Lie-operator rings and noncommutative affine schemes

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One of the principal foundations of noncommutative algebraic geometry is to extend the concept of affine schemes to noncommutative rings. As in the commutative case the main motivation for this is to represent a noncommutative ring as the ring of "functions" over its spectrum. The classical result, which is due to I. M. Gelfand, asserts that a commutative Banach algebra can be realized as the algebra of continuous functions over the space of all its maximal ideals modulo its Jacobson radical. For commutative rings the assertion is generalized in the following elegant way, which is due to A. Grothendieck. A commutative ring A is the ring of all global sections $\Gamma(\mathrm{Spec}(A), \mathcal{O})$ of the structure sheaf $\mathcal{O}$ of the ring A over its scheme $\mathrm{Spec}(A)$ (the space of all prime ideals of A) up to an isomorphism. The noncommutative ring would generalize here a commutative ring of regular functions on a commutative scheme. In the Banach algebra context the indicated direction is closely related to the noncommutative functional calculus problem and noncommutative spectral theory [1-4]. The Lie algebra methods allow us to handle the problem and formulate the relevant restrictions for noncommutative functional calculus to be built up. As shown in [5-7] the most reliable case of an operator family which admits a noncommutative spectrum and noncommutative (Taylor) functional calculus is a family generating a Lie-nilpotent algebra. Based on this result from analysis, it is reasonable to expect good behavior of Lie-nilpotent rings in noncommutative algebraic geometry. This proposal has been partially supported in Kapranov's theory of NC-schemes [8]. The NC-algebras present an interesting subclass of Lie-nilpotent algebras.

Another motivation for the present talk is to use a purely operator approach to noncommutative schemes without any sheaf constructions as classically done. The operator realization of many noncommutative algebras is the well known fact. For example, each C^*-algebra A admits an exact C^*-representation in $\mathcal{B}(H)$ for some Hilbert space H, due to Gelfand-Naimark theorem. Furthermore each abstract operator space V is a subspace of $\mathcal{B}(H)$ up to a matrix isometry, due to Ruan's representation theorem. Each quantum space [9] V turns out to be a concrete quantum space on a quantum domain X, which consists of unbounded operators.

In the present talk we propose a new approach to noncommutative schemes of Lie-complete associative rings, which is based on the quantum space constructions [10-12]). These noncommutative rings are represented as linear (unbounded) operators over the quantum domains rather than regular functions (or global sections) over schemes. Commutative functional rings are replaced by operator rings, whose localizations are reduced to the operators over invariant submodules. We prove that the formal spectrum $\mathrm{Spf}(\mathcal{A})$ of a Lie-operator ring $\mathcal{A}$ over the domain X is the Kolmogorov completion of X. Moreover, each Lie-complete ring A turns out to be a Lie-operator ring $\mathcal{A}$ over the formal inflation X of $\mathrm{Spf}(A)$ up to a topological isomorphism. The framework proposed present an interest in the commutative case as well.

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