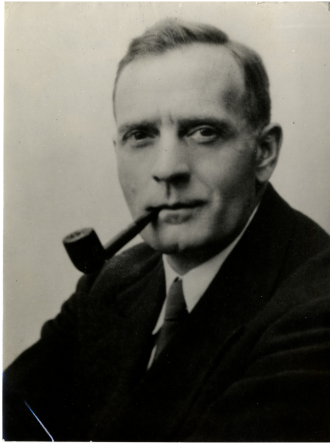
Content

1. What is spacetime geometry ?
2. Cosmological models
3. Special time functions
4. The geometry of the initial singularity

Edwin Powell Hubble (20/11/1889 - 28/09/1953)



The history of astronomy is a history of receding horizons.

Content

1. What is spacetime geometry ?
   - General Notions
   - Expansion of the universe

Space-time geometry

$M = S\times]a, b[$ with Lorentzian metric :

$$g := g_t(.) - \mathbf{N}dt^2$$

g_t : 1-parameter family of Riemannian metrics ;
$\mathbf{N}$: « lapse » function
S compact $\rightarrow$ *globally hyperbolic spatially compact* (GHC)

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Maximality condition (completeness) $\to$ MGHC spacetimes.
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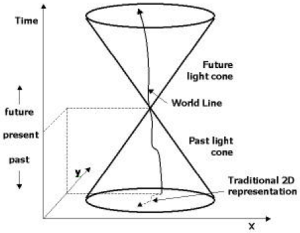
Behavior of g_t when $t \rightarrow a, b$?

Causality

- Non-differentiable causal curve :

$$\forall t,\ \exists \epsilon, U \ \ st.\ \ \forall s \in]0, \epsilon[\ c(t+s) \in J^+_U(c(t))$$

Figure 1. Minkowski's light cone



- **Inextendible** causal curve
- **Proper time** $T(c)$.

Time functions

- A time function on M is a map $t : M \to]a, b[\subset \mathbb{R}$ increasing along every causal curve.

Time functions

- A time function on M is a map $t : M \to]a, b[\subset \mathbb{R}$ increasing along every causal curve.
- If M is GHC, then any time function is a **Cauchy time function,** ie. the restriction to every inextensible causal curve is onto. The spacetime then admits a splitting $M = S \times]a, b[$ with a Lorentzian metric of the form :

$$g_t - \mathbf{N}^2 dt^2$$

where the coordinate t is the given time function.

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What means expansion ?

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A special time :

Definition (*"The time from the Big-Bang"*)

The *cosmological time* is the map $\tau : M \to]0, +\infty]$ defined by :

$\sup\{T(c)/c : [0,1] \to M$ future oriented causal curve s.t. $c(1) = p\}$

Content

2 Cosmological models
- Models
- Friedman-Lemaître-Robertson-Walker spacetimes
- Locally flat spacetimes

$$g_t - \mathbf{N}^2 dt^2$$

$\mathfrak{g}_\kappa$: Riemannian metric of constant sectional curvature $\kappa = -1, 0$ or $+1$.

- **Minkowski** : $g_t = \mathfrak{g}_0$, $\mathbf{N} = 1$
- **Schwarzschild** : $g_t = g_0 = \mathbf{N}^{-1} dr^2 + r^2 \mathfrak{g}_1$, $\mathbf{N} = \sqrt{1 - (2M/r)}$,
- **Anti-de Sitter** : $g_t = \mathfrak{g}_{-1}$ on $\mathbb{H}^n$, $\cosh(\mathbf{N}) = d_{hyp}(., 0)$.

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All these examples are static.

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$$\begin{array}{rcl} \mathbf{N} & = & 1 \\ g_t & = & f(t)\mathfrak{g}_\kappa \end{array}$$

$$\begin{array}{lrcl} \textbf{Pressure } p\textbf{ :} & -8\pi p & = & 2\frac{f''}{f} + (\frac{f'}{f})^2 + \frac{\kappa}{f^2} \\ \textbf{Energy } \rho\textbf{ :} & 8\pi\frac{\rho}{3} & = & (\frac{f'}{f})^2 + \frac{\kappa}{f^2} \\ \textbf{Hubble « Constant » :} & H(t) & = & \frac{f'}{f} \end{array}$$

«Physical» cases : $\rho + 3p > 0$, and $H(t_0) > 0$ for some t_0.

Theorem (Big-Bang)

f vanishes at some t_*. Besides, $f'(t_*) = +\infty$.

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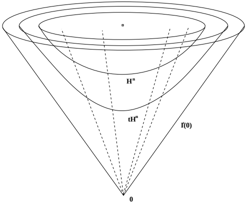
Nota Bene : Expansion !

Particular cases (unphysical)

- **De Sitter space :** $\mathbf{N} = 1$, $g_t = \cosh^2(t)\mathfrak{g}_1$
- **«Standard» spacetime :** $\mathbf{N} = 1$, $g_t = t^2\mathfrak{g}_{-1}$

This spacetime is locally flat, isometric to the future of a point in Minkowski space !



Let Γ be a cocompact lattice of $SO_0(1, n)$: the quotient is MGHC, diffeomorphic to $S \times \mathbb{R}$ where $S = \Gamma\backslash\mathbb{H}^n$.

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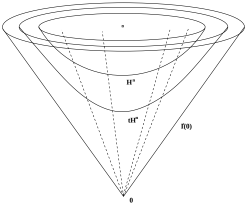

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Nota Bene : Expansion !

All these examples are **conformally** static :

$$g_t = f(t) g_0$$

Examples where the conformal class $[g_t]$ changes with t ?

Content

2 Cosmological models
- Models
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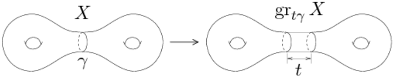
Grafting

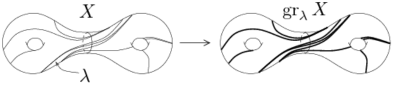
– Grafting –

Start with X, a closed hyperbolic surface, and γ, a simple closed hyperbolic geodesic. Cut X along γ and insert a Euclidean cylinder of length t.



The result is $\mathrm{gr}_{t\gamma} X$, the **grafting** of X along $t\gamma$.

Grafting extends continuously to limits of weighted geodesics, i.e. **measured laminations**. Intuitively, grafting replaces λ with a thickened version that has a Euclidean metric. [Thurston; Kamishima-Tan]



For every measured geodesic lamination λ on a hyperbolic surface S the **grafting** $\mathrm{gr}_\lambda S$ is a Riemannian surface with $C^{1,1}$ regularity (Thurston's metric).

Theorem (Mess, 1990)

The product $S \times]0, +\infty[$ equipped with the metric $t^2 \, \mathrm{gr}_{\lambda/t^2} S - dt^2$ is locally isometric to the Minkowski space.

$\Longrightarrow$ Flat cosmological spacetime $M_\lambda(S)$.

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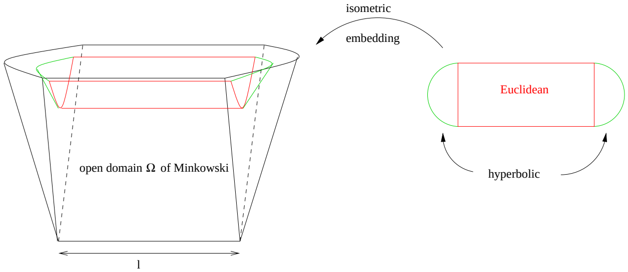
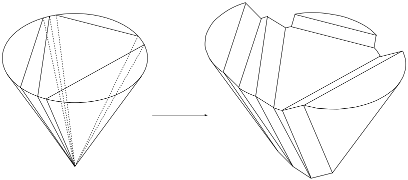
Nota Bene : Expansion !

Insertion of Misner strips

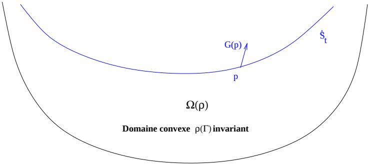


This construction extends to general measured geodesic laminations (we insert Misner strips, with null length for non-isolated leaves), in a Γ-equivariant way if λ is Γ-invariant.



Regular convex domains

$M_\lambda(S)$ is the quotient of a convex open domain $\Omega(\rho)$ in $\mathbb{R}^{1,2}$ preserved by the image of a representation $\rho : \Gamma = \pi_1(S) \to SO(1,2) \ltimes \mathbb{R}^{1,2}$



The boundary $\partial\Omega(\rho)$ also has a geometry : one can compute the length of curves inside $\partial\Omega(\rho)$, hence the (pseudo)distance between point in $\partial\Omega(\rho)$.

Content

3 Special time functions
- Cosmological time
- CMC time
- K-time

Cosmological time in spacetimes of constant curvature

Definition

The cosmological time is the map $\tau : M \to]0, +\infty]$ defined by :

$\sup\{T(c)/c : [0,1] \to M$ future oriented causal curve s.t. $c(1) = p\}$

Theorem

Up to time reversal, the cosmological time of (non-elementary) GHC spacetimes of constant sectional curvature is in expansion, at least near the initial singularity.

Elementary cases are easy to deal with «by hand»...

Content

3 Special time functions
- Cosmological time
- CMC time
- K-time

Definition

A **CMC time** is a time function $t_{CMC} : M \to \mathbb{R}$ such that every level set $t_{CMC}^{-1}(H)$ has constant mean curvature H.

According to the maximum principle, every CMC spacelike hypersurface in a spacetime admitting a CMC time function is a level set of this time function. In particular, CMC time, if any, is unique.

Theorem (Barbot, Béguin, Zeghib (2005) ; + Andersson (2007))

Regular MGHC spacetimes of constant sectional curvature $k \leq 0$ admit CMC time functions. In the de Sitter case $k = +1$, this is also true in dimension $2 + 1$, or «near» conformally static spacetimes, but not necessarily in other cases.

Content

3. Special time functions
   - Cosmological time
   - CMC time
   - K-time

From now we only consider the dimension $2+1$.

Definition

A **K-time** is a time function $t_K : M \to \mathbb{R}$ such that :
– every level set $t_K^{-1}(k)$ has constant Gauss curvature k (minus the product of the principal curvatures),
– t_K is **convex** (the future of every $t_K^{-1}(k)$ is geodesically convex).

Once more, by maximum principle, K-time, if any, is unique.

Theorem (Barbot, Béguin, Zeghib (2007))

Every regular MGHC spacetimes of constant sectional curvature ≥ 0 admits a K-time. In anti-de Sitter case $k=-1$, the K-time exists, but only in the past of the ***convex core.***

Content

4. The geometry of the initial singularity
   - The $2+1$-dimensional case
   - The general case
   - Recent results and open questions

Geometry of the Big-Bang ?

- (Benedetti-Guadagnini ; Bonsante) For $t \to 0$, the cosmological levels $\mathrm{gr}_{t\lambda} S$ converge in the Haudorff-Gromov equivariant topology to the real tree dual to λ.
- (Andersson) If λ is a multicurve, the CMC levels $t_{CMC}^{-1}(H)$ for $H \to -\infty$ converge in the Haudorff-Gromov equivariant topology to the real tree dual to λ.

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General λ ? K-levels ?

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Flat spacetimes in expansion

Let M be a MGHC spacetime, with Cauchy hypersurface S, and let $T : M \to]0, +\infty[$ be a time function. Let $g_t^T - Ndt^2$ be the splitting of the metric (every line $\{x\} \times]0, +\infty[$ is a gradient line of T).

Definition

The time function $T : M \to]0, +\infty[$ is **in expansion** if the metrics g_t^T increases with t.

In particular, the second fundamental form is positive. In the flat case, it implies that every (S, g_t^T) is CAT(0).

Family of decreasing spaces

Proposition

Let $(S, d_t)_{(t>0)}$ be a family of metric spaces such that, for every x, y in S, and every positive real numbers s, t we have :

$$s \leq t \implies d_s(x,y) \leq d_t(x,y)$$

Then, $d_0(x,y) = \lim_{t \to 0} d_t(x,y)$ is a well-defined pseudo-distance on S.

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Definition

The **cleaning** of (S, d_0) is the metric space $(\bar{S}, \bar{d}_0)$ where $\bar{S}$ is the quotient of S by the relation identifying x with y if $d_0(x, y) = 0$. The distance $\bar{d}_0(\bar{x}, \bar{y})$ is defined as $d_0(x, y)$ if x, y are any representants of $\bar{x}$, $\bar{y}$.

Family of decreasing CAT(0) spaces

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- (S, d_t) proper $\not\Rightarrow$ $(\bar{S}, \bar{d}_0)$ proper.

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- (S, d_t) complete $\not\Rightarrow$ $(\bar{S}, \bar{d}_0)$ complete.
- (S, d_t) proper $\not\Rightarrow$ $(\bar{S}, \bar{d}_0)$ proper.

However :

Theorem (Ascoli-Arzela in non-proper metric limit spaces)

Let $(f_n)_{n\in\mathbb{N}}$ a family of C-Lipschitz maps $f_n : S \to \mathbb{R}$. Then, up to a subsequence, f_n converges in the compact-open topology.

Preuve : For every $t > 0$, and every x, y in S we have $|f_n(x) - f_n(y)| \leq Cd_0(x, y) \leq Cd_t(x, y)$. Hence, up to a subsequence, f_n converges towards $f_\infty : S \to \mathbb{R}$. This map is C-Lipschitz with respect to d_t for every $t > 0$, hence for d_0. $\square$

Flat spacetime : quotient of $\Omega \subset \mathbb{R}^{1,n}$ by $\Gamma \subset \mathrm{SO}(1,n) \ltimes \mathbb{R}^{1,n}$
Retraction map $r : \Omega \to \Sigma \subset \partial\Omega$
Induced (pseudo)metric d_Σ on Σ
Let l_1, l_2 be two gradient lines for the cosmological time. Let $l_i(t)$ be the unique point in l_i of cosmological time t.

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Theorem (Bonsante)

$$d_\Sigma(r(l_1), r(l_2)) = \lim_{t\to 0} d_t(l_1(t), l_2(t))$$

Therefore, (Σ, d_Σ) is isometric to the initial singularity $(\bar{S}, \bar{d}_0^\tau)$ defined by the cosmological time τ.

Content

4 The geometry of the initial singularity
- The $2+1$-dimensional case
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Uniqueness of the geometry in the $2+1$-dimensional case

Theorem (Belraouti, 2011)

Let $M_\lambda(S)$ be a $2+1$-dimensional MGHC flat spacetime. Let $T : M \to]0, +\infty[$ be a time function in expansion. Then, the associated family of metric spaces (S, g_t^T) converge for the equivariant Gromov-Hausdorff topology to the real tree dual to λ.

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As a corollary, Benedetti-Guadagnini Conjecture is true : CMC levels converge to the real tree. K-levels too.

Nota Bene

The spectral convergence is an intermediate "obvious" result.

Metric properties of the initial singularity in higher dimensions.

Theorem (Belraouti, 2012)

Let M be a flat MGHC regular spacetime. Then the initial singularity $(\bar{S}, \bar{d}_0^\tau)$ is geodesic and $CAT(0)$.

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Let M be a flat MGHC regular spacetime. Then the initial singularity $(\bar{S}, \bar{d}_0^\tau)$ is geodesic and $CAT(0)$. Moreover, if $T : M \to]0, +\infty[$ is a time function in expansion, then the associated family of metric spaces (S, g_t^T) converge for the equivariant Gromov-Hausdorff topology to $(\bar{S}, \bar{d}_0^\tau)$.

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Let M be a flat MGHC regular spacetime. Then the initial singularity $(\bar{S}, \bar{d}_0^{\tau})$ is geodesic and CAT(0). Moreover, if $T : M \to]0, +\infty[$ is a time function in expansion, then the associated family of metric spaces (S, g_t^T) converge for the equivariant Gromov-Hausdorff topology to $(\bar{S}, \bar{d}_0^{\tau})$.

Questions and work in progress :

- Study the isometry group the CAT(0)-space $(\bar{S}, \bar{d}_0^T)$.
- Find an example of flat MGHC spacetime for which $(\bar{S}, \bar{d}_0^{\tau})$ is not a real tree.
- Study the (Hausdorff-Gromov)-limit of $(S, \sigma(t)^2 g_t^T)$ for some renormalization function σ.

Conclusion

THANKS FOR YOUR ATTENTION !