Double coset varieties and actions of maximal torus on spherical homogeneous spaces

Artem Anisimov

Askoldfest, June 2012

Let G be a complex affine algebraic group and $F, H \subseteq G$ be two subgroups of it. Consider the following action of $F \times H$ on G: $(f, h) \circ g = fgh^{-1}$.

Definition

The *double coset variety* $F\backslash\backslash G/\!/H$ is the underlying variety of the categorical quotient for the described action $H \times F : G$.

Two questions

Question

Does the variety $F\backslash\!\backslash G/\!/H$ exist if one of the subgroups F or H is not reductive?

Question

Can the variety $F\backslash\!\backslash G/\!/H$ be isomorphic to an affine space?

If the subgroups F and H are reductive then the second question can be stated this way: can the algebra ${}^{F}\mathbb{C}[G]^{H}$ of $F \times H$-invariant regular functions on G be free?

There is no analogue of the Chevalley Theorem

Example

There exists a reductive group G and two subgroups $F, H \subset G$ such that the variety $F\backslash\backslash G/\!/H$ does not exist.

Example

There exists a reductive group G and two subgroups $F, H \subset G$ such that the algebra $R = {}^F\mathbb{C}[G]^H$ is finitely generated and the natural morphism $\pi : G \to \operatorname{Spec} R$ is surjective, but π is not a categorical quotient.

Idea behind the examples

1. Take an action $F : X$ which is known to have a desired “bad” property (e. g. $X/\!/F$ does not exist).
2. Represent X as a homogeneous space of some group G: $X = G/H$.
3. Find a subgroup $F' \subset G$ such that the action $F' : G/H$ is isomorphic to $F : X$.
4. Profit! The variety $F'\backslash\backslash G/\!/H$ has the same bad property as $X/\!/F$.

Construction of the first example

Fact

Let $U = (\mathbb{C}, +)$ *and* $X = \mathrm{Mat}_{2\times 2}(\mathbb{C})$. *Consider the action*

$$a \circ X = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot X.$$

The action $U : X$ *admits no categorical quotient in the category of algebraic varieties.*

Construction of the first example

Fact

Let $U = (\mathbb{C}, +)$ *and* $X = \mathrm{Mat}_{2\times 2}(\mathbb{C})$. *Consider the action*

$$a \circ X = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot X.$$

The action $U : X$ *admits no categorical quotient in the category of algebraic varieties.*

Use

$$X \cong \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & * & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

Construction of the first example

Fact

Let $U = (\mathbb{C}, +)$ *and* $X = \mathrm{Mat}_{2\times 2}(\mathbb{C})$. *Consider the action*

$$a \circ X = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot X.$$

The action $U : X$ *admits no categorical quotient in the category of algebraic varieties.*

Use

$$\begin{pmatrix} 1 & * & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} : X \cong \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & * & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

A better framework for double coset varieties

Theorem

Let G be a connected affine algebraic group and $F, H \subset G$ be closed connected subgroups with trivial character groups. Suppose that the algebra ${}^F\mathbb{C}[G]^H$ is finitely generated and let $\pi : G \to \mathrm{Spec}({}^F\mathbb{C}[G]^H)$ be the canonical morphism. Then $F\backslash\backslash G/\!/H$ exists as a constructible space and the map $\pi : G \to \pi(G)$ is the constructible quotient for the action of $F \times H$ on G.

An observation

Fact (Onishchik; Kraft and Popov)

Let G be a reductive group and $H \subsetneq G$ be a closed proper subgroup. Then the homogeneous space G/H is not isomorphic to an affine space (for a reductive subgroup H: the algebra $\mathbb{C}[G]^H$ is not free).

Fact

$T\backslash\backslash \mathrm{SL}_2/\!/T \cong \mathbb{A}^1$ and $T\backslash\backslash \mathrm{SL}_4/\!/\mathrm{Sp}_4 \cong \mathbb{A}^2$ (T denotes a maximal torus in SL_2 or in SL_4).

An observation

Fact (Onishchik; Kraft and Popov)

Let G be a reductive group and $H \subsetneq G$ be a closed proper subgroup. Then the homogeneous space G/H is not isomorphic to an affine space (for a reductive subgroup H: the algebra $\mathbb{C}[G]^H$ is not free).

Fact

$T\backslash\!\backslash \mathrm{SL}_2/\!\!/T \cong \mathbb{A}^1$ and $T\backslash\!\backslash \mathrm{SL}_4/\!\!/\mathrm{Sp}_4 \cong \mathbb{A}^2$ (T denotes a maximal torus in SL_2 or in SL_4).

Remark

$T \subset \mathrm{SL}_2$ and $\mathrm{Sp}_4 \subset \mathrm{SL}_4$ are spherical pairs.

Question

How to find all pairs $H \subset G$ with G classical and H connected spherical reductive such that $T\backslash\!\!\backslash G/\!\!/H$ is an affine space, or, equivalently, the algebra ${}^T\mathbb{C}[G]^H$ is free?

Theorem

Let G be a classical algebraic group, $T \subset G$ be a maximal torus, $H \subset G$ be a connected spherical reductive subgroup and let $\pi : G \to T\backslash\backslash G/\!/H$ be the quotient morphism. Then the double coset variety $T\backslash\backslash G/\!/H$ is an affine space if and only if the image $\pi(e)$ of the identity element is a regular point.

Theorem

The double coset variety $T\backslash\backslash G/\!/H$ is an affine space if and only if the groups H and G are listed in the following table:

G	H	$\dim T\backslash\backslash G/\!/H$
SL_{n+1}	$\mathrm{S}(\mathrm{GL}_n \times \mathrm{GL}_1)$	n
SL_4	Sp_4	2
SO_{2n+1}	SO_{2n}	n
SO_{2n}	SO_{2n-1}	$n-1$
SO_4	GL_2	1
SO_8	Spin_7	3
SO_6	GL_3	3
SO_4	$\mathrm{SO}_2 \times \mathrm{SO}_2$	2
SO_3	GL_1	1
Sp_4	$\mathrm{Sp}_2 \times \mathrm{Sp}_2$	2

Sketch of the proof

The main Lemma

Let $H \subseteq G$ be a reductive subgroup and $\pi : G \to T\backslash\backslash G/\!/H$ be the quotient morphism. Let Z be the categorical quotient for the action $T \cap H : \mathrm{Lie}\, G / (\mathrm{Lie}\, T + \mathrm{Lie}\, H)$ induced by the adjoint action of $T \cap H$ on $\mathrm{Lie}\, G$. Then the point $\pi(e) \in T\backslash\backslash G/\!/H$ is regular if and only if Z is an affine space.

This lemma helps to rule out those pairs $H \subset G$ that can not have $T\backslash\backslash G/\!/H$ isomorphic to an affine space.

For the rest of spherical pairs $H \subset G$ it turns out that $T\backslash\backslash G/\!/H$ is an affine space.

An open question

Observation

If a pair $H \subset G$ is in the table above and G is simple then $\operatorname{rk}\Lambda_+(G/H) = 1$.

An open question

Observation

If a pair $H \subset G$ is in the table above and G is simple then $\operatorname{rk} \Lambda_+(G/H) = 1$.

Conjecture (Panyushev)

Let G be a simple algebraic group and $H \subset G$ a connected spherical reductive subgroup. If $T\backslash\!\backslash G /\!\!/ H$ is an affine space then $\operatorname{rk} \Lambda_+(G/H) = 1$.

If the conjecture of Panyushev is true then

Theorem

Let G be a **classical** algebraic group, $T \subset G$ be a maximal torus, $H \subset G$ be a connected spherical reductive subgroup and let $\pi : G \to T\backslash\backslash G/\!/H$ be the quotient morphism. Then the double coset variety $T\backslash\backslash G/\!/H$ is an affine space if and only if the image $\pi(e)$ of the identity element is a regular point.

can be replaced with

Theorem

Let G be a **simple** algebraic group, $T \subset G$ be a maximal torus, $H \subset G$ be a connected spherical reductive subgroup and let $\pi : G \to T\backslash\backslash G/\!/H$ be the quotient morphism. Then the double coset variety $T\backslash\backslash G/\!/H$ is an affine space if and only if the image $\pi(e)$ of the identity element is a regular point.